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Some Estimates of The Symbol and The Symbol Calculus

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Summary

When a differential operator on the Schwartz class is given, on the assumption that the adjoint of the symbol corresponding to the operator possesses a series expansion, namely, its adjoint symbol admits an asymptotic sum, if the operator given is a pseudo-differential operator, then its adjoint operator becomes the same type of pseudo-differential operator. Consequently, the original operator can be extended to an operator on the space of tempered distributions. In order to realize the above-mentioned program, we need some precise estimates of the symbol. We will note some crucial technical points behind the proof.

Key Words: pseudo-differential operator, symbol, the symbol calculus, the Schwartz class, tempered distribution, asymptotic sum.

1. Introduction and notation

In this section we shall first explain the notation used throughout this article. Let α be a multiindex, namely, $\alpha \in \mathbb{Z}_+^n$, $\alpha = (\alpha_1,...,\alpha_n)$, and $\alpha_j \in \mathbb{Z}_+$ for any $j = 1, 2,...,n$. We denote $\partial_x = \frac{\partial}{\partial x}$ and $D_x = \frac{1}{i}\partial_x$, where $i = \sqrt{-1}$. For $x = (x_1,...,x_n) \in \mathbb{R}^n$, we mean $\partial_x = \partial_1\partial_2\cdots\partial_n$ and $\partial_j = \partial_{xj}$, and $\partial_x^\alpha = \partial_x^{\alpha 1} \cdots \partial_n^{\alpha n}$. In particular, we use the following simple notation

$$D_x^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \left(\frac{1}{i}\right)^n \frac{\partial^m}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \tag{1}$$

when $|\alpha| = \alpha_1 + \cdots + \alpha_n = m$.

The space $\mathcal{S} \equiv \mathcal{S}\,(\mathbb{R}^n)$ is the totality of C^∞ functions $u = u(x) = u(x_1,...,x_n)$ defined on $\mathbb{R}^n$ which decrease rapidly, namely, for any $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+^n$,

$$\lim_{|x|\to\infty} |x^\alpha \partial_x^\beta u(x)| = 0 \tag{2}$$

holds. The space $\mathcal{S}$ is also called the Schwartz class. The semi-norms on $\mathcal{S}$ is given by

$$|u|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| < +\infty \tag{3}$$

for any α, β in $\mathbb{Z}_+^n$. A linear form on $\mathcal{S}$ which is continuous for the semi-norms $|\cdot|_{\alpha,\beta}$, is said to be a tempered distribution on $\mathbb{R}^n$, and the space of tempered distributions is denoted by $\mathcal{S}'$. On the other hand, the space of distributions with compact support in $\mathbb{R}^n$ is denoted by $\mathcal{E}'(\mathbb{R}^n)$, which is identified with the space of linear forms on $C^\infty(\mathbb{R}^n)$ being continuous for the topology defined by the semi-norms

$$|u|_{K,m} := \sup_{x\in K} \sup_{|\alpha|\leqslant m} |\partial_x^\alpha u(x)|, \tag{4}$$

where K runs over the compact subsets of $\mathbb{R}^n$ and m runs over the integers. Note that $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'$ and $\mathcal{S} \subset \mathcal{S}'$. Since $\mathcal{D} = C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}$, it is easy to see that $\mathcal{S}$ is dense in $\mathcal{S}'$ with the topology of simple convergence on $\mathcal{S}$. The Fourier transformation $\mathcal{F}$ is a continuous linear mapping

$$\mathcal{F} : \mathcal{S} \ni u \mapsto \mathcal{F}u = \hat{u} \in \mathcal{S}, \tag{5}$$

defined by

$$(\mathcal{F}u)(\xi) \equiv \hat{u}(\xi) := \int e^{-ix\cdot\xi} u(x)dx \tag{6}$$

for $u \in \mathcal{S}$ and $\xi \in \mathbb{R}^n$. Its inverse mapping $\mathcal{F}^{-1}$ is given by

$$(\mathcal{F}^{-1}\hat{u})(x) = u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \hat{u}(\xi)d\xi. \tag{7}$$

