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The Spectrum of Second Order Elliptic Operator and Useful Integral Inequality

DÔKU, Isamu
Faculty of Education, Saitama University

Summary

We consider the second order elliptic equation $A(\mathrm{x}, \partial)\mathrm{u}(\mathrm{x})+q(\mathrm{x})\mathrm{u}(\mathrm{x})=f(\mathrm{x})$, and also consider the integral inequality that provides us with a lower bound estimate of the associated quadratic form. In so doing, we need to introduce newly a positive continuous function $\lambda(\mathrm{x})$ on the region in question. This type of estimate is quite very useful if we apply the result to the elliptic equation to obtain estimates of eigenfunctions for the equation. The principal purpose of this article is to settle down the positive continuous function $\lambda(\mathrm{x})$ in a concrete manner. It suffices to involve the spectra of operator and the compactness argument in functional spaces, in order to realize the above-mentioned program.

Key Words: spectra, second order elliptic operator, elliptic equation, integral inequality, compactness argument.

1. Introduction and notation

In this section we shall first explain the notation used throughout this article. We consider the second order elliptic operator

$$A \equiv A(x, \partial) = -\sum_{i,j=1}^{n} \partial_j a^{ij}(x)\partial_i = -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_j} a^{ij}(x) \frac{\partial}{\partial x_i}. \tag{1}$$

We assume that the coefficients $a^{ij}(x)$ of A are continuous bounded functions on Ω. The space $M_{loc}(\Omega)$ consists of the functions $u \in L^1_{loc}(\Omega)$ such that

$$\lim_{r\to 0} \int_{B_r(x^0)\cap\Omega} |u(x)| \cdot \Phi(x)\, dx = 0, \tag{2}$$

where we set $\Phi(x) = |x - x^0|^{2-(n+\delta)}$ and $B_r(y)$ denotes the ball of radius r centered at y. For $u, v \in H^1_{loc}(\Omega)$, we define

$$\nabla_A u(x) \cdot \nabla_A v(x) = \sum_{i,j=1}^{n} a^{ij}(x)\partial_i u(x)\partial_j v(x) \tag{3}$$

$$|\nabla_A u(x)|^2 = \nabla_A u(x) \cdot \nabla_A \bar{u}(x) = \sum_{i,j=1}^{n} a^{ij}(x)\partial_i u(x)\partial_j \bar{u}(x). \tag{4}$$

The elliptic equation

$$A(x,\partial)u(x)+q(x)u(x)=f(x) \tag{5}$$

is considered here, for complex valued functions $q \in L^1_{loc}(\Omega)$ and $f \in L^2_{loc}(\Omega)$. While, a function $u \in H^1_{loc}(\Omega)$ is said to be a solution to (5) if $qu \in L^1_{loc}(\Omega)$ and

$$\int_\Omega (\nabla_A u(x)\cdot\nabla_A\varphi(x)+q(x)u(x)\varphi(x))dx=\int_\Omega f(x)g(x)dx \tag{6}$$

for every $\varphi \in C_0^\infty(\Omega)$. Let $q \in L^1_{loc}(\mathbb{R}^n)$ and $q_-(x) := \max(0, -Re\{q(x)\}) \in M_{loc}(\mathbb{R}^n)$ and assume that q is real valued. Then we set

$$P \equiv P(x,\partial)=A(x,\partial)+q(x) \tag{7}$$

and define

$$(Pu,u)=\int_\Omega \left(|\nabla_A u(x)|^2+q(x)|u(x)|^2\right)dx \tag{8}$$

for all $u \in H^1_{loc}(\Omega)$. Notice that the integral (8) makes sense under these assumptions.

Our main aim of this article is to find out continuous functions $\lambda(x)$ on $\mathbb{R}^n$, not necessarily positive but as large as possible, such that the following inequality

$$(P\varphi,\varphi)\geq\int_\Omega \lambda(x)\cdot|\varphi(x)|^2dx \tag{9}$$

holds for all $\varphi \in C_0^\infty(\Omega)$. In what follows, the norm $\|\cdot\|=\|\cdot\|_2$ denotes the norm in the usual space $L^2(\mathbb{R}^n)$. For any $y \in \mathbb{R}^n$ and $R > 0$, we define

$$\Lambda_R(y;P):=\inf\left\{\frac{(P\varphi,\varphi)}{\|\varphi\|^2}\,:\,\varphi\in C_0^\infty(B_R(y)),\varphi\neq 0\right\}. \tag{10}$$

It is interesting to note that $\Lambda_R(y; P)$ can be identified with the lowest eigenvalue of the self-adjoint realization of P in $L^2(B_R(y))$ under zero Dirichlet boundary conditions.

