A geometric representation of the generalized mean curvature

by Kouta Takano
Graduate School of Science
Saitama University, Japan

Abstract

A varifold is a generalization of a differential manifold using Radon measures. The theory of varifolds is a central topic in geometric measure theory. Any varifold possesses a notion similar to “the area”, and the generalized mean curvature is defined through the first variation of “the area”. If a varifold has C^2 regularity, then the generalized mean curvature coincides with the classical mean curvature. Furthermore, if the generalized mean curvature vector has some integrablity, then we obtain some regularity of the varifold. In this sense the generalized mean curvature contains information concerning its shape. However, it is not known that generalized mean curvature vector is represented without the first variation. In this paper, under the $C^{1,\alpha}$ regularity condition, for $\alpha > 1/3$, we give a geometric representation of the generalized mean curvature using a limit of integral averages suggested by the Menger curvature.

Contents

Chapter 1

Introduction

The theory of varifolds is a branch of geometric measure theory. It is a generalized notion of differentiable submanifolds using Radon measures, and, for example, we know the monotonicity formula, the compactness theorem and the isoperimetric inequality for varifolds. Today, varifold theory is used for the study of minimal surfaces as well as mean curvature flows, and some regularity theorems are known.

A varifold V is a Radon measure on the product topological space of $\mathbb{R}^n$ and the Grassmannian $\mathrm{G}(n,k)$. We consider a differentiable submanifold M on $\mathbb{R}^n$ as a varifold $\mathrm{v}(M)$ defined by

$$\mathrm{v}(M)(f) = \int_M f(x, T_x M)\, d\mathscr{H}^k x, \tag{1.1}$$

for a continuous function f on $\mathbb{R}^n \times \mathrm{G}(n,k)$ with compact support. Here $\mathscr{H}^k$ is the k-dimensional Hausdorff measure. For a varifold V, we can define the first variation $\delta V(g)$ by

$$\delta V(g) = \int Dg(x) \cdot S\, dV(x, S),$$

where g is a C^1 vector field. In particular, if the total variation measure of δV is absolutely continuous with respect to the area of V, then there exists a V measurable vector field $\mathrm{h}(V, \cdot)$ such that

$$\delta V(g) = -\int \mathrm{h}(V, x) \cdot g(x)\, dV(x, S) \tag{1.2}$$

for any C^1 vector field g. The vector field $\mathrm{h}(V,\cdot)$ is called the generalized mean curvature vector of V, and coincides with the classical mean curvature vector when V has C^2 regularity.

In [1], Allard showed the regularity theorem which says that if the generalized mean curvature vector has the L^p integrability, then the varifold can be locally written by the graph of a $C^{1,1-k/p}$ function, where p is greater than dimension of the varifold. The monotonicity formula and the isoperimetric inequality mentioned above are represented by the first variation and the generalized mean curvature. A varifold is called *integral* if it is represented by a countable summation of (1.1) type varifolds, and we know that the generalized mean curvature vector of such an integral varifold is contained in the orthogonal space almost everywhere ([4]). Hence the generalized mean curvature reflects the geometric shape of varifolds in this sense. Other than (1.2), we are not aware of different representations of the generalized mean curvature.

In this paper, we give a representation of the generalized mean curvature vector using a limit of integral averages of a discretization of the classical mean curvature vector. Roughly speaking, the assertion of our main theorem is as follows. If a varifold V is locally $C^{1,\alpha}$ with $\alpha > 1/3$, then the generalized mean curvature vector satisfies

$$\begin{aligned}&\frac{1}{k}\operatorname{Tan}^k(\|V\|,a)^{\perp}(\mathrm{h}(V,a))\\ =\ &\lim_{R\downarrow 0}\frac{2}{\|V\|\mathrm{B}^n(a,R)}\int_{\mathrm{B}^n(a,R)}\frac{\operatorname{Tan}^k(\|V\|,a)^{\perp}(x-a)}{|x-a|_n^2}\,d\|V\|x,\end{aligned}$$

where $|\cdot|_n$ is the Euclidean norm of $\mathbb{R}^n$, $\mathrm{B}^n(a,r)$ is the n-dimensional closed ball with radius r and center a. In paticular, if $V=\mathrm{v}(M)$, then we have

$$\frac{1}{k}\mathrm{h}(\mathrm{v}(M),a)=\lim_{R\downarrow 0}\frac{2}{\omega_k R^k}\int_{\mathrm{B}^n(a,R)\cap M}\frac{T_aM^{\perp}(x-a)}{|x-a|_n^2}\,d\mathscr{H}^k x, \tag{1.3}$$

where $\omega_k=\mathscr{H}^k(\mathrm{B}^k(0,1))$. The norm of the integrand of (1.3) is the inverse of the radius of the k-dimensional sphere which is tangent to T_aM at a and passes through x. That is, it is just the Menger curvature. For details, see chapter 4. Hence, we expect to obtain the quantity of mean curvature by some limiting procedure. The main theorem realizes it by use of the limit of integral averages. The author hopes that our theorem contributes to understanding and development of [1], [10] and related works in this field.

In chapter 2, we prepare definitions and theorems for the proof of the main theorem. In chapter 3, we state the main theorem (Theorem 3.1) and prove it. In chapter 4, we explain a vector-valued version of the inverse of a tangent-point radius and provide a geometric meaning of the main theorem, as well as some examples. The notion of the classical mean curvature has been generalized in several ways, and their representations are known. We will give two of them, and compare them with (1.3) in the final chapter.

Chapter 2

Preliminaries

We refer the reader to [1] and [4] for facts given in this chapter. Throughout this chapter, k and n are always integers satisfying $2 \leq k \leq n$.

2.1 Some notations

For r, $s \geq 0$, we use the notation

$$r \lesssim s$$

when there exists $C > 0$ independent of r and s such that $r \leq Cs$. Let

$$\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R}^n)$$

be the space of linear mappings from $\mathbb{R}^n$ to itself, and the inner product on $\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R}^n)$ is defined by

$$A \cdot B = \mathrm{Tr}\,(A^* \circ B)$$

for A, $B \in \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R}^n)$. Let

$$\mathrm{G}(n, k)$$

be the space of k-dimensional subspaces of $\mathbb{R}^n$. If $S \in \mathrm{G}(n, k)$, we also use "S" to denote the orthogonal projection from $\mathbb{R}^n$ onto S. That is, $S \in \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R}^n)$ is characterized by the conditions

$$S \circ S = S, \; S^* = S \; \text{ and } \; \mathrm{Im} S = S.$$

Whenever U is topological space, let

$$\mathscr{K}(U)$$

be the space of continuous functions on U. Let

$$\mathscr{X}(\mathbb{R}^n)$$

be the vector space of smooth mappings $g : \mathbb{R}^n \to \mathbb{R}^n$. For $m \in \mathbb{N}$, let

$$|\cdot|_m$$

be the Euclidean norm of $\mathbb{R}^m$. Whenever $a \in \mathbb{R}^m$, $r > 0$, let

$$\mathrm{B}^m(a,r) = \{ x \in \mathbb{R}^m : |x-a|_m \le r \},$$

$$\mathrm{U}^m(a,r) = \{ x \in \mathbb{R}^m : |x-a|_m < r \}.$$

Suppose V and W are finite-dimensional linear spaces, with $\dim V = m$, $\dim W < \infty$. Let

$$\Lambda_k V$$

be the k-th exterior power of V. The inner product on $\Lambda_k V$ is induced from V when V has an inner product as follows. For $u_1 \wedge \cdots \wedge u_k$, $v_1 \wedge \cdots \wedge v_k \in \Lambda_k V$, we define the inner product between them by

$$(u_1 \wedge \cdots \wedge u_k) \cdot (v_1 \wedge \cdots \wedge v_k) = \sum_{\sigma \in \mathrm{S}_k} \operatorname{sgn} \sigma \prod_{i=1}^{k} u_i \cdot v_{\sigma(i)}.$$

For $f \in \mathrm{Hom}(V, W)$, we define

$$\Lambda_k f \in \mathrm{Hom}(\Lambda_k V, \Lambda_k W)$$

by

$$\Lambda_k f(u_1 \wedge \cdots \wedge u_k) = f(u_1) \wedge \cdots \wedge f(u_k),$$

when $k \ge 2$, and

$$\Lambda_0 f = 1_V,\ \ \Lambda_1 f = f.$$

Whenever μ is a measure on U and A, $B \subset U$, let

$$(\mu \llcorner A)(B) = \mu(A \cap B).$$

Let

$$\omega_k$$

be the k-dimensional area of the unit ball in $\mathbb{R}^k$. $\mathscr{H}^k$ be the k-dimensional Hausdorff measure on $\mathbb{R}^n$.

2.2 Densities and approximate tangent spaces

Defitition 2.1 is a generalization of the tangent space to a subset of $\mathbb{R}^n$.

Definition 2.1 (see [7, 3.1.21]) *Whenever $S \subset \mathbb{R}^n$, $a \in \bar{S}$, let*

$$\operatorname{Tan}(S,a) = \bigcap_{\varepsilon>0} \bigcup_{x\in S} \bigcup_{r>0} \{\, v \in \mathbb{R}^n \,:\, |x-a|_n < \varepsilon,\ |r(x-a)-v|_n < \varepsilon \,\}.$$

Definition 2.2 (see [7, 3.2.16]) *If μ is a Radon measure on U, $U \subset \mathbb{R}^n$ and $a \in U$, let*

$$\Theta^k(\mu,a) = \lim_{r\downarrow 0} \frac{\mu(\mathrm{B}^n(a,r))}{\omega_k r^k}$$

and

$$\operatorname{Tan}^k(\mu,a) = \bigcap \left\{ \operatorname{Tan}(S,a) \,:\, S \subset U,\ \Theta^k(\mu \llcorner U \setminus S, a) = 0 \right\}.$$

Remark 2.3 $\Theta^k(\mu,a)$ and $\operatorname{Tan}^k(\mu,a)$ in Definition 2.2 reflect a geometric sharp of the support of μ around a. In particular, if M is a k-dimensional submanifold in $\mathbb{R}^n$ and $\mu = \mathscr{H}^k \llcorner M$, then $\Theta^k(\mu,a) = 1$ and $\operatorname{Tan}^k(\mu,a) = T_aM$ for any $a \in M$ (see [7, 3.2.19]).

2.3 Varifolds

In this section, we give some definitions and properties of varifolds; we refer the reader to [1] for further details.

Definition 2.4 (see [1, 3.1 and 3.5]) We say that V is a *k-dimensional varifold* in $\mathbb{R}^n$ if V is a Radon measure on $\mathbb{R}^n \times \mathrm{G}(n,k)$. Let

$$\mathrm{V}_k(\mathbb{R}^n)$$

be the weakly topologized space of k-dimensional varifolds in $\mathbb{R}^n$. Whenever $V \in \mathrm{V}_k(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$, let

$$\|V\|(A) = V(A \times \mathrm{G}(n,k)).$$

Whenever M is C^1 submanifold of $\mathbb{R}^n$, let

$$\mathrm{v}(M) \in \mathrm{V}_k(\mathbb{R}^n)$$

be defined by

$$\mathrm{v}(M)(f) = \int f(x, T_x M)\, d(\mathscr{H}^k \llcorner M)x$$

for $f \in \mathscr{K}(\mathbb{R}^n)$. We say that V is a *k-dimensional rectifiable varifold* in $\mathbb{R}^n$ if there exist a positive real number sequence $(a_l)_{l=1}^{\infty}$ and k-dimensional C^1 submanifolds $(M_l)_{l=1}^{\infty}$ in $\mathbb{R}^n$ such that

$$V = \sum_{l=1}^{\infty} a_l \mathrm{v}(M_l).$$

If the a_l may be taken to be positive integers, we say that V is an *integral varifold* in $\mathbb{R}^n$. Let

$$\mathrm{RV}_k(\mathbb{R}^n) \quad \text{and} \quad \mathrm{IV}_k(\mathbb{R}^n)$$

be the spaces of k-dimensional rectifiable varifolds in $\mathbb{R}^n$ and k-dimensional integral varifolds in $\mathbb{R}^n$, respectively.