In the last we shall give you a rough idea about what this article was written for. When a differential operator A on the Schwartz class $\mathcal{S}$ is given, on the assumption that the adjoint $a^*(x, \xi)$ of the symbol $a(x, \xi)$ corresponding to the operator A possesses a series expansion $\sum_j a_j^*$, namely, its adjoint symbol a^* admits an asymptotic sum $\sum_{j=1}^k a_j^*$, if the operator A given is a pseudo-differential operator, then its adjoint operator A^* becomes the same type of pseudo-differential operator. Consequently, the original operator A can be extended to an operator on the space $\mathcal{S}'$ of tempered distributions. In order to realize the above-mentioned program, we need some precise estimates of the symbol. We will note some crucial technical points behind the proof.

Here is the principal assertion in this article, which provides us with a fundamental result of the symbol calculus.

Theorem 1. *Suppose that* $a \equiv a(x, \xi) \in S^m$.
(a) *Then*

$$a^*(x,\xi) = \frac{1}{(2\pi)^n} \int \exp\{-iy\cdot\eta\} \cdot \bar{a}(x-y, \xi-\eta)dyd\eta \tag{8}$$

belongs to S^m.
(b) *The symbol* $a^*(x, \xi)$ *satisfies the asymptotic formula*

$$a^*(x,\xi) \sim \sum_{\alpha\geq 0} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{a}(x,\xi). \tag{9}$$

(c) *If* $A = Op(a)$ *is a pseudo-differential operator of order* m, *then* $A= Op(a^*)$ *is a pseudo-differential operator of order* m.
(d) *Consequently*, A *extends to an operator from* $\mathcal{S}'(\mathbb{R}^n)$ *to* $\mathcal{S}'(\mathbb{R}^n)$.

2. Symbol and pseudo-differential operator

We shall start with defining the symbol class.

Definition 2. For $m \in \mathbb{R}$, let $S^m = S^m(\mathbb{R}^n\times\mathbb{R}^n)$ denote the set of $a \equiv a(x, \xi) \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ such that, for $\forall\alpha, \beta \in \mathbb{Z}_+^n$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \tag{10}$$

holds. An element $a \in S^m$ is called a symbol of order m. We also denote

$$S^{-\infty} = \bigcap_m S^m.$$

□

Let us define the semi-norms on S^m by

$$|a|_{\alpha,\beta}^m := \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} \left\{(1+|\xi|)^{-(m-|\beta|)}|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|\right\}. \tag{11}$$

The convergence $a_n \to a$ in S^m means that

$$\text{for} \quad \forall\alpha, \forall\beta, \qquad |a_n - a|_{\alpha,\beta}^m \to 0 \qquad (\text{as} \quad n \to \infty), \tag{12}$$

which determines the topology for the space S^m. Then S^m becomes a Fréchet space with respect to the metric induced by the semi-norms $|\cdot|_{\alpha,\beta}^m$. We write

$$a(x,\xi) \;\sim\; \sum_j a_j(x,\xi) \qquad (\text{when} \quad |\xi| \to \infty) \tag{13}$$

where the above expression means that for $\forall k \geq 0$,

$$a(x,\xi) - \sum_{j=0}^{k} a_j(x,\xi) \in S^{m_{k+1}} \tag{14}$$

with $a_j = a_j(x, \xi) \in S^{mj}$, $(\forall j \in \mathbb{N})$ and the sequence $\{m_j\}$ satisfies $m_j \to -\infty$.

For $a \in S^m$and $u \in \mathcal{S}$, the formula

$$Op(a)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi \tag{15}$$

defines a function of $\mathcal{S}$, and the mapping: $(a,u) \mapsto Op(a)u$ is continuous. For $a \in S^m$, the operator $Op(a)$ is the pseudo-differential operator with symbol $a = a(x, \xi)$. A pseudo-differential operator is said to be of order m if its symbol belongs to S^m. By convention we often denote $Op(a) = a(x, D)$ and $A = Op(a)$.