2. Principal result

We are going to prove a theorem which enables us to obtain non-constant λ functions, quite similar as in (9) in §1 or similar as in (12) in the proceeding section §3. In Theorem 1 the functions $\lambda(x)$ in question depend only on the direction x. As is well known, this type of assertion would be in particular useful in applications of series of inequalities stated in §3, by which we can get eigenfunction estimates of multiparticle Schrödinger operators.

THEOREM 1. *Let $g(\omega)$ be a continuous function on S^{n-1} such that $g(\omega) < K(\omega) = K(\omega; P)$ for all $\omega \in S^{n-1}$. Then there exists $C > 0$ such that*

$$\int_{\Omega_C}\left(|\nabla_A\varphi(x)|^2+g(x)|\varphi(x)|^2\right)dx\geq\int_{\Omega_C} g\left(\frac{x}{|x|}\right)|\varphi(x)|^2dx \tag{11}$$

holds for all $\varphi \in C_0^\infty(\Omega_C)$ where $\Omega_C := \{x : |x| > C\}$.

3. Preliminaries and technical key lemmas

When Ω is a connected open subset in $\mathbb{R}^n$, let us define $q(x)$ as a complex valued function on Ω such that (i) $q \in L^1_{loc}(\Omega)$; (ii) $q_- \in M_{loc}(\Omega)$. Suppose that there exists a positive continuous function $\lambda(x)$ on Ω such that

$$\int_\Omega \lambda(x)|\varphi(x)|^2\,dx \leqslant Re\int_\Omega \left(|\nabla_A\varphi(x)|^2 + q(x)|\varphi(x)|^2\right)\,dx \tag{12}$$

for every $\varphi \in C_0^\infty(\Omega)$. Let $\rho_\lambda(x, y)$ be the geodesic distance in Ω between the points x and y in the Riemannian metric

$$ds^2_\lambda = \lambda(x)\sum_{i,j=1}^n a_{ij}(x)dx_i dx_j, \tag{13}$$

where a_{ij} is an element of the inverse $(a_{ij}) = (a^{ij})^{-1}$. Let h be a real valued Lipschitz function on Ω such that $|\nabla_A h(x)|^2 < \lambda(x)$ holds a.e. If we suppose that u is a function in $H^1_{loc}(\Omega)$ which satisfies the differential equation $Au + qu = f$ in the sense that $qu \in L^1_{loc}(\Omega)$, $f \in L^2_{loc}(\Omega)$, and

$$((P-\mu)\varphi,\varphi) \geq \int_{\{|x|>R\}} \lambda(x)|\varphi(x)|^2\,dx \tag{14}$$

holds for all $\varphi \in C_0^\infty(\Omega_R)$ and $\lambda \in C^+(\mathbb{R}^n)$, and $\Omega_R = \{x \in \mathbb{R}^n\colon |x| > R\}$ (where note that μ is an eigenvalue of P, i.e., $Pu = \mu u$), and in addition, if we assume that

$$\int_\Omega |u(x)|^2\lambda(x)\cdot\exp\{-2(1-\delta)\rho_\lambda(x,y^0)\}\,dx < \infty \tag{15}$$

for some $\delta > 0$ and fixed point $y^0 \in \Omega$, then the following inequalities are valid: that is to say, if Ω is complete in the metric ρ_λ, then

$$\int_\Omega |u(x)|^2\left(\lambda(x) - |\nabla_A h(x)|^2\right)e^{2h(x)}\,dx \leqslant \int_\Omega |f(x)|^2\left(\lambda(x) - |\nabla_A h(x)|^2\right)^{-1}e^{2h(x)}\,dx \tag{16}$$