Definition 2.5 (see [1, 3.2]) Suppose that k, l and m are integers with $0 < k \leq l$, m. Let $V \in \mathrm{V}_k(\mathbb{R}^l)$ and let $F : \mathbb{R}^l \to \mathbb{R}^m$ be continuous differentiable. Then

$$F_\sharp V \in \mathrm{V}_k(\mathbb{R}^m)$$

is defined by

$$F_\sharp V(f) = \int f(F(x), DF(x)(S)) |\Lambda_k DF(x) \circ S|\, dV(x, S)$$

for $f \in \mathscr{K}(\mathbb{R}^m)$.

Definition 2.6 (see [1, 4.1]) Suppose $V \in \mathrm{V}_k(\mathbb{R}^n)$. We define a linear functional

$$\delta V : \mathscr{X}(\mathbb{R}^n) \to \mathbb{R},$$

called the *first variation* of V, by

$$\delta V(g) = \int Dg(x) \cdot S\, dV(x, S)$$

for $g \in \mathscr{X}(\mathbb{R}^n)$. The *total variation* of V is defined by

$$\|\delta V\|(G) = \sup \{\, \delta V(g) \,:\, g \in \mathscr{X}(\mathbb{R}^n),\ \operatorname{spt} g \subset G,\ |g|_n \leq 1 \,\}$$

whenever $G \subset \mathbb{R}^n$ is an open subset. For $A \subset \mathbb{R}^n$, not necessarily an open set, we define $\|\delta V\|(A)$ by

$$\|\delta V\|(A) = \inf \{ \|\delta V\|(G) \, : \, A \subset G \text{ is open } \} .$$

Theorem 2.7 (see [1, 4.2]) *Suppose that $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. Then there exists a locally $\|V\|$ summable vector field* $\mathrm{h}(V, \cdot)$ *such that*

$$\delta V(g) = -\int \mathrm{h}(V,x) \cdot g(x) \, d\|V\|x \tag{2.1}$$

for $g \in \mathscr{X}(\mathbb{R}^n)$.

Definition 2.8 We call $\mathrm{h}(V, \cdot)$ in Theorem 2.7 the *generalized mean curvature vector.*

Proposition 2.9 is the first variation formula of a varifold restricted to a ball; it follows from [1, 4.10(1)].

Proposition 2.9 (see [1, 4.10(1)]) *Suppose $V \in \mathrm{V}_k(\mathbb{R}^n)$, $g \in \mathscr{X}(\mathbb{R}^n)$, $R > 0$. Then it holds that*

$$\begin{aligned} \delta V(\chi_{\mathrm{B}^n(0,R)} g) \;&=\; \delta(V \llcorner \mathrm{B}^n(0,R) \times \mathrm{G}(n,k))(g) \\ &\qquad - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\mathrm{B}^n(0,R+\varepsilon) \setminus \mathrm{B}^n(0,R) \times \mathrm{G}(n,k)} \frac{S(g(x)) \cdot x}{|x|_n} \, dV(x,S). \end{aligned} \tag{2.2}$$

Proposition 2.10 is concerned with the direction of the generalized mean curvature vector.

Proposition 2.10 (see [4, 5.8]) *Suppose that $V \in \mathrm{IV}_k(\mathbb{R}^n)$ and that $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$. Then*

$$S^{\perp}(\mathrm{h}(V,x)) = \mathrm{h}(V,x) \tag{2.3}$$

holds for V a.e. $(x, S) \in \mathbb{R}^n \times \mathrm{G}(n,k)$.

Proposition 2.11 is an elementary but important fact; use this result in the proof of Lemma 3.5 below.

Proposition 2.11 *Suppose that $f : \mathbb{R}^k \to \mathbb{R}$ is a $\mathscr{H}^k$-summable function. Then we have*

$$\int f(x)\, d\mathscr{H}^k x = \int_{\partial \mathrm{B}^k(0,1)} \int_0^\infty f(r\omega) r^{k-1}\, dr\, d\mathscr{H}^{k-1}\omega. \tag{2.4}$$

Proof. We define $\mu : \mathbb{R}^k \to \mathbb{R}$ by $\mu(x) = |x|_k$. We use the coarea formula (see [7, 3.2.22]), we have

$$\begin{aligned} \int f(x)\, d\mathscr{H}^k x &= \int_{\mathbb{R}} \int_{\mu^{-1}(r)} f(y)\, d\mathscr{H}^{k-1} y\, dr \\ &= \int_0^\infty \int_{\partial \mathrm{B}^k(0,r)} f(y)\, d\mathscr{H}^{k-1} y\, dr. \end{aligned} \tag{2.5}$$

By a properties of the Hausdorff measure, we have

$$\int_{\partial \mathrm{B}^k(0,r)} f(y)\, d\mathscr{H}^{k-1} y = \int_{\partial \mathrm{B}^k(0,1)} f(r\omega) r^{k-1}\, d\mathscr{H}^{k-1}\omega \tag{2.6}$$

for any $r > 0$. Therefore we have (2.4) by (2.5), (2.6) and Fubini's theorem. $\square$

Chapter 3

The main theorem and its proof

Throughout this chapter, k and n are integers satisfying $2 \leq k < n$ and α is a real number satisfying

$$\alpha > 1/3. \tag{3.1}$$

In this chapter, we prove the following main theorem in this paper.

3.1 The assertion of the main theorem

Theorem 3.1 *Suppose that $V \in \mathrm{RV}_k(\mathbb{R}^n)$, $T \in \mathrm{G}(n,k)$, $a \in T$. Let $f : T \to T^{\perp}$ be a continuous differentiable map, and let $F : T \to \mathbb{R}^n$. Assume that there exists $C_1 > 0$ and $\delta > 0$ such that $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ on $\mathrm{U}^n(a,\delta)$, and*

$$\mathrm{Im}F \cap \mathrm{U}^n(a,\delta) = \mathrm{spt}\,\|V\| \cap \mathrm{U}^n(a,\delta)\,, \tag{3.2}$$

$$T(F(x)) = x,\ T^{\perp}(F(x)) = f(x),\ \|\nabla f(x) - \nabla f(a)\| \leq C_1 |x-a|_n^{\alpha} \tag{3.3}$$

for $x \in T \cap \mathrm{U}^n(a,\delta)$. Let $a \in T$ be a Lebesgue point of $\mathrm{h}(V,\cdot)$ and $\Theta^k(\|V\|,\cdot)$ with respect to $\|V\|$; that is,

$$\begin{gathered} 0 < \Theta^k(\|V\|,a) < \infty,\ \mathrm{h}(V,a) = \lim_{r \downarrow 0} \frac{1}{\|V\|\mathrm{B}^n(a,r)} \int_{\mathrm{B}^n(a,r)} \mathrm{h}(V,x)\, d\|V\|x, \\ \Theta^k(\|V\|,a) = \lim_{r \downarrow 0} \frac{1}{\|V\|\mathrm{B}^n(a,r)} \int_{\mathrm{B}^n(a,r)} \Theta^k(\|V\|,x)\, d\|V\|x. \end{gathered} \tag{3.4}$$

For $R > 0$, we assume that

$$\left|\Theta^k(\|V\|, x) - \Theta^k(\|V\|, a)\right| \leq C_1 R^{1-\alpha} \tag{3.5}$$

for $\|V\|$ almost every $x \in \mathrm{U}^n(a, R)$. Then the normal component of $\mathrm{h}(V, a)$ *is given by*

$$\begin{aligned} &\frac{1}{k}\mathrm{h}(V, a)\cdot v \\ = \;& \lim_{R\downarrow 0} \frac{2}{\|V\|\mathrm{B}^n(a, R)} \int_{\mathrm{B}^n(a,R)} \frac{\operatorname{Tan}^k(\|V\|, a)^\perp (x-a)\cdot v}{|x-a|_n^2}\, d\|V\|x \end{aligned} \tag{3.6}$$

for $v \in \operatorname{Tan}^k(\|V\|, a)^\perp$.

Remark 3.2 When f is in the class of C^2 and $T = \operatorname{Tan}^k(\|V\|, a)$, it is known that

$$\begin{aligned} &\frac{1}{k}\mathrm{h}(V, a)\cdot v \\ = \;& \lim_{R\downarrow 0} \frac{2}{\mathscr{H}^k(T \cap \mathrm{B}^n(b, R))} \int_{T\cap \mathrm{B}^n(b,R)} \frac{(F(y) - F(b))\cdot v}{|y-b|_k^2}\, d\mathscr{H}^k y, \end{aligned} \tag{3.7}$$

where $b = T(a)$. (3.6) implies (3.7) and vice versa if $f \in C^2$. In this sense (3.6) is a generalization of (3.7) under less reguralitty condition.

Remark 3.3 We prepare a few lemmas which we need for the proof of main theorem. By use of $f(\cdot\,) - f(a)$, if necessary, we may assume

$$f(a) = 0. \tag{3.8}$$

Moreover, by translation, we may assume

$$a = 0. \tag{3.9}$$

The map f is defined on T, but not necessary on $\mathrm{Im}DF(0)$. However, we regard the map f as a map defined on $\mathrm{Im}DF(0)$ provided δ sufficiently small. Next lemma is a rigorous statement of this fact.