3. Adjoint and conjugate

Let A be an arbitrary operator from $\mathcal{S}$ to $\mathcal{S}$. Then we seek an operator A^*: $\mathcal{S} \to \mathcal{S}$, such that

$$(Au, v) = (u, A^*v), \qquad \forall u, v \in \mathcal{S}. \tag{16}$$

Note that, if A^* exists, then it is unique. Then A^* is called the adjoint of A. While, the existence of A^* allows us to define an extension $A : \mathcal{S}' \to \mathcal{S}'$ by the formula

$$(Au, v) = (u, A^*v), \qquad \text{for} \quad \forall u \in \mathcal{S}', \quad \forall v \in \mathcal{S}. \tag{17}$$

Notice that, if $u \in \mathcal{S}'$ and $v \in \mathcal{S}$, then the inner product (or Hermitian product) (u, v) denotes the duality bracket $\langle u, \overline{v}\rangle$. So that, we have

$$\langle Au, v\rangle = \langle u, \overline{A^*\overline{v}}\rangle. \tag{18}$$

Verification. Since we have $(u, v) = \langle u, \bar{v}\rangle$, we can get $(Au, v) = \langle Au, \bar{v}\rangle$. By employing this result, it is easy to see that

$$\langle Au, v\rangle = \langle Au, \bar{\bar{v}}\rangle = (Au, \bar{v}) = (u, A^*\bar{v}) = \langle u, \overline{A^*\bar{v}}\rangle. \tag{19}$$

□

If $P = \Sigma_{|\alpha|\leqslant m} a_\alpha(x) D^\alpha$ is a differential operator with slowly increasing C^∞ coefficients, then or all functions u, v in $\mathcal{S}$ we have

$$(Pu, v) = (u, P^*v), \qquad \text{where} \quad P^*v = \sum_{|\alpha|\leqslant m} D^\alpha(\bar{a}_\alpha v). \tag{20}$$

Note that P^* is a differential operator with slowly increasing coefficients in C^∞. It is interesting to note that its principal symbol is merely the conjugate of the principal symbol of P. As a matter of fact, we obtain the following lemma.

Lemma 3. *When $a(D)$ denotes a differential operator with slowly increasing coefficients in C^∞, then we have*

$$a(D)^* = \bar{a}(D). \tag{21}$$

Proof. From the definition of adjoint operator, we have

$$(a(D)u, v) = (u, a(D)^*v). \tag{22}$$

On the other hand, by virtue of the Fourier integral theory an easy computation with u and v in $\mathcal{S}$

$$(a(D)u, v) = \frac{1}{(2\pi)^n}(a(D)\hat{u}, \hat{v}) = \frac{1}{(2\pi)^n}(\hat{u}, \bar{a}(D)\hat{v}) = (u, \bar{a}(D)v) \tag{23}$$

yields to the equality $a(D)^* = \bar{a}(D)$, if we compare (20) to (21). □

In the above two cases on P and $a(D)$, the adjoint of the pseudo-differential operator in question is a pseudo-differential operator of the same order. This is true in general.

4. Kernel and pseudo-differential operator

Let Ω_1 and Ω_2 be open subsets of $\mathbb{R}^n$. For a differential operator $A = A_K$, the distribution $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ satisfies the equation

$$\langle A_K v, u\rangle = \langle K, u \otimes v\rangle, \tag{24}$$

where $u \in C_0^\infty(\Omega_1)$ and $v \in C_0^\infty(\Omega_2)$. The tensor product means here that

$$u \otimes v(x_1, x_2) = u(x_1)v(x_2). \tag{25}$$

Suppose now that the differential operator

$$A_K : C_0^\infty(\Omega_2) \to \mathcal{D}'(\Omega_1) \tag{26}$$

associated with K is linear and continuous. By virtue of the continuity of A_K, we have the following estimate: for any $u \in C_0^\infty(\Omega_1)$ and any compact subset $\widetilde{K}$ of Ω_1, there exist a positive constant