holds. In general, if Ω is not necessarily complete, then

$$\begin{aligned}\int_{\Omega_d} |u(x)|^2\left(\lambda(x) - |\nabla_A h(x)|^2\right)\cdot e^{2h(x)}\,dx \leqslant &\int_\Omega |f(x)|^2\left(\lambda(x) - |\nabla_A h(x)|^2\right)^{-1}\cdot e^{2h(x)}\,dx\\ &+ C(d)\int_{\Omega\setminus\Omega_d} |u(x)|^2\lambda(x)e^{2h(x)}\,dx\end{aligned} \tag{17}$$

holds, where $d > 0$, $C(d) := 2(1 + 2d)/d^2$, and

$$\Omega_d := \{x \in \Omega :\ \rho_\lambda(x, \{\infty\}) > d\,\}, \tag{18}$$

and $\rho_\lambda(x, \{\infty\})$ is defined by

$$\rho_\lambda(x,y) = \inf_\gamma \int_0^1 \{\lambda(\gamma(t))\cdot\sum_{i,j} a_{ij}(\gamma(t))\cdot\gamma_i(t)\gamma_j(t)\}^{1/2}\,dt, \tag{19}$$

and the infimum is taken over all absolutely continuous paths $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = y$ and $\gamma(1) = x$. For $E \subset \Omega$, let $\rho_\lambda(x, E) := \inf\{\rho_\lambda(x, y): y \in E\}$. Then if $\Omega \cup \{\infty\}$ is the one point compactification on Ω, we define for $x \in \Omega$,

$$\rho_\lambda(x, \{\infty\}) := \sup_K \{\rho_\lambda(x, \Omega \setminus K) : \ K \text{ is a compact subset of } \Omega\}. \tag{20}$$

We may apply the above-mentioned inequality results to measure the decay of solutions to $Au + qu = 0$. If $q(x) = q_1(x) - \mu$, then we can thereby obtain estimates for eigenfunctions of $A + q_1$ with eigenvalue μ. However, in order to apply the aforementioned results, the real part of the quadratic form associated with the operator $A + q$ need be positive on $C_0^\infty(\Omega)$. Moreover, we must find a positive continuous function $\lambda(x)$ such that the quadratic form in question is strongly positive on $C_0^\infty(\Omega)$ in the sense that

$$\int_\Omega \lambda(x)|\varphi(x)|^2 \, dx \leqslant Re \int_\Omega \left(|\nabla_A \varphi(x)|^2 + q(x)|\varphi(x)|^2\right) \, dx \tag{21}$$

holds for all $\varphi \in C_0^\infty(\Omega)$. That is why we have the following technical key lemmas.

LEMMA 2. $\Lambda_R(x; P)$ *is a continuous function of* (x, R) *on* $\mathbb{R}^n \times \mathbb{R}_+$.

Furthermore, we can get another assertion.

LEMMA 3. $\Lambda_R(x; P) = \Lambda_R(x; A + q)$ *is also continuous in* (a^{ij}) *in the sense that if* $A_m = -\sum_{i,j=1}^n \partial_j a_m^{ij} \partial_i$ *(where* $(a_m^{ij}(x))$ *has all the properties of* $(a^{ij}(x))$ *for* $m = 1, 2, ...$*), and if*

$$\lim_{m\to\infty} |a^{ij}(x) - a_m^{ij}(x)| = 0 \tag{22}$$

uniformly on compact sets, then $\Lambda_R(x; A_m + q)$ *converges to* $\Lambda_R(x; A + q)$ *uniformly for* x *in compact subsets.*

4. Proofs of key lemmas

Proof of Lemma 2. First of all we shall show that $\Lambda_R(x) \equiv \Lambda_R(x; A + q)$ is an everywhere finite upper semicontinuous function. According to Schechter's lemma [15] (1971) (cf. Theorem 7.3, p.138), when $q \in M_{\delta,loc}(\Omega)$, for every $\varepsilon > 0$ and every compact subset K of Ω, there exists a constant $C(\varepsilon, K) > 0$ such that for $\theta = 1 - (\delta/2)$

$$\| \, |g|^{1/2}\varphi \, \| \leqslant \varepsilon \|\Lambda^\theta \varphi\| + C(\varepsilon, K)\|\varphi\| \tag{23}$$

holds for all $\varphi \in C_0^\infty(\Omega)$ with supp$\varphi \subset K$. As a simple corollary for the case $\delta = 0$, we have the following estimates.