3.2 Lemmas

Lemma 3.4 *Let δ, f and F satisfy (3.2) and (3.3) with $a = 0$. Then there exist $\tilde{\delta} > 0$, $C_2 > 0$, continuous differentiable maps $\tilde{f} : \mathrm{Im}DF(0) \to \mathrm{Im}DF(0)^\perp$, and $\tilde{F} : \mathrm{Im}DF(0) \to \mathbb{R}^n$ such that*

$$\mathrm{Im}\tilde{F} \cap \mathrm{U}^n(0,\tilde{\delta}) = \mathrm{spt}\,\|V\| \cap \mathrm{U}^n(0,\tilde{\delta}) \tag{3.10}$$

and

$$\begin{gathered}\mathrm{Im}DF(0)(\tilde{F}(x)) = x,\ \mathrm{Im}DF(0)^\perp(\tilde{F}(x)) = \tilde{f}(x),\ |\tilde{f}(0)| = \|\nabla\tilde{f}(0)\| = 0, \\ |\tilde{f}(x)|_n \le C_2|x|_n^{1+\alpha},\ \|\nabla\tilde{f}(x) - \nabla\tilde{f}(y)\| \le C_2|x-y|_n^\alpha.\end{gathered} \tag{3.11}$$

Proof. Let $S = \mathrm{Im}DF(0)$. Take $0 < \tilde{\delta} < \delta$. Let $\bar{x}$, $\bar{y} \in \mathrm{spt}\,\|V\| \cap \mathrm{U}^n(0,\tilde{\delta})$ with $\bar{x} \ne \bar{y}$. By (3.10), there exist x, $y \in T \cap \mathrm{U}^n(0,\tilde{\delta})$ uniquely such that $x \ne y$, and

$$F(x) = \bar{x},\ F(y) = \bar{y}. \tag{3.12}$$

Let $X = (x, x\cdot\nabla f(0))$ and $Y = (y, y\cdot\nabla f(0))$. Then

$$|S(\bar{x}) - S(\bar{y})|_n \ge |X - Y|_n - |(X-Y) - (S(\bar{x}) - S(\bar{y}))|_n. \tag{3.13}$$

Since

$$X - Y \in S,$$

and since

$$|\bar{x} - \bar{y} - (S(\bar{x}) - S(\bar{y}))|_n = \mathrm{dist}\,(\bar{x}-\bar{y}, S),$$

we have

$$\begin{aligned} & |(X-Y) - (S(\bar{x}) - S(\bar{y}))|_n \\ = \ & |(X-Y) - (\bar{x}-\bar{y}) + (\bar{x}-\bar{y}) - (S(\bar{x}) - S(\bar{y}))|_n \\ \le \ & 2|(X-Y) - (\bar{x}-\bar{y})|_n \\ = \ & 2|(x-y, (x-y)\cdot\nabla f(0)) - (x-y, f(x)-f(y))|_n \\ = \ & 2|(x-y)\cdot\nabla f(0) - (f(x)-f(y))|_{n-k} \\ = \ & 2\left|\int_0^1 (\nabla f(y+t(x-y)) - \nabla f(0))\cdot(x-y)\,dt\right|_{n-k} \\ \le \ & 2C_1\tilde{\delta}^\alpha|x-y|_k. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13), we have

$$|S(\bar{x}) - S(\bar{y})|_n \geq (1 - 2C_1\tilde{\delta}^\alpha)|x - y|_k. \tag{3.15}$$

Hence, we obtain

$$|S(\bar{x}) - S(\bar{y})|_n > 0 \tag{3.16}$$

for sufficiently small $\tilde{\delta}$. Then, whenever $\bar{x} \in \operatorname{spt}\|V\| \cap \mathrm{U}^n(0, \tilde{\delta})$ there exists uniquely

$$\tilde{x} \in S \cap \mathrm{U}^n(0, \tilde{\delta})$$

such that

$$F(\tilde{x}) = \bar{x}.$$

Using this correspondence, we define $\tilde{F}$ by

$$\tilde{F}(\tilde{x}) = \bar{x} \tag{3.17}$$

and observe that, by definition, $\tilde{F}$ satisfies (3.10). Furthermore $\tilde{F}$ is continuously differentiable by the continuous differentiablity of F. Using [1, 8.9(5)], (3.10) and (3.15), we have

$$\begin{aligned}
& \left(1 - \|\mathrm{Im}D\tilde{F}(\tilde{x}) - \mathrm{Im}D\tilde{F}(\tilde{y})\|^2\right) \|D\tilde{f}(\tilde{x}) - D\tilde{f}(\tilde{y})\|^2 \\
\leq \; & \|\mathrm{Im}D\tilde{F}(\tilde{x}) - \mathrm{Im}D\tilde{F}(\tilde{y})\|^2 \\
= \; & \|\mathrm{Im}DF(x) - \mathrm{Im}DF(y)\|^2 \\
= \; & \|Df(x) - Df(y)\|^2 \\
\leq \; & C_1^2|x - y|_k^{2\alpha} \\
\leq \; & C_1^2 \left(1 - 2C_1\tilde{\delta}^\alpha\right)^{-2\alpha} |\tilde{x} - \tilde{y}|_n^{2\alpha}
\end{aligned}$$

for $\tilde{x}$ and $\tilde{y} \in S \cap \mathrm{U}^n(0, \tilde{\delta})$. Consequently, if $\tilde{\delta}$ is sufficiently small, then we have (3.11). $\square$

By Lemma 3.4, we may assume (3.10) and (3.11) with

$$\tilde{f} = F, \;\; \tilde{F} = F, \;\; \mathrm{Im}DF(0) = \mathbb{R}^k \times \{0\}. \tag{3.18}$$

In what follows, we always assume (3.9) and (3.18). Furthermore, we identify $\mathbb{R}^k \times \{0\}$ with $\mathbb{R}^k$.

Lemma 3.5 *Let f and F be functions as in Theorem 3.1. Then there exist $R_0 > 0$ and $C_3 > 0$ satisfying the following properties. Whenever $R \in (0, R_0)$ and $\omega \in \partial \mathrm{B}^k(0,1)$, there exists*

$$r(R,\omega) \in (0, R]$$

such that

$$\begin{aligned} r(R,\omega)^2 + |f(r(R,\omega)\omega)|_{n-k}^2 = R^2, \ R \le C_3 r(R,\omega), \\ 2(r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega) \ge r(R,\omega). \end{aligned} \tag{3.19}$$

The function $r(R,\omega)$ is continuously differentiable with respect to R, and satisfies

$$\frac{\partial}{\partial R} r(R,\omega) = \frac{R}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega}. \tag{3.20}$$

Furthemore for any continuous function $G : \mathbb{R}^k \to \mathbb{R}$, we have

$$\int_{F^{-1}(\mathrm{B}^n(0,R))} G(x)\, d\mathscr{H}^k x = \int_{\partial \mathrm{B}^k(0,1)} \int_0^{r(R,\omega)} G(r\omega) r^{k-1}\, dr\, d\mathscr{H}^{k-1}\omega, \tag{3.21}$$

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)) \setminus F^{-1}(\mathrm{B}^n(0,R))} G(x)\, d\mathscr{H}^k x \\ = &\int_{\partial \mathrm{B}^k(0,1)} G(r(R,\omega)\omega) \frac{r(R,\omega)^{k-1} R}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega}\, d\mathscr{H}^{k-1}\omega. \end{aligned} \tag{3.22}$$

Proof. By (3.11),

$$\frac{\partial}{\partial r}\left(r^2 + |f(r\omega)|_{n-k}^2\right) = 2\left(r + f(r\omega)\cdot \nabla f(r\omega)\cdot \omega\right) \ge 2r\left(1 - C_2^2 r^{2\alpha}\right).$$

By (3.1), there exists $R_0 > 0$ such that

$$\frac{\partial}{\partial r}\left(r^2 + |f(r\omega)|_{n-k}^2\right) > 0 \tag{3.23}$$

for $r \in (0, R_0)$. Note that R_0 is independent of ω. Hence, for $R \in (0, R_0)$ and $\omega \in \partial \mathrm{B}(0,1)$, there exists

$$r(R,\omega) \in (0, R] \tag{3.24}$$

uniquely such that

$$r(R,\omega)^2 + |f(r(R,\omega)\omega)|_{n-k}^2 = R^2. \tag{3.25}$$

By the implicit function theorem, we find that $r(R,\omega)$ is continuously differentiable with respect to R. Hence, there exists $C_3 > 0$ such that

$$\begin{aligned} & R^2 = r(R,\omega)^2 + |f(r(R,\omega)\omega)|_{n-k}^2 \\ \leq \ & r(R,\omega)^2 \left(1 + C_2^2 r(R,\omega)^{2\alpha}\right) \\ \leq \ & r(R,\omega)^2 \left(1 + C_2^2 R_0^{2\alpha}\right) \\ \leq \ & C_3^2 r(R,\omega)^2. \end{aligned}$$

Since

$$r(R,\omega) \leq R, \tag{3.26}$$

for sufficiently small $R_0 > 0$, we have

$$1 - C_2^2 r(R,\omega)^{2\alpha} > r(R,\omega)/2 \tag{3.27}$$

whenever $0 < R < R_0$, and

$$\begin{aligned} & r(R,\omega) + f(r(R,\omega)\omega) \cdot \nabla f(r(R,\omega)\omega)\omega \\ \geq \ & r(R,\omega)\left(1 - C_2^2 r(R,\omega)^{2\alpha}\right) \\ \geq \ & r(R,\omega)/2. \end{aligned}$$

Consequently, (3.19) holds. (3.20) is obtained by differentiating (3.25) with respect to R. By Proposition 2.11, we have

$$\begin{aligned} & \int_{F^{-1}(\mathrm{B}^n(0,R))} G(x)\, d\mathscr{H}^k x \\ = \ & \int_{\partial \mathrm{B}^k(0,1)} \int_0^\infty \chi_{\left\{\, x \,:\, |x|_k^2 + |f(x)|_{n-k}^2 \leq R^2 \,\right\}}(r\omega) G(r\omega) r^{k-1}\, dr\, d\mathscr{H}^{k-1}\omega. \end{aligned} \tag{3.28}$$

Hence, we have (3.21) by (3.25) and (3.28). If $\varepsilon > 0$ is sufficiently small, then

$$R + \varepsilon < R_0. \tag{3.29}$$

Hence, by (3.21),

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)) \setminus F^{-1}(\mathrm{B}^n(0,R))} G(x)\, d\mathscr{H}^k x \\ = \ & \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\partial \mathrm{B}^k(0,1)} \int_{r(R,\omega)}^{r(R+\varepsilon,\omega)} G(r\omega) r^{k-1}\, dr\, d\mathscr{H}^{k-1}\omega, \end{aligned} \tag{3.30}$$

and we use Lebesgue's convergence theorem to obtain

$$\begin{aligned}
(3.30) &= \int_{\partial \mathrm{B}^k(0,1)} \frac{\partial}{\partial R} \int_0^{r(R,\omega)} G(r\omega) r^{k-1}\, dr\, d\mathscr{H}^{k-1}\omega \\
&= \int_{\partial \mathrm{B}^k(0,1)} G(r(R,\omega)\omega) r(R,\omega)^{k-1} \frac{\partial}{\partial R} r(R,\omega)\, dr\, d\mathscr{H}^{k-1}\omega. \quad (3.31)
\end{aligned}$$

We have

$$\begin{aligned}
(3.31) = \int_{\partial \mathrm{B}^k(0,1)} \Bigg\{ & G(r(R,\omega)\omega) r(R,\omega)^{k-1} \\
& \times \frac{R}{r(R,\omega) + f(r(R,\omega)\omega) \cdot \nabla f(r(R,\omega)\omega)\omega} \Bigg\} d\mathscr{H}^{k-1}\omega
\end{aligned}$$

by (3.20). Thus (3.22) is proved. □

3.3 Proof of the main theorem

In this section, we prove Theorem 3.1.