$c > 0$ and an integer $m \in \mathbb{N}$ such that

$$|\langle A_K v, u\rangle| \leqslant c \cdot \sup_{x \in \tilde{K}} \sup_{|\alpha| \leqslant m} |\partial^\alpha v(x)| \tag{27}$$

holds for any $v \in C_0^\infty(\Omega_2)$ with supp $v \subset \widetilde{K}$. If K is contained in $L^1_{loc}(\Omega_1 \times \Omega_2)$, then the dual pairing relation $\langle A_K v, u\rangle = \langle K, u \otimes v\rangle$ yields to an explicit expression

$$A_K v(x_1) = \int K(x_1, x_2) v(x_2) dx_2 \tag{28}$$

where $A_K v(x_1) \in \mathcal{D}'(\Omega_1)$. Because we have from the Fubini theorem

$$\langle A_K v, u\rangle = \langle K, u \otimes v\rangle = \iint K(x_1, x_2) u \otimes v(x_1, x_2) dx_1 dx_2 \tag{29}$$

$$= \int \left(\int K(x_1, x_2) v(x_2) dx_2 \right) u(x_1) dx_1 = \langle \int K(x_1, x_2) v(x_2) dx_2, u\rangle. \tag{30}$$

Let us observe that if the adjoint operator A^* exists, there is a simple expression for its kernel K^* employing the kernel K of A. As a matter of fact, we can get

$$\langle K(x, y), u(y) v(x)\rangle = \langle Au, v\rangle = \langle u, \overline{A^* \bar{v}}\rangle = \overline{\langle \bar{u}, A^* \bar{v}\rangle} \tag{31}$$

$$= \langle K^*(y, x), \bar{v}(x) \bar{u}(y)\rangle. \tag{32}$$

Finally, it follows from (32) that

$$K^*(y, x) = \overline{K(x, y)}. \tag{33}$$

Now we are going to consider seeking for a concrete expression of the adjoint of a pseudo-differential operator given. We suppose that the pseudo-differential operator A has a symbol $a \in S^m$. We would like to determine the adjoint A^* of A on the assumption that it exists. As we have seen in the above, it suffices to verify that if K is the kernel of the symbol $a = a(x, \xi)$ given by

$$K(x, y) = \frac{1}{(2\pi)^n} (\mathcal{F}_\xi a)(x, y - x), \tag{34}$$

where $\mathcal{F}_\xi a$ denotes the Fourier transform of a with respect to the variable ξ in $\mathcal{S}'(\mathbb{R}^{2n}) = \mathcal{S}'(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, then the operator with kernel K^* given by (33), sends $\mathcal{S}$ to $\mathcal{S}$.

LEMMA 4. *When $a \in S^{-\infty}$, its pseudo-differential operator $Op(a)$ possesses the kernel K given by*

$$K(x, y) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} a(x, \xi) d\xi. \tag{35}$$

Proof. Since we have

$$Op(a)u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} a(x, \xi) \hat{u}(\xi) d\xi \tag{36}$$

for any u in $\mathcal{S}$, from the definition of kernel K, we can get

$$Op(a)u(x) = \int K(x, y) u(y) dy. \tag{37}$$

On the other hand, it is easy to see that

$$Op(a)u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} a(x,\xi) d\xi \int e^{-iy\cdot\xi} u(y) dy \tag{38}$$

$$= \frac{1}{(2\pi)^n} \int u(y) dy \int e^{i(x-y)\cdot\xi} a(x,\xi) d\xi \tag{39}$$

$$= \int \left(\frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} a(x,\xi) d\xi \right) u(y) dy \tag{40}$$

where we have used the Fubini theorem. From (37) and (40), we finally obtain the desired equality (35). □

The Fourier inversion formula gives

$$a(x,\xi) = \mathcal{F}_{y\to\xi}[K(x, x-y)]. \tag{41}$$

LEMMA 5. *The adjoint a^* of symbol $a \in S^m$ is given as a function of a by*

$$a^*(x,\xi) = \frac{1}{(2\pi)^n} \int e^{-iy\cdot\eta} \bar{a}(x-y, \xi-\eta) dy d\eta. \tag{42}$$

Proof. Since we have $K^*(y, x) = \overline{K(x, y)}$, we can get easily

$$K^*(y,x) = \overline{K(x,y)} = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} \overline{a(y,\xi)} d\xi \tag{43}$$

by employing the integral representation of kernel K of $Op(a)$. Hence, an elementary calculation yields to

$$a^*(x,\xi) = \int K^*(x, x-y) e^{-iy\cdot\xi} dy = \frac{1}{(2\pi)^n} \iint e^{iy\cdot(\eta-\xi)} \bar{a}(x-y, \eta) dy d\eta \tag{44}$$

$$= \frac{1}{(2\pi)^n} \iint e^{-iy\cdot\eta} \bar{a}(x-y, \xi-\eta) dy d\eta. \tag{45}$$

□

5. Oscillatory integral

This section treats a class of oscillatory integrals of the type

$$\int_{\mathbb{R}^N} e^{i\varphi(\theta)} a(\theta) d\theta, \tag{46}$$

where φ is a rapidly varying function at infinity, and a is regular with essentially polynomial growth. We need the following technical lemma, which is closely related to the stationary phase.

LEMMA 6. *Let K be a compact subset of $\mathbb{R}^N$, and φ be a real-valued function in $C^\infty(\mathbb{R}^N)$ satisfying that*

$$|\varphi'(\theta)| \geq c_0 \geq 0 \quad \text{on} \quad K \tag{47}$$

for some positive constant c_0. Then for any function $a = a(\theta) \in C_0^\infty(K)$, for any $k \in \mathbb{N}$, *the oscillatory integral*

$$\int_{\mathbb{R}^N} e^{i\lambda\varphi(\theta)} a(\theta) d\theta \tag{48}$$

admits the following estimate:

$$\lambda^k \left| \int e^{i\lambda\varphi(\theta)} a(\theta) d\theta \right| \leqslant C_{k+1}(\varphi) \cdot C(c_0, K) \cdot \sup_{|\alpha| \leqslant k} |\partial^\alpha a(\theta)| \tag{49}$$

for every $\lambda \geq 1$, *where the constant* $C_{k+1}(\varphi)$ *remains bounded when* φ *remains bounded in the space* $C^{k+1}(K)$.

Proof. When we set

$$L := -i|\varphi'|^{-2} \sum_{j=1}^N \frac{\partial \varphi}{\partial \theta_j} \frac{\partial}{\partial \theta_j}, \tag{50}$$

we can get easily the equality

$$L(e^{i\lambda\varphi}) = \lambda e^{i\lambda\varphi}. \tag{51}$$

Hence, it follows immediately from (51) that

$$\lambda^k \int e^{i\lambda\varphi(\theta)} a(\theta) d\theta = \lambda^{k-1} \int \lambda e^{i\lambda\varphi(\theta)} a(\theta) d\theta = \lambda^{k-1} \int L(e^{i\lambda\varphi}) a(\theta) d\theta \tag{52}$$

$$= \lambda^{k-2} \int L(\lambda e^{i\lambda\varphi}) a(\theta) d\theta = \lambda^{k-2} \int L^2(e^{i\lambda\varphi}) a(\theta) d\theta = \cdots \quad \text{by induction} \quad \cdots \tag{53}$$

$$= \lambda \int L^{k-1}(e^{i\lambda\varphi}) a(\theta) d\theta = \int L^k(e^{i\lambda\varphi}) a(\theta) d\theta = \langle L^k(e^{i\lambda\varphi}), a \rangle \tag{54}$$

$$= \langle e^{i\lambda\varphi}, ({}^tL)^k a \rangle = \int e^{i\lambda\varphi(\theta)} ({}^tL)^k a(\theta) d\theta. \tag{55}$$