Proof. We assume that

$$f(a) = 0,\ \nabla f(a) = 0,\ a = 0 \ \text{ and } \ T = \mathbb{R}^k \times \{0\} \in \mathrm{G}(n,k). \qquad (3.32)$$

Let

$$(e_i)_{i=1}^n \qquad (3.33)$$

be an orthonormal basis of $\mathbb{R}^n$, with $(e_i)_{i=1}^k$ being an orthonormal basis of $\mathbb{R}^k \times \{0\}$. Since

$$\frac{1}{\|V\|\mathrm{B}^n(0,R)} = \frac{R^k}{\|V\|\mathrm{B}^n(0,R)} \cdot \frac{1}{R^k}$$

by (3.4), it is enough to show

$$\begin{aligned}
& \lim_{R \downarrow 0} \frac{1}{R^k} \left\{ 2 \int_{\mathrm{B}^n(0,R)} \frac{T^\perp(x) \cdot e_l}{|x|_n^2}\, d\|V\|x - \frac{1}{k} \int_{\mathrm{B}^n(0,R)} \mathrm{h}(V,x) \cdot e_l\, d\|V\|x \right\} \\
= \ & 0 \qquad (3.34)
\end{aligned}$$

for any natural number l with $k+1 \le l \le n$. By (3.11), there exists
$$J : T \to \mathbb{R}$$
such that
$$|\Lambda_k DF(y)| = \sqrt{1+J(y)},\ \ J(y) \lesssim \|\nabla f(y)\|^2. \tag{3.35}$$
For simplicity, we write $\Theta(x) = \Theta^k(\|V\|, x)$. By (2.1), we have
$$\int_{\mathrm{B}^n(0,R)} \mathrm{h}(V,x)\cdot e_l \, d\|V\|x = -\delta V(\chi_{\mathrm{B}^n(0,R)} e_l). \tag{3.36}$$
We use Proposition 2.9 for (3.36), we have
$$\begin{aligned}
-\delta V(\chi_{\mathrm{B}^n(0,R)} e_l) &= -\delta(V \llcorner \mathrm{B}^n(0,R) \times \mathrm{G}(n,k))(e_l) \\
&\quad + \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R) \times \mathrm{G}(n,k)} \frac{S(e_l)\cdot x}{|x|_n} \, dV(x,S) \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R) \times \mathrm{G}(n,k)} \frac{S(e_l)\cdot x}{|x|_n} \, dV(x,S),
\end{aligned} \tag{3.37}$$
where
$$\delta(V \llcorner \mathrm{B}^n(0,R) \times \mathrm{G}(n,k))(e_l) = 0$$
by the definition of the first variation of $V \llcorner \mathrm{B}^n(0,R) \times \mathrm{G}(n,k)$. By (3.10) and the area formura ([1, 2.8(6)]), we have
$$\begin{aligned}
&(V \llcorner \mathrm{U}^n(0,R) \times \mathrm{G}(n,k))(a) \\
= &\int_{F^{-1}(\mathrm{B}^n(0,R))} a(F(x), \mathrm{Im} F(x)) \Theta(F(x)) |\Lambda_k DF(x) \circ T| \, d\mathscr{H}^k x
\end{aligned} \tag{3.38}$$
for $a \in \mathscr{K}(\mathbb{R}^n \times \mathrm{G}(n,k))$ and $0 < R < \delta$. Hence, using (3.38) and (3.35) for (3.37), we have
$$\begin{aligned}
(3.37) \;=\; &\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R))} \frac{g^{ij}(y)(e_l\cdot \partial_i F(y))(\partial_j F(y)\cdot F(y))}{|F(y)|_n} \\
&\qquad \times |\Lambda_k DF(x) \circ T| \Theta(F(y)) \, d\mathscr{H}^k y \\
=\; &\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus \mathrm{B}^n(0,R))} \frac{g^{ij}\partial_i f_l(y)(y_j + \partial_j f(y)\cdot f(y))}{\sqrt{|y|_k^2 + |f(y)|_{n-k}^2}} \\
&\qquad \times \sqrt{1+J(y)}\Theta(F(y)) \, d\mathscr{H}^k y,
\end{aligned} \tag{3.39}$$

where

$$f_l = f \cdot e_l$$

and

$$g^{ij}$$

is the (i,j) element of the inverse matrix of $(\partial_i F \cdot \partial_j F)_{i,j=1}^k$, and we sum i, j over repeated indices from 1 to k. Since

$$\partial_i F \cdot \partial_j F = \delta_{ij} + \partial_i f \cdot \partial_j f, \tag{3.40}$$

and since

$$\|(\partial_i f \cdot \partial_j f)\| \lesssim \|\nabla f\|^2, \tag{3.41}$$

we have

$$\begin{aligned} \|(g^{ij}) - (\delta^{ij})\| &\leq \frac{\|\nabla f\|^2}{1 - \|\nabla f\|^2} \\ &\lesssim \|\nabla f\|^2 \end{aligned} \tag{3.42}$$

provided $\|\nabla f\| < 1$. Using this and (3.11), we have

$$\begin{aligned} &\left| \frac{(g^{ij} - \delta^{ij}) \partial_i f_l(y)(y_j + \partial_j f(y) \cdot f(y))}{\sqrt{|y|_k^2 + |f(y)|_{n-k}^2}} \sqrt{1 + J(y)} \right| \\ \lesssim\ & \|\nabla f(y)\|^3 \frac{|y|_k + \|\nabla f(y)\| |f(y)|}{|y|_k} \\ \lesssim\ & |y|_k^{3\alpha}(1 + |y|_k^{2\alpha}). \end{aligned} \tag{3.43}$$

By (3.22),

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon) \setminus \mathrm{B}^n(0,R))} |y|_k^{3\alpha}(1 + |y|_k^{2\alpha}) \, d\mathscr{H}^k y \\ =\ & \int_{\partial \mathrm{B}^k(0,1)} \frac{|r(R,\omega)\omega|_k^{3\alpha}(1 + |r(R,\omega)\omega|_k^{2\alpha}) r(R,\omega)^{k-1} R}{r(R,\omega) + f(r(R,\omega)\omega) \cdot \nabla f(r(R,\omega)\omega)\omega} \, d\mathscr{H}^{k-1}\omega, \end{aligned} \tag{3.44}$$

and using (3.19) for (3.44), we have

$$(3.44) \lesssim R^{3\alpha - 1 + k}. \tag{3.45}$$

Note that

$$3\alpha - 1 > 0$$

when (3.1). Consequently we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus\mathrm{B}^n(0,R))}\frac{(g^{ij}-\delta^{ij})\partial_i f_l(y)(y_j+\partial_j f(y)\cdot f(y))}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\\
&\qquad\qquad\times\sqrt{1+J(y)}\Theta(F(y))\,d\mathscr{H}^k y\\
=&\quad o(R^k)\quad\text{as } R\downarrow 0. \qquad (3.46)
\end{aligned}$$

Hence, we use (3.46), then (3.39) can be written as

$$\begin{aligned}
(3.39) \;=&\;\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus\mathrm{B}^n(0,R))}\frac{\delta^{ij}\partial_i f_l(y)(y_j+\partial_j f(y)\cdot f(y))}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\\
&\qquad\qquad\times\sqrt{1+J(y)}\Theta(F(y))\,d\mathscr{H}^k y\\
&+\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus\mathrm{B}^n(0,R))}\frac{(g^{ij}-\delta^{ij})\partial_i f_l(y)(y_j+\partial_j f(y)\cdot f(y))}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\\
&\qquad\qquad\times\sqrt{1+J(y)}\Theta(F(y))\,d\mathscr{H}^k y\\
=&\;\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus\mathrm{B}^n(0,R))}\frac{(\nabla f_l(y)\cdot y+\nabla f_l(y)\cdot\nabla(f^2)(y)/2)}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\\
&\qquad\qquad\times\sqrt{1+J(y)}\Theta(F(y))\,d\mathscr{H}^k y\\
&+\;o(R^k)\quad\text{as } R\downarrow 0. \qquad (3.47)
\end{aligned}$$

By (3.11), we have

$$\begin{aligned}
&\left|\frac{\nabla f_l(y)\cdot \nabla(f^2)(y)}{2\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\right| \\
=&\left|\sum_{i=1}^{k}\frac{(\partial_i f_l(y)-\partial_i f_l(0))(\partial_i f(y)-\partial_i f(0))\cdot (f(y)-f(0))}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\right| \\
\lesssim&\ \frac{\|\nabla f(y)-\nabla f(0)\|^2|f(y)-f(0)|_{n-k}}{|y|_k} \\
\lesssim&\ |y|_k^{3\alpha}. \qquad (3.48)
\end{aligned}$$

We use (3.22) for (3.48), we have

$$\begin{aligned}
&\left|\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus\mathrm{B}^n(0,R))}\frac{\nabla f_l(y)\cdot \nabla(f^2)(y)}{2\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\,d\mathscr{H}^k y\right| \\
\lesssim&\ \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus\mathrm{B}^n(0,R))}|y|_k^{3\alpha}d\mathscr{H}^k y \\
=&\ \int_{\partial\mathrm{B}^k(0,1)}\frac{|r(R,\omega)\omega|_k^{3\alpha}r(R,\omega)^{k-1}R}{r(R,\omega)+f(r(R,\omega)\omega)\cdot\nabla f(r(R,\omega)\omega)\omega}\,d\mathscr{H}^{k-1}\omega. \qquad (3.49)
\end{aligned}$$

Using (3.19) for (3.49), we have

$$(3.49)\lesssim R^{3\alpha-1+k}, \qquad (3.50)$$

and noting that

$$3\alpha-1>0$$

when (3.1), we have

$$\begin{aligned}
(3.47)=&\ \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus\mathrm{B}^n(0,R))}\frac{\nabla f_l(y)\cdot y\sqrt{1+J(y)}}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\Theta(F(y))\,d\mathscr{H}^k y \\
&+\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus\mathrm{B}^n(0,R))}\frac{\nabla f_l(y)\cdot \nabla(f^2)(y)}{2\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\,d\mathscr{H}^k y \\
=&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{F^{-1}(\mathrm{B}^n(0,R+\varepsilon)\setminus\mathrm{B}^n(0,R))}\frac{\nabla f_l(y)\cdot y\sqrt{1+J(y)}}{\sqrt{|y|_k^2+|f(y)|_{n-k}^2}}\Theta(F(y))\,d\mathscr{H}^k y \\
&+o(R^k)\quad \text{as } R\downarrow 0. \qquad (3.51)
\end{aligned}$$

Using (3.22), we have

$$
\begin{aligned}
&(3.51) \\
= \quad & \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R\Theta(F(r(R,\omega)\omega))}{(r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega)} \\
& \qquad \times \frac{r(R,\omega)^k\sqrt{1+J(r(R,\omega)\omega)}}{\sqrt{r(R,\omega)^2+|f(r(R,\omega)\omega)|_{n-k}^2}}\, d\mathscr{H}^{k-1}\omega \\
& + o(R^k) \quad \text{as } R\downarrow 0. \qquad (3.52)
\end{aligned}
$$

Since

$$
\begin{aligned}
& \frac{r(R,\omega)^k\sqrt{1+J(r(R,\omega)\omega)}}{\sqrt{r(R,\omega)^2+|f(r(R,\omega)\omega)|_{n-k}^2}} \\
= \quad & \frac{r(R,\omega)^k(\sqrt{1+J(r(R,\omega)\omega)}-1)}{\sqrt{r(R,\omega)^2+|f(r(R,\omega)\omega)|_{n-k}^2}} \\
& + \frac{r(R,\omega)^k}{\sqrt{r(R,\omega)^2+|f(r(R,\omega)\omega)|_{n-k}^2}}, \qquad (3.53)
\end{aligned}
$$

using the fundamental theorem of calculus, we have

$$
\begin{aligned}
\left|\sqrt{1+J(r(R,\omega)\omega)}-1\right| \quad &= \quad \left|\int_0^{J(r(R,\omega)\omega)} \frac{1}{2\sqrt{1+t}}\, dt\right| \\
&\lesssim \quad J(r(R,\omega)\omega). \qquad (3.54)
\end{aligned}
$$