Moreover, we obtain by integration by parts

$$\langle L(e^{i\lambda\varphi}), a \rangle = \int_K (-i)|\varphi'|^{-2} \sum_{j=1}^N \frac{\partial \varphi}{\partial \theta_j} \frac{\partial}{\partial \theta_j}(e^{i\lambda\varphi}) \cdot a(\theta) d\theta \tag{56}$$

$$= (-i) \sum_{j=1}^N \int_K |\varphi'|^{-2} \frac{\partial \varphi}{\partial \theta_j} \frac{\partial}{\partial \theta_j}(e^{i\lambda\varphi}) \cdot a(\theta) d\theta \tag{57}$$

$$\leqslant (-i) c_0^{-2} \sum_{j=1}^N \int_K \frac{\partial \varphi}{\partial \theta_j} \frac{\partial}{\partial \theta_j}(e^{i\lambda\varphi}) \cdot a(\theta) d\theta \tag{58}$$

$$= (-i) c_0^{-2} \sum_{j=1}^N \int_K \frac{\partial}{\partial \theta_j}(e^{i\lambda\varphi}) \cdot \frac{\partial \varphi}{\partial \theta_j} a(\theta) d\theta \tag{59}$$

$$= i c_0^{-2} \sum_{j=1}^N \int_K e^{i\lambda\varphi} \frac{\partial}{\partial \theta_j} \left(\frac{\partial \varphi}{\partial \theta_j} a(\theta) \right) d\theta \tag{60}$$

$$= i c_0^{-2} \sum_{j=1}^N \int_K \left(e^{i\lambda\varphi} \frac{\partial^2 \varphi}{\partial \theta_j^2} a(\theta) + \frac{\partial \varphi}{\partial \theta_j} \frac{\partial a}{\partial \theta_j}(\theta) \right) d\theta \tag{61}$$

$$\leqslant C(c_0, K) \cdot C_2(\varphi) \cdot \sup_{|\alpha| \leqslant 1} |\partial^\alpha a|, \tag{62}$$

where there exist some proper positive constants $C(c_0,K)$, $C_2(\varphi)$. As we have seen in the above computation like (62), we may repeat the same procedure k times just like in (54), to obtain the desired result (49). □

DEFINITION 7. (Class of amplitude) For $\rho \in (-\infty,1]$ and $m \in \mathbb{R}$, the class of amplitude $A^m_\rho(\mathbb{R}^N)$ is the totality of functions $a = a(\theta) \in C^\infty(\mathbb{R}^N)$ satisfying that

$$|\partial^\alpha a(\theta)| \leqslant C(\alpha) \cdot (1 + |\theta|)^{m-\rho|\alpha|} \tag{63}$$

holds for any $\theta \in \mathbb{R}^N$and any $\alpha \in \mathbb{Z}^N_+$. Furthermore,

$$A^{+\infty}_\rho = \bigcup_{m\in\mathbb{R}} A^m_\rho. \tag{64}$$

As for S^m, we define a natural structure of a complete space on $A^m_\rho(\rho \leqslant 1)$ by using the semi-norms:

$$N^m_{\rho,k}(a) := \sup_{|\alpha|\leqslant k, \theta\in\mathbb{R}^N} (1 + |\theta|)^{-m+\rho|\alpha|} |\partial^\alpha a(\theta)|. \tag{65}$$

6. Outline of proof for the principal assertion

For the oscillatory integrals, the following result is obtained.