Substituting (3.35) into (3.54), we have

$$
(3.54) \lesssim \|\nabla f(r(R,\omega)\omega)\|^2, \qquad (3.55)
$$

and substituting (3.11) and (3.19) into (3.55), we have

$$
(3.55) \lesssim r(R,\omega)^{2\alpha} \lesssim R^{2\alpha}. \qquad (3.56)
$$

Substituting (3.53) into (3.52), we have

$$
\begin{aligned}
(3.52) \quad = \quad & \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R \Theta(F(r(R,\omega)\omega))}{(r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega)} \\
& \qquad \times \frac{r(R,\omega)^k(\sqrt{1+J(r(R,\omega)\omega)}-1)}{\sqrt{r(R,\omega)^2+|f(r(R,\omega)\omega)|_{n-k}^2}}\, d\mathscr{H}^{k-1}\omega \\
& + \int_{\partial \mathrm{B}^k(0,1)} \left\{ \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R r(R,\omega)^k \Theta(F(r(R,\omega)\omega))}{(r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega)} \right. \\
& \qquad \left. \times \frac{1}{\sqrt{r(R,\omega)^2+|f(r(R,\omega)\omega)|_{n-k}^2}} \right\} d\mathscr{H}^{k-1}\omega \\
& + o(R^k) \quad \text{as } R \downarrow 0. \qquad (3.57)
\end{aligned}
$$

Substituting (3.56) into (3.57), we have

$$
\begin{aligned}
(3.57) \quad = \quad & \int_{\partial \mathrm{B}^k(0,1)} \left\{ \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R r(R,\omega)^k \Theta(F(r(R,\omega)\omega))}{(r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega)} \right. \\
& \qquad \left. \times \frac{1}{\sqrt{r(R,\omega)^2+|f(r(R,\omega)\omega)|_{n-k}^2}} \right\} d\mathscr{H}^{k-1}\omega, \\
& + o(R^k) \quad \text{as } R \downarrow 0 \\
= \quad & + \int_{\partial \mathrm{B}^k(0,1)} \left\{ \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R r(R,\omega)^k \Theta(F(r(R,\omega)\omega))}{(r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega)} \right. \\
& \qquad \left. \times \left(\frac{1}{\sqrt{r(R,\omega)^2+|f(r(R,\omega)\omega)|_{n-k}^2}} - \frac{1}{r(R,\omega)} \right) \right\} d\mathscr{H}^{k-1}\omega \\
& + \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R \Theta(F(r(R,\omega)\omega))}{(r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega)} \\
& \qquad \times r(R,\omega)^{k-1}\, d\mathscr{H}^{k-1}\omega \\
& + o(R^k) \quad \text{as } R \downarrow 0. \qquad (3.58)
\end{aligned}
$$

Using (3.11) and (3.19), we have

$$\begin{aligned}
& \left| \frac{1}{\sqrt{r(R,\omega)^2 + |f(r(R,\omega)\omega)|_{n-k}^2}} - \frac{1}{r(R,\omega)} \right| \\
\lesssim \;& \left| \int_0^{|f(r(R,\omega)\omega)|_{n-k}} \frac{t}{(r(R,\omega)^2 + t^2)^{\frac{3}{2}}}\, dt \right| \\
\lesssim \;& r(R,\omega)^{2\alpha-1} \\
\lesssim \;& R^{2\alpha-1}. \qquad (3.59)
\end{aligned}$$

Substituting (3.59) into (3.58), we have

$$\begin{aligned}
& (3.58) \\
= \;& \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega) \cdot \omega R \Theta(F(r(R,\omega)\omega))}{(r(R,\omega) + f(r(R,\omega)\omega) \cdot \nabla f(r(R,\omega)\omega)\omega)} \\
& \qquad \times r(R,\omega)^{k-1}\, d\mathscr{H}^{k-1}\omega \\
& + o(R^k) \quad \text{as } R \downarrow 0 \\
= \;& \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega) \cdot \omega R \Theta(F(r(R,\omega)\omega))}{(r(R,\omega) + f(r(R,\omega)\omega) \cdot \nabla f(r(R,\omega)\omega)\omega)} \\
& \qquad \times \left(r(R,\omega)^{k-1} - R^{k-1} \right)\, d\mathscr{H}^{k-1}\omega \\
& + \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega) \cdot \omega R^k \Theta(F(r(R,\omega)\omega))}{(r(R,\omega) + f(r(R,\omega)\omega) \cdot \nabla f(r(R,\omega)\omega)\omega)}\, d\mathscr{H}^{k-1}\omega \\
& + o(R^k) \quad \text{as } R \downarrow 0. \qquad (3.60)
\end{aligned}$$

Using (3.11) and (3.19), we have

$$\begin{aligned}
\left| r(R,\omega)^{k-1} - R^{k-1} \right| &= \left| (k-1) \int_{r(R,\omega)}^{R} t^{k-2}\, dt \right| \\
&\lesssim R^{k-3} \int_{r(R,\omega)}^{R} t\, dt \\
&\lesssim R^{k-3} (R^2 - r(R,\omega)^2) \\
&= R^{k-3} |f(r(R,\omega)\omega)|_{n-k}^2 \\
&\lesssim R^{2\alpha-1+k}. \qquad (3.61)
\end{aligned}$$

Substituting (3.61) into (3.60), we have

$$\begin{aligned}
(3.60) \quad = \quad & \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R^k \Theta(F(r(R,\omega)\omega))}{(r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega)}\, d\mathscr{H}^{k-1}\omega \\
& + o(R^k) \quad \text{as } R\downarrow 0 \\
= \quad & \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R^k (\Theta(F(r(R,\omega)\omega))-\Theta(0))}{r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\cdot \omega}\, d\mathscr{H}^{k-1}\omega \\
& + \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R^k \Theta(0)}{r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\cdot \omega}\, d\mathscr{H}^{k-1}\omega \\
& + o(R^k) \quad \text{as } R\downarrow 0. \qquad (3.62)
\end{aligned}$$

Using (3.5) for (3.62), we have

$$\begin{aligned}
(3.62) \quad = \quad & \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega R^k \Theta(0)}{r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\cdot \omega}\, d\mathscr{H}^{k-1}\omega \\
& + o(R^k) \quad \text{as } R\downarrow 0. \qquad (3.63)
\end{aligned}$$

We divide both sides of (3.63) by R^k, and take the limit as $R\downarrow 0$. It follows from (3.4), (3.5) and (3.19) that

$$\lim_{R\downarrow 0} \int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega \Theta(0)}{r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\omega}\, d\mathscr{H}^{k-1}\omega \in \mathbb{R}. \quad (3.64)$$

By L'Hospital's rule and (3.20), we have

$$\begin{aligned}
& \lim_{R\downarrow 0} \frac{2}{R^2} \int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)\Theta(0)\, d\mathscr{H}^{k-1}\omega \\
= \quad & \lim_{R\downarrow 0} \frac{1}{R} \int_{\partial \mathrm{B}^k(0,1)} \nabla f_l(r(R,\omega)\omega)\cdot \omega \Theta(0) \frac{\partial r(R,\omega)}{\partial R}\, d\mathscr{H}^{k-1}\omega \\
= \quad & (3.64). \qquad (3.65)
\end{aligned}$$

On the other hand, using (3.38), we have

$$\begin{aligned}
&\int_{\mathrm{B}^n(0,R)} \frac{T^\perp(x)\cdot e_l}{|x|_n^2}\, d\|V\|x \\
=\ & \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{f_l(y)}{|y|_k^2+|f(y)|_{n-k}^2} \sqrt{1+J(y)}\Theta(F(y))\, d\mathscr{H}^k y \\
=\ & \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{f_l(y)}{|y|_k^2+|f(y)|_{n-k}^2} \Theta(F(y))\, d\mathscr{H}^k y \\
& + \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{f_l(y)}{|y|_k^2+|f(y)|_{n-k}^2} \\
& \qquad \times (\sqrt{1+J(y)}-1)\Theta(F(y))\, d\mathscr{H}^k y. \qquad (3.66)
\end{aligned}$$

Using (3.56), we have

$$\begin{aligned}
&\left| \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{f_l(y)(\sqrt{1+J(y)}-1)\Theta(F(y))}{|y|_k^2+|f(y)|_{n-k}^2}\, d\mathscr{H}^k y \right| \\
\lesssim\ & \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{|f_l(y)|_{n-k}\|\nabla f(y)\|^2}{|y|_k^2}\, d\mathscr{H}^k y. \qquad (3.67)
\end{aligned}$$

Then we substitute (3.21) into (3.67) to obtain

$$\begin{aligned}
(3.67) &= \int_{\partial \mathrm{B}^k(0,1)} \int_0^{r(R,\omega)} \frac{|f_l(r\omega)|_{n-k}\|\nabla f(r\omega)\|^2}{|r\omega|_k^2} r^{k-1}\, dr\, d\mathscr{H}^{k-1}\omega \\
&= \int_{\partial \mathrm{B}^k(0,1)} \int_0^{r(R,\omega)} |f_l(r\omega)|_{n-k}\|\nabla f(r\omega)\|^2 r^{k-3}\, dr\, d\mathscr{H}^{k-1}\omega. \qquad (3.68)
\end{aligned}$$

Substituting (3.11) into (3.68), we have

$$\begin{aligned}
(3.68) &\lesssim \int_{\partial \mathrm{B}^k(0,1)} \int_0^{r(R,\omega)} r^{3\alpha k-2}\, dr\, d\mathscr{H}^{k-1}\omega \\
&\lesssim r(R,\omega)^{3\alpha-1+k}, \qquad (3.69)
\end{aligned}$$

and substituting (3.69) into (3.66), we have

$$
\begin{aligned}
(3.66) \quad = \quad & \int_{F^{-1}(\mathrm{B}^n(0,R))} \frac{f_l(y)}{|y|_k^2 + |f(y)|_{n-k}^2} \Theta(F(y))\, d\mathscr{H}^k y \\
& + o(R^k) \quad \text{as } R \downarrow 0 \\
= \quad & \int_{F^{-1}(\mathrm{B}^n(0,R))} f_l(y) \frac{1}{|y|_k^2} \Theta(F(y))\, d\mathscr{H}^k y \\
& + \int_{F^{-1}(\mathrm{B}^n(0,R))} f_l(y) \left(\frac{1}{|y|_k^2 + |f(y)|_{n-k}^2} - \frac{1}{|y|_k^2} \right) \Theta(F(y))\, d\mathscr{H}^k y \\
& + o(R^k) \quad \text{as } R \downarrow 0. \qquad (3.70)
\end{aligned}
$$

Since

$$
\begin{aligned}
& \left| \frac{f_l(y)}{|y|_k^2 + |f(y)|_{n-k}^2} - \frac{f_l(y)}{|y|_k^2} \right| \\
= \quad & |f_l(y)| \left| \int_0^{|f(y)|_{n-k}} \frac{d}{dt} \left(\frac{1}{|y|_k^2 + t^2} \right) dt \right| \\
= \quad & |f_l(y)| \left| \int_0^{|f(y)|_{n-k}} \frac{2t}{(|y|_k^2 + t^2)^2}\, dt \right| \\
\lesssim \quad & \frac{|f_l(y)||f(y)|_{n-k}^2}{|y|_k^4} \\
\lesssim \quad & |y|_k^{3\alpha - 1}, \qquad (3.71)
\end{aligned}
$$

and since

$$
\begin{aligned}
& \int_{F^{-1}(\mathrm{B}^k(0,R))} |y|_k^{3\alpha-1}\, d\mathscr{H}^k y \\
= \quad & \int_{\partial \mathrm{B}^k(0,1)} \int_0^{r(R,\omega)} |r\omega|_k^{3\alpha-1} r^{k-1}\, dr\, d\mathscr{H}^{k-1}\omega \\
\lesssim \quad & R^{3\alpha-1+k}, \qquad (3.72)
\end{aligned}
$$

we have

$$
\begin{aligned}
(3.70) \quad = \quad & \int_{F^{-1}(\mathrm{B}^n(0,R))} f_l(y) \frac{1}{|y|_k^2} \Theta(F(y)) \, d\mathscr{H}^k y \\
& + o(R^k) \quad \text{as } R \downarrow 0 \\
= \quad & \int_{F^{-1}(\mathrm{B}^n(0,R))} f_l(y) \frac{1}{|y|_k^2} \Theta(F(0)) \, d\mathscr{H}^k y \\
& + \int_{F^{-1}(\mathrm{B}^n(0,R))} f_l(y) \frac{1}{|y|_k^2} (\Theta(F(y)) - \Theta(F(0))) \, d\mathscr{H}^k y \\
& + o(R^k) \quad \text{as } R \downarrow 0. \qquad (3.73)
\end{aligned}
$$

Using (3.5) for the second term of the right-hand side in (3.73), we have

$$
\begin{aligned}
(3.73) \quad = \quad & \int_{F^{-1}(\mathrm{B}^n(0,R))} f_l(y) \frac{1}{|y|_k^2} \Theta(F(0)) \, d\mathscr{H}^k y \\
& + o(R^k) \quad \text{as } R \downarrow 0. \qquad (3.74)
\end{aligned}
$$

Using (3.21) for (3.74), we have

$$
\begin{aligned}
(3.74) \quad = \quad & \int_{\partial \mathrm{B}^k(0,1)} \int_0^{r(R,\omega)} \frac{f_l(r\omega)}{|r\omega|_k^2} r^{k-1} \Theta(0) \, dr \, d\mathscr{H}^{k-1} \omega \\
& + o(R^k) \quad \text{as } R \downarrow 0 \\
= \quad & \int_{\partial \mathrm{B}^k(0,1)} \int_0^{r(R,\omega)} f_l(r\omega) r^{k-3} \Theta(0) \, dr \, d\mathscr{H}^{k-1} \omega \\
& + o(R^k) \quad \text{as } R \downarrow 0. \qquad (3.75)
\end{aligned}
$$

The derivative of the first term of the right-hand side in (3.75) with respect to R is

$$\begin{aligned}
&\int_{\partial \mathrm{B}^k(0,1)} \frac{f_l(r(R,\omega)\omega)r(R,\omega)^{k-3}R\Theta(0)}{r(R,\omega)+f(r(R,\omega))\cdot \nabla f(r(R,\omega)\omega)\omega}\, d\mathscr{H}^{k-1}\omega \\
= &\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)r(R,\omega)^{k-4}R\Theta(0)\, d\mathscr{H}^{k-1}\omega \\
&+\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)r(R,\omega)^{k-3}R \\
&\qquad\times\left(\frac{\Theta(0)}{r(R,\omega)+f(r(R,\omega))\cdot \nabla f(r(R,\omega)\omega)\omega}\right. \\
&\qquad\qquad\left.-\frac{\Theta(0)}{r(R,\omega)}\right) d\mathscr{H}^{k-1}\omega. \qquad (3.76)
\end{aligned}$$

Since

$$\begin{aligned}
&\left|\frac{1}{r(R,\omega)+f(r(R,\omega))\cdot \nabla f(r(R,\omega)\omega)\omega}-\frac{1}{r(R,\omega)}\right| \\
= &\left|\int_0^{f(r(R,\omega))\cdot \nabla f(r(R,\omega)\omega)\omega} \frac{d}{dt}\left(\frac{1}{r(R,\omega)+t}\right) dt\right| \\
= &\int_0^{f(r(R,\omega))\cdot \nabla f(r(R,\omega)\omega)\omega} \frac{1}{(r(R,\omega)+t)^2}\, dt \\
\lesssim &\left|\frac{f(r(R,\omega))\cdot \nabla f(r(R,\omega)\omega)\omega}{r(R,\omega)^2}\right| \\
\lesssim &\ r(R,\omega)^{2\alpha}, \qquad (3.77)
\end{aligned}$$

we have

$$\begin{aligned}
(3.76) = &\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)r(R,\omega)^{k-4}R\Theta(0)\, d\mathscr{H}^{k-1}\omega \\
&+o(R^{k-1}) \quad \text{as } R\downarrow 0 \\
= &\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)R^{k-3}\Theta(0)\, d\mathscr{H}^{k-1}\omega \\
&+\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)(r(R,\omega)^{k-4}-R^{k-4})R\Theta(0)\, d\mathscr{H}^{k-1}\omega \\
&+o(R^{k-1}) \quad \text{as } R\downarrow 0 \qquad (3.78)
\end{aligned}$$

by the substituting of (3.77) into (3.76), we have

$$\left|r(R,\omega)^{k-4}-R^{k-4}\right| \lesssim \left|\int_{r(R,\omega)}^{R} t^{k-5}dt\right|. \tag{3.79}$$

Using (3.19) for (3.79), we find that

$$\begin{aligned}(3.79) &\lesssim \left|R^{k-6}\int_{r(R,\omega)}^{R} t\,dt\right| \\ &\lesssim \left|R^{k-6}(R^2-r(R,\omega)^2)\right| \\ &\lesssim R^{k-6}\left|f(r(R,\omega))\right|^2 \\ &\lesssim R^{2\alpha-6+k}. \end{aligned} \tag{3.80}$$

Substituting (3.80) into (3.78), we have

$$\begin{aligned}(3.78) &= R^{k-3}\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)\Theta(0)\,d\mathscr{H}^{k-1}\omega \\ &\quad +o(R^{k-1}) \quad \text{as } R\downarrow 0. \end{aligned} \tag{3.81}$$

We divide both sides of (3.75) by R^k, and use L'Hospital's rule. Then we have

$$\begin{aligned}&\lim_{R\downarrow 0}\frac{(3.75)}{R^k} \\ &=\lim_{R\downarrow 0}\frac{1}{R^k}\int_{\partial \mathrm{B}^k(0,1)}\int_0^{r(R,\omega)} f_l(r\omega)r^{k-3}\Theta(0)\,dr\,d\mathscr{H}^{k-1}\omega \\ &=\lim_{R\downarrow 0}\frac{1}{kR^{k-1}}\frac{d}{dR}\left(\int_{\partial \mathrm{B}^k(0,1)}\int_0^{r(R,\omega)} f_l(r\omega)r^{k-3}\Theta(0)\,dr\,d\mathscr{H}^{k-1}\omega\right). \end{aligned} \tag{3.82}$$

Substituting (3.81) into (3.82), we have

$$\begin{aligned}(3.82) &= \lim_{R\downarrow 0}\frac{1}{kR^{k-1}}R^{k-3}\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)\Theta(0)\,d\mathscr{H}^{k-1}\omega \\ &= \lim_{R\downarrow 0}\frac{1}{kR^2}\int_{\partial \mathrm{B}^k(0,1)} f_l(r(R,\omega)\omega)\Theta(0)\,d\mathscr{H}^{k-1}\omega. \end{aligned} \tag{3.83}$$

The assertion of Theorem 3.1 follows from (3.65) and (3.83). □

3.4 Comparison with the Laplacian of a graph

If the varifold has higher regularity, then the assertion of Theorem 3.1 becomes simpler. Corollary 3.6 associates the Laplacian of a graph which represents a varifold with the generalized mean curvature of the varifold.

Corollary 3.6 *Let f and F satisfy (3.2) and (3.3). We assume that $\alpha > 1/2$. Then*

$$\lim_{R\downarrow 0}\frac{1}{R}\int_{\partial\mathrm{B}^k(0,1)}\nabla F(R\omega)\cdot\omega\Theta(0)\,d\mathscr{H}^{k-1}\omega\in\mathbb{R}^n \tag{3.84}$$

and this value coincides with the generalized mean curvature.

Proof. We rewrite the integrand of (3.64) as

$$\begin{aligned}
&\frac{\nabla f_l(r(R,\omega)\omega)\cdot\omega}{r(R,\omega)+f(r(R,\omega)\omega)\cdot\nabla f(r(R,\omega)\omega)\omega}\\
=\;&\nabla f_l(r(R,\omega)\omega)\cdot\omega\\
&\times\left(\frac{1}{r(R,\omega)+f(r(R,\omega)\omega)\cdot\nabla f(r(R,\omega)\omega)\omega}-\frac{1}{r(R,\omega)}\right)\\
&+\nabla f_l(r(R,\omega)\omega)\cdot\omega\left(\frac{1}{r(R,\omega)}-\frac{1}{R}\right)\\
&+\frac{1}{R}\left(\nabla f_l(r(R,\omega)\omega)-\nabla f_l(R\omega)\right)\cdot\omega+\frac{\nabla f_l(R\omega)\cdot\omega}{R}.
\end{aligned}\tag{3.85}$$

Using (3.11) and (3.19) for the third term of (3.85), we have

$$\begin{aligned}
\frac{1}{R}\left|\nabla f_l(r(R,\omega)\omega)-\nabla f_l(R\omega)\right| &\leq \frac{C_2}{R}\left|r(R,\omega)\omega-R\omega\right|^\alpha\\
&= \frac{C_2}{R}\left(\frac{|f(r(R,\omega)\omega)|^2}{r(R,\omega)+R}\right)^\alpha\\
&\lesssim R^{2\alpha^2+\alpha-1},
\end{aligned}\tag{3.86}$$

and we note that

$$2\alpha^2+\alpha-1>0$$

is positive when $\alpha > 1/2$. Then we have

$$\begin{aligned}
(3.85) \quad = \quad & \nabla f_l(r(R,\omega)\omega)\cdot \omega \\
& \times \left(\frac{1}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\cdot \omega} - \frac{1}{r(R,\omega)} \right) \\
& + \nabla f_l(r(R,\omega)\omega)\cdot \omega \left(\frac{1}{r(R,\omega)} - \frac{1}{R} \right) \\
& + \frac{\nabla f_l(R\omega)\cdot \omega}{R} + o(1) \quad \text{as } R \downarrow 0. \qquad (3.87)
\end{aligned}$$

Substituting (3.77) into the first term of the right-hand side in (3.87), we have

$$\begin{aligned}
& |\nabla f_l(r(R,\omega)\omega)\cdot \omega| \\
& \times \left| \frac{1}{r(R,\omega) + f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\cdot \omega} - \frac{1}{r(R,\omega)} \right| \\
\lesssim \quad & R^{3\alpha}. \qquad (3.88)
\end{aligned}$$

And using (3.88), we have

$$\begin{aligned}
(3.87) \quad = \quad & \nabla f_l(r(R,\omega)\omega)\cdot \omega \left(\frac{1}{r(R,\omega)} - \frac{1}{R} \right) + \frac{\nabla f_l(R\omega)\cdot \omega}{R} \\
& + o(1) \quad \text{as } R \downarrow 0. \qquad (3.89)
\end{aligned}$$

It holds that

$$\begin{aligned}
\left| \frac{1}{r(R,\omega)} - \frac{1}{R} \right| \quad & = \quad \frac{R - r(R,\omega)}{r(R,\omega)R} \\
& = \quad \frac{R^2 - r(R,\omega)^2}{(R + r(R,\omega))r(R,\omega)R)}. \qquad (3.90)
\end{aligned}$$

Substituting (3.11) and (3.19) into (3.90), we have

$$\begin{aligned}
(3.90) \quad & \lesssim \quad \frac{|f(r(R,\omega)\omega)|^2_{n-k}}{R^3} \\
& \lesssim \quad R^{2\alpha - 1}. \qquad (3.91)
\end{aligned}$$

Substituting (3.91) into (3.89), we have

$$(3.89) \quad = \quad \frac{\nabla f_l(R\omega)\cdot \omega}{R} + o(1) \quad \text{as } R \downarrow 0. \qquad (3.92)$$

Hence, we integrate (3.92), and take the limit for its as $R \downarrow 0$. Then we have

$$\begin{aligned}
&\int_{\partial \mathrm{B}^k(0,1)} \frac{\nabla f_l(r(R,\omega)\omega)\cdot \omega \Theta(0)}{r(R,\omega)+f(r(R,\omega)\omega)\cdot \nabla f(r(R,\omega)\omega)\cdot \omega}\, d\mathscr{H}^{k-1}\omega \\
= \quad & \frac{1}{R}\int_{\partial \mathrm{B}^k(0,1)} \nabla f_l(R\omega)\cdot \omega \Theta(0)\, d\mathscr{H}^{k-1}\omega \qquad (3.93)\\
& +o(1) \quad \text{as } R \downarrow 0. \qquad (3.94)
\end{aligned}$$

$\square$

Chapter 4

Inverse of a tangent-point radius and some examples

In this chapter, we explain the vector-valued tangent-point radius, and we introduce a generalization of the mean curvature vector different from [1]. Theorem 3.1 says that this coincides with that of [1] when the varifold V is locally the graph of a $C^{1,\alpha}$ function with $\alpha > 1/3$. Furthermore we give some examples.