LEMMA 8. *Let $\varphi \in C^\infty(\mathbb{R}^N \setminus \{0\})$ is a real-valued and homogeneous of degree $\mu > 0$. If $\mu + \rho > 1$, then the oscillatory integral*

$$I_\varphi(a) := \int_{\mathbb{R}^N} e^{i\varphi(\theta)} a(\theta) d\theta \tag{66}$$

can be extended by continuity to all the A^m_ρ, $m \in \mathbb{R}$. This extension is unique, taking into account the density of $\mathcal{S}$ in these spaces.

We set $\varphi(y, \eta) = -y\eta$, which is a non-degenerate quadratic form on $\mathbb{R}^{2n}$. Thus it satisfies the hypotheses of Lemma 8 with $\mu = 2$. Since the function $\overline{a}(x - y, \xi - \eta)$ belongs to $A^{m+}_0(\mathbb{R}^{2n})$ with

$$m+ := \max(m, 0) \tag{67}$$

for fixed (x, ξ), Lemma 8 enables us to define the integral

$$\frac{1}{(2\pi)^n} \int e^{-iy\cdot\eta} \bar{a}(x - y, \xi - \eta) dy d\eta \tag{68}$$

as an oscillating integral. Note that this quantity can only be $a^*(x, \xi)$ by continuity. We may apply Lemma 8 to get the following estimate

$$|a^*(x, \xi)| \leqslant C \cdot N^{m+}_{0,k}(\bar{a}(x - \cdot, \xi - \cdot)) \tag{69}$$

for some k and some positive constant $C > 0$. Hence, this estimate gives the result for $m \geq 0$. By choosing $\mu = |m|$, we can get easily

$$|a^*(x, \xi)| \leqslant C(1 + |\xi|)^m, \tag{70}$$

where we have employed Peetre's inequality. Repeating the same discussion for the term $\partial^\alpha_x \partial^\beta_\xi$

$a(x, \xi)$ instead of $a = a(x, \xi)$, we deduce that $a^* \in S^m$.

As for the asymptotic expansion, an application of Taylor's formula with integral remainder for the function

$$g(t) = \bar{a}(x + ty, \xi + t\eta) \tag{71}$$

to obtain

$$a^*(x, \xi) = \frac{1}{(2\pi)^n} \sum_{|\alpha|+|\beta| \leqslant 2k+1} \frac{(-1)^{|\alpha|+|\beta|}}{\alpha!\beta!} \left(\int e^{-iy\eta} y^\alpha \eta^\beta dy d\eta \right) \partial_x^\alpha \partial_\xi^\beta \bar{a}(x, \xi) + R_k(x, \xi) \tag{72}$$

with

$$g^{(k)}(t) = \sum_{|\alpha|+|\beta|=k} \frac{k!}{\alpha!\beta!} y^\alpha \eta^\beta \partial_x^\alpha \partial_\xi^\beta \bar{a}(x + ty, \xi + t\eta), \tag{73}$$

where

$$\begin{aligned} R_k(x, \xi) = {} & \frac{1}{(2\pi)^n} \int_0^1 (1-t)^{2k+1} dt \int e^{-iy\eta} \cdot \\ & \times \sum_{|\alpha|+|\beta|=2k+2} (-1)^{|\alpha|+|\beta|} \frac{2k+2}{\alpha!\beta!} \partial_y^\alpha \partial_\eta^\beta \bar{a}(x - ty, \xi - t\eta) y^\alpha \eta^\beta dy d\eta. \end{aligned} \tag{74}$$

Finally the proof would be finished by calculating the general integral term in (72). In fact, it suffices to show that for any $\alpha \in \mathbb{N}^n$and any $\beta \in \mathbb{N}^n$

$$\frac{1}{(2\pi)^n} \int e^{-iy\cdot\eta} y^\alpha \eta^\beta dy d\eta = (-i)^{|\alpha|} \alpha! \delta_{\alpha\beta} \tag{75}$$

where the symbol $\delta_{\alpha\beta}$ denotes the Kronecker delta, and it indicates that $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$, and $\delta_{\alpha\beta}=$ 0 if $\alpha \neq \beta$.

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