4.1 The explanation of the geometric meaning of the main theorem

Suppose that $S \in \mathrm{G}(n,k)$, $x \in \mathbb{R}^n$ and $a \in S$. Then we say

$$\frac{|x-a|_n^2}{2|S^\perp(x-a)|_n}$$

is the tangent-point radius. We consider a curve which is tangent to S at a. If a point x on the curve approaches a along the curve, then inverse of the tangent-point radius tends to its curvature. The quantity

$$\frac{2S^\perp(x-a)}{|x-a|_n^2}$$

in the integrand of (3.6) corresponds to the vector-valued inverse of the tangent-point radius. Hence its integral average might approximate the mean curvature, which is the heart of (3.6). By Theorem 3.1, the classical mean curvature is represented by (3.6). It is a generalization of the generalized mean curvature without using the variation, and this is different from the corresponding generalization in [1].

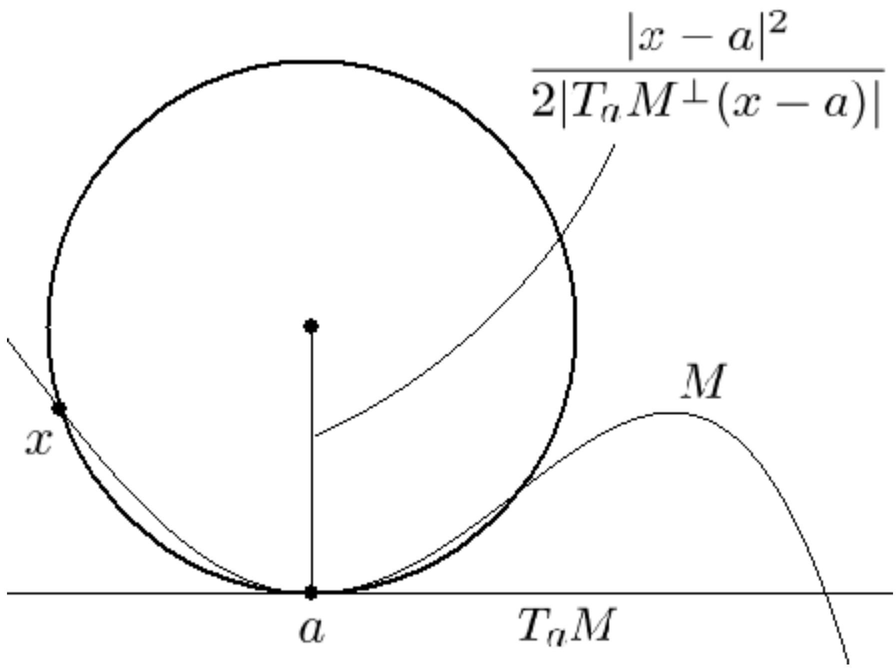

Figure 4.1: The figure of the tangent-point radius.

4.2 Examples

In this section, we present examples with the help of (3.6); it is easier to calculate than the generalization of the mean curvature in [1]. For these examples, we prepare Propositon 4.1 below.

Proposition 4.1 *Under the same assumptions as Theorem 3.1 with* $T = \mathrm{Im}DF(0) = \mathbb{R}^k \times \{0\}$, $F(0) = 0$. *Then we have*

$$\mathrm{h}(V,0) = \lim_{R \downarrow 0} \frac{2k}{\omega_k R^k} \int_{\mathrm{B}^n(0,R)} \frac{T^\perp(F(y))}{|y|_k^2} \, d\mathscr{H}^k y. \tag{4.1}$$

Proof. We rewrite the integrand in (3.75) as

$$\begin{aligned}
&\int_0^{r(R,\omega)} f_l(r\omega) r^{k-3}\Theta(0)\, dr \\
= \;& -\int_{r(R,\omega)}^R f_l(r\omega) r^{k-3}\Theta(0)\, dr + \int_0^R f_l(r\omega) r^{k-3}\Theta(0)\, dr. \qquad (4.2)
\end{aligned}$$

Using (3.11) and (3.19) for the first term of the right-hand side in (4.2), we have

$$\begin{aligned}
\left| \int_{r(R,\omega)}^R f_l(r\omega) r^{k-3}\, dr \right| &\lesssim R^{k-4+\alpha+1} \int_{r(R,\omega)}^R r\, dr \\
&\lesssim R^{\alpha-3+k}(R^2 - r(R,\omega)^2) \\
&= R^{\alpha-3-k} |f(r(R,\omega)\omega)|^2 \\
&\lesssim R^{3\alpha-1+k}. \qquad (4.3)
\end{aligned}$$

By Theorem 3.1 and (3.75), we have

$$\mathrm{h}(V,0)\cdot e_l = \lim_{R\downarrow 0} \frac{2k}{\omega_k R^k} \int_{\partial \mathrm{B}^k(0,1)} \int_0^{r(R,\omega)} f_l(r\omega) r^{k-3}\, dr\, d\mathscr{H}^{k-1}\omega. \qquad (4.4)$$

Substituting (4.2) into (4.4), we have

$$\begin{aligned}
(4.4) = \lim_{R\downarrow 0} \frac{2k}{\omega_k R^k} \int_{\partial \mathrm{B}^k(0,1)} \Bigg\{ & -\int_{r(R,\omega)}^R f_l(r\omega) r^{k-3}\, dr \\
& + \int_0^R f_l(r\omega) r^{k-3}\, dr \Bigg\}\, d\mathscr{H}^{k-1}\omega. \qquad (4.5)
\end{aligned}$$

Substituting (4.3) into (4.5), we have

$$(4.5) = \lim_{R\downarrow 0} \frac{2k}{\omega_k R^k} \int_{\partial \mathrm{B}^k(0,1)} \int_0^R f_l(r\omega) r^{k-3}\, dr\, d\mathscr{H}^{k-1}\omega. \qquad (4.6)$$

Finally, substituting (2.11) into (4.6), we have

$$(4.6) = \lim_{R\downarrow 0} \frac{2k}{\omega_k R^k} \int_{\mathrm{B}^n(0,R)} \frac{F(y)\cdot e_l}{|y|_k^2}\, d\mathscr{H}^k y. \qquad (4.7)$$

□

We assume that

$$0 < \beta < 1 \tag{4.8}$$

in the following examples.

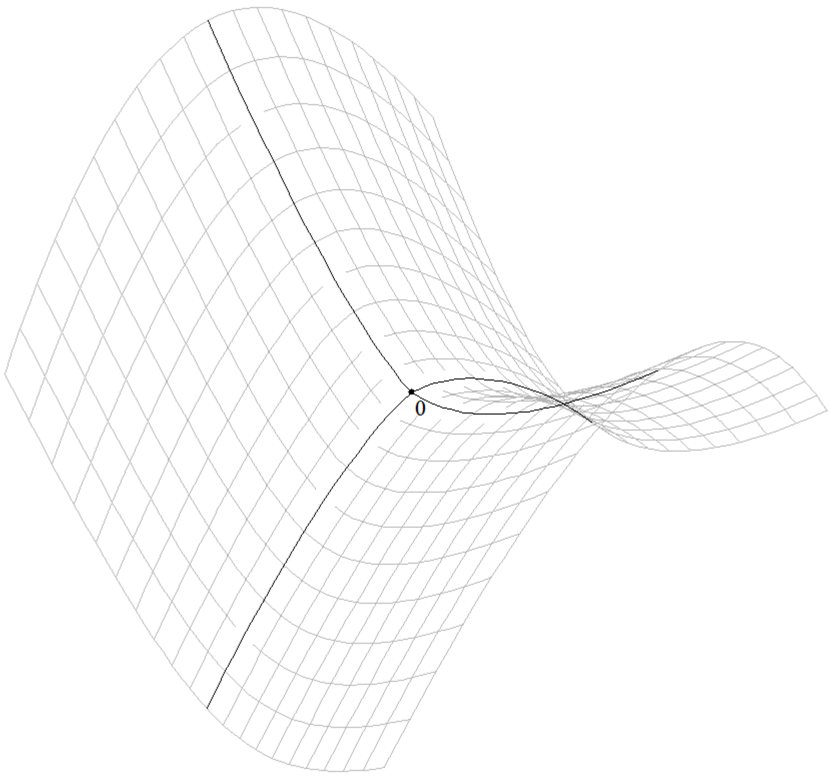

Figure 4.2: The figure of Example 4.2.

Example 4.2 *Suppose*

$$n = k + 1,\ \ y = (y_1, \ldots, y_k) \in \mathbb{R}^k,\ \ |y|_k = r \quad and \quad y_1 = r\sigma_1.$$

We let

$$f(y) = r^{1+\beta}\sigma_1.$$

Then $f \notin C^2$, and the mean curvature in the classical sense of f cannot be defined. However, by Proposition 2.11, we have

$$\begin{aligned}\frac{1}{R^k}\int_{\mathrm{B}^k(0,R)} \frac{F(y)\cdot e_{k+1}}{|y|_k^2}\,d\mathscr{H}^k y &= \frac{1}{R^k}\int_{\mathrm{B}^k(0,R)} \frac{f(y)}{|y|_k^2}\,d\mathscr{H}^k y \\ &= \frac{1}{R^k}\int_0^R r^{k+\beta-2}\,dr \int_{\partial\mathrm{B}^k(0,1)} \sigma_1\,d\mathscr{H}^{k-1}\sigma = 0,\end{aligned}$$

and the generalization of mean curvature exists.

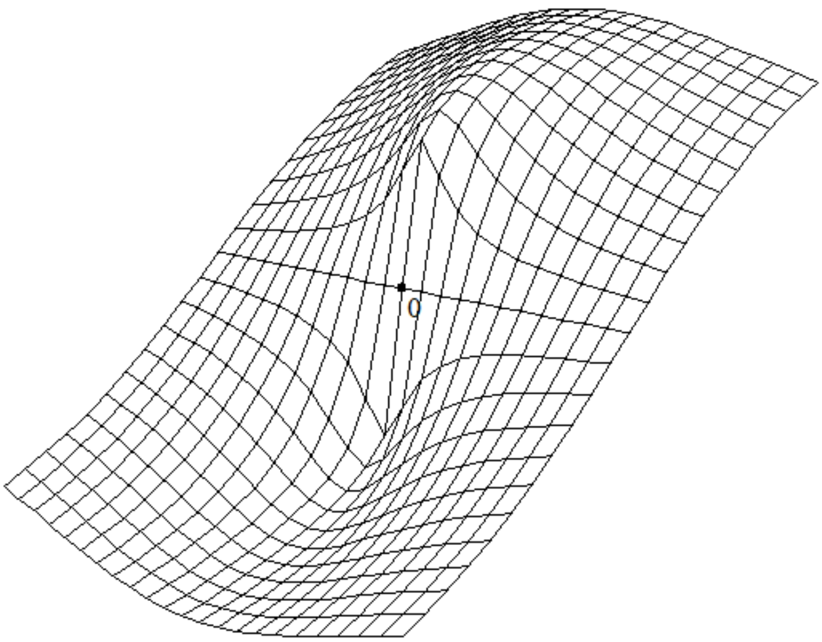

Figure 4.3: The figure of Example 4.3.

Example 4.3 *Suppose that*

$$k = 2,\ n = 3 \ \ \textit{and set} \ \ y_1 = r\cos\theta.$$

We let

$$f(y) = r^{1+\beta}\cos 2\theta.$$

Then $f \notin C^2$, and the mean curvature in the classical sense of f cannot be defined. However, by Proposition 2.11, we have

$$\frac{1}{R^2}\int_{\mathrm{B}^2(0,R)} \frac{f(y)}{|y|_2^2}\, d\mathscr{H}^2 x = \frac{1}{R^2}\int_0^R r^\beta\, dr \int_0^{2\pi} \cos 2\theta\, d\theta = 0$$

and the generalization of mean curvature exists.

Chapter 5

Comparisons with other generalizations of mean curvature

In this chapter, we introduce generalizations of the mean curvature different from Allard's one and we make a comparison with (3.6).

5.1 Comparison with curvature measures

In this section, we give a generalization of the mean curvature using Steiner's formula.

Definition 5.1 For a convex body $K \subset \mathbb{R}^n$, let $p(K, \cdot\,) : \mathbb{R}^n \to K$ be the nearest point map for K and define

$$u(K, x) = \frac{p(K, x) - x}{|p(K, x) - x|_n} \tag{5.1}$$

for any $x \in \mathbb{R}^n$. For $\varepsilon > 0$, we let

$$K_\varepsilon = \{\, x \in \mathbb{R}^n \,:\, \mathrm{dist}\,(K, x) < \varepsilon \,\}\,. \tag{5.2}$$

Also, for a Borel subset $E \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^n \times S^{n-1}$, we set

$$A_\varepsilon(K, E) = \{\, x \in K_\varepsilon \,:\, p(K, x) \in E \,\} \tag{5.3}$$

and

$$M_\varepsilon(K,A) = \{\, x \in K_\varepsilon \setminus K \;:\; (p(K,x), u(K,x)) \in A \,\}\,. \tag{5.4}$$

Next theorem is called Steiner's formula, see [8], [6] and [7].

Theorem 5.2 *For any convex body $K \subset \mathbb{R}^n$, there exist Radon measures*

$$\phi_0(K,\cdot\,),\ \ldots,\ \phi_n(K\cdot\,) \tag{5.5}$$

on $\mathbb{R}^n$, and Radon measures

$$\theta_0(K,\cdot\,),\ \ldots,\ \theta_{n-1}(K,\cdot\,) \tag{5.6}$$

on $\mathbb{R}^n \times S^{n-1}$ such that

$$\mathscr{H}^n(A_\varepsilon(K,E)) = \sum_{l=0}^{n} \varepsilon^{n-l}\omega_{n-l}\phi_l(K,E) \tag{5.7}$$

and

$$\mathscr{H}^n(M_\varepsilon(K,E)) = \sum_{l=0}^{n-1} \varepsilon^{n-l}\omega_{n-l}\theta_l(K,A) \tag{5.8}$$

for any Borel subset $E \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^n \times S^{n-1}$.

Definition 5.3 We call $\phi_i(K,\cdot\,)$ *Federer's curvature measure* ($i = 0, \ldots, n$), and we call $\theta_j(K,\cdot\,)$ the *generalized curvature measure* ($j = 0, \ldots, n-1$). In particular, we call $\phi_{n-2}(K,\cdot\,)$ the *mean curvature measure.*

The next theorem is a comparison between the generalized mean curvature vector and the mean curvature measure. See [8].

Theorem 5.4 *For $\phi_{n-2}(K,\cdot\,)$-almost every $x \in \mathbb{R}^n$, there exists a Radon measure λ_x on S^{n-1} such that*

$$\begin{aligned}
&\int_{\mathbb{R}^n \times S^{n-1}} g(x,y)\, d\theta_{n-2}(K,\cdot\,)(x,y) \\
=\;& \int_{\mathbb{R}^n} \left(\int_{S^{n-1}} g(x,y)\, d\lambda_x y \right) d\phi_{n-2}(K,\cdot\,)x
\end{aligned} \tag{5.9}$$

for any $g \in \mathscr{K}(\mathbb{R}^n \times S^{n-1})$*. Moreover, for any Borel subset* $E \subset \mathbb{R}^n$*, we have*

$$\begin{aligned} &\frac{n-1}{2\pi} \int_E \mathrm{h}(\mathrm{v}(\partial K), x)\, d\mathscr{H} \llcorner \partial K x \\ = &\int_{\partial K \cap E} \left(\int_{S^{n-1}} y\, d\lambda_x y \right) \phi_{n-2}(K, \cdot\,)x. \end{aligned} \tag{5.10}$$

By this theorem and Theorem 3.1, we can consider the representation in (3.6) is a part of Steiner's formula, that is, (3.6) represents a perturbation of the convex body K if ∂K is sufficiently smooth. Moreover, we can expect a representation of other curvature measures using formulas like (3.6).

The mean curvature measure $\phi_{n-2}(K, \cdot\,)$ is defined for a convex body, and gives the representation (5.10) of the generalized mean curvature of the boundary of the convex body. We need not assume such convexity for our representation (3.6); see Examples 4.2–4.3.

5.2 Comparison with the variational mean curvature

In this section, we introduce a generalization of the mean curvature using a minimizer of some functional.

Definition 5.5 Let $W \subset \mathbb{R}^n$ be open. For $v : W \to \mathbb{R}$, $E \subset W$ and an open subset $U \subset W$, set

$$\|Dv\|(U) = \sup \left\{ \int v(x) Dg(x) \cdot \mathbb{R}^n\, dx \ :\ g \in \mathscr{X}(W),\ |g|_n \leq 1 \right\}, \tag{5.11}$$

$$\|\partial E\|(U) = \sup \left\{ \int_E Dg(x) \cdot \mathbb{R}^n\, dx \ :\ g \in \mathscr{X}(W),\ |g|_n \leq 1 \right\}. \tag{5.12}$$

For a summable function $H : W \to \mathbb{R}$, we set

$$\mathscr{F}_H(E, U) = \|\partial E\|(U) + \int_E H(x)\, dx. \tag{5.13}$$

Using the above notation, a set E is said to have the *variational mean curvature* H in W if

$$\|\partial E\|(U) < \infty \tag{5.14}$$

for any open subset $U \subset W$ whose closure is compact, and

$$\mathscr{F}_H(E,U) \le \mathscr{F}_H(F,U) \tag{5.15}$$

for any open subsets $U \subset W$ and $F \subset W$ such that $\mathrm{Closure}((E \setminus F) \cup (F \setminus E)) \subset U$ is compact.

The next theorem is a comparison between the variational mean curvature and the generalized mean curvature, proved in [3].

Theorem 5.6 *Let $\Omega \subset \mathbb{R}^{n-1}$ be an open subset and $f : \Omega \to \mathbb{R}$ be a $C^{1,\alpha}$ function $(0 < \alpha < 1)$. Also, we set*

$$E = \{\, (y,z) \in \Omega \times \mathbb{R} : \ \ z \le f(y) \,\} . \tag{5.16}$$

Suppose

$$\int_\Omega \sqrt{1 + |Df(y)|_{n-1}^2}\, dy \le \int_\Omega \sqrt{1 + |Dv(y)|_{n-1}^2}\, dy \tag{5.17}$$

for any $v : \Omega \to \mathbb{R}$ such that $\|Dv\|(\Omega) < \infty$, $\mathrm{Closure}(\mathrm{spt}\,(v - f)) \subset \Omega$ and $f \le v$. Then E has the variational mean curvature in $\Omega \times \mathbb{R}$. Moreover, the generalized mean curvature of the graph of f exists, and it coincides with the variational mean curvature of E.

By this theorem and Theorem 3.1, we can consider (3.6) as a representation of a minimizer of (5.13). Hence we can expect a representation of minimizers of functionals like (5.13) using (3.6).

The variational mean curvature in Definition 5.5 is defined for the hypersurfaces in Examples 4.2–4.3. However, since the variational mean curvature is given as an L^1-function, that is, since it is defined only at almost every point, it is difficult to see the geometric meaning pointwisely around a given point. On the other hand, we can see the geometric meaning through (3.6) as the limit of integral averages of the vector-valued inverse of the tangent-point radius.

Acknowledgement. The author would like to thank Prof. Neal Bez for English language editing.

Bibliography

[1] Allard, W. K., *On the first variation of a varifold*, Ann. of Math. (2) **95** (1978), 417–491.

[2] Almgren, F. J. Jr., “The Theory of Varifolds”, Mimeographed notes, Princeton, 1965.

[3] Barozzi, E., Gonzalez, E. & U. Massari, *The mean curvature of a Lipshitz continuous manifold*, Rend. Mat. Acc. Lincei. s. 9, **14** (2003), 257–277.

[4] Brakke, K. A., “The Motion of a Surface by its Mean Curvature”, Mathematical notes **20**, Princeton University Press, Princeton, 1978.

[5] Evans, L. C. & R. F. Gariepy, “Measure Theory and Fine Properties of Functions”, Studies in Avanced Mathematics, CRC Press, Boca Ratin, Ann Arbor, London, 1992.

[6] Federer, H., *Curvature Measures*, Trans. Amer. Mat. Soc. **93** (3) (1959) 418–491.

[7] Federer, H., “Geometric Measure Theory”, Die Grundlehren der mathematischen Wissenschaften **53**, Springer-Verlag, New York, 1969.

[8] Ossanna, E., *Comparison between the generalized mean curvature according to Allard and Federer's mean curvature measure*, Rend. Sem. Mat. Univ. Padova, **88** (1992), 221–227.

[9] L. Simon, Lecture on geometric measure theory, Proceedings of the Center for Mathematical Analysis, Australian National University, vol. 3, Australian National University Center for Mathematical Analysis, Canberra, 1983.

[10] Kasai, K. & Y. Tonegawa, *A general regularity theory for weak mean curvature flow*, Car. Var. Partial Differential Equations **50** (2014), 1–68.

[11] Menne, U., *Second order rectifiability of integral varifolds of locally bounded first variation*, J. Geom. Anal. (2) **23** (2013), 709–763.

[12] Strzelecki, P. & H. von der Mosel, *Tangent-point repulsive potentials for a class of non-smooth m-dimensional sets in $\mathbb{R}^n$. Part I: Smoothing and self-avoidance effects*, Geom. Anal. **23** (2013), 1085–1139.

[13] Takano, K., *A geometric representation of the generalized mean curvature*, submitted.