LEMMA 4. *Let* $g \in M_{loc}(\Omega)$. *Then for any* $\varepsilon > 0$ *and every compact subset* K of Ω, *there exist positive constants* $C_1(\varepsilon, K)$ *and* $C_2(\varepsilon, K)$ *such that*

$$\| \, |g|^{1/2}\varphi \, \| \leqslant \varepsilon \|\nabla \varphi\| + C_1(\varepsilon, K)\|\varphi\|, \tag{24}$$

$$\int_\Omega |g(x)| \cdot |\varphi(x)|^2 \, dx \leqslant \varepsilon \int_\Omega \sum_{i=1}^n |\partial_i \varphi(x)|^2 \, dx + C_2(\varepsilon, K) \int_\Omega |\varphi(x)|^2 \, dx \tag{25}$$

hold for all $\varphi \in C_0^\infty(\Omega)$ *with* supp $\varphi \subset$ K. *Here we have*

$$\|\nabla u\| = \left(\int_\Omega \sum_{i,j=1}^{n} |\partial_i u(x)|^2 \, dx \right)^{1/2} \tag{26}$$

Consequently, it follows immediately from the above lemma 4 that there exists a constant $C_\Omega = C_\Omega(\Omega, A) > 0$ such that

$$\int_\Omega q_-(x)|\varphi(x)|^2 \, dx \leqslant \frac{1}{2} \int_\Omega |\nabla_A \varphi(x)|^2 \, dx + C(\Omega, A) \int_\Omega |\varphi(x)|^2 \, dx \tag{27}$$

holds for every $\varphi \in C_0^\infty(\Omega)$, where $\varepsilon = \min(1/2, \delta/2)$ with $\delta > 0$. This integral inequality yields to

$$\begin{aligned}(P\varphi, \varphi) &= \int_\Omega \left(|\nabla_A \varphi(x)|^2 + q_+(x)|\varphi(x)|^2 - q_-(x)|\varphi(x)|^2 \right) \, dx \\ &\geq \frac{1}{2} \int_\Omega \left(|\nabla_A \varphi(x)|^2 + q_+(x)|\varphi(x)|^2 \right) \, dx - C(\Omega, A) \int_\Omega |\varphi(x)|^2 \, dx \end{aligned} \tag{28}$$

for every $\varphi \in C_0^\infty(\Omega)$. The above inequality (28) implies in particular that $\Lambda_R(\mathrm{x}) \geq -C(\Omega, A)$ whenever $B_R(x) \subset \Omega$. Since Ω is an arbitrary bounded open set, for a fixed point (x^0, R_0) in $\mathbb{R}^n \times \mathbb{R}_+$, a routine work leads with ease to a fundamental estimate

$$(P\psi, \psi) \leqslant \Lambda_{R_0}(x^0, A + q) + \varepsilon \tag{29}$$

for a properly chosen function $\psi \in C_0^\infty(B_{R0}(x^0))$ and a number $\varepsilon > 0$. With ψ defined as zero in $\mathbb{R}^n \setminus B_{R0}(x^0)$, it is clear that supp $\psi \subset B_{Rj}(x^j)$ for j large enough, with respect to a sequence $\{(x^j, R_j)\}_j$ satisfying $(x^j, R_j) \to (x^0, R_0)$ as $j \to \infty$. Hence, it follows that

$$\begin{aligned}\Lambda_{R_j}(x^j; A + q) &= \inf\{(P\varphi, \varphi);\ \varphi \in C_0^\infty(B_{R_j}(x^j)),\ \|\varphi\| = 1\} \\ &\leqslant (P\psi, \psi) \leqslant \Lambda_{R_0}(x^0, A + q) + \varepsilon.\end{aligned} \tag{30}$$

Letting $j \to \infty$ and then $\varepsilon \to 0$, we can finally get

$$\limsup_{j \to \infty} \Lambda_{R_j}(x^j; A + q) \leqslant \Lambda_{R_0}(x^0; A + q), \tag{31}$$

which proves that $\Lambda_R(x; A + q)$ is an upper semicontinuous function.

Next we shall show that $\Lambda_R(x; A + q)$ is a lower semicontinuous function. The proof of this part goes almost similarly as mentioned above. The only difference consists of the following point, namely, compactness argument in functional spaces. We need to introduce some functional spaces. $H^1_{00}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in the $H^1(\Omega)$ norm: i.e.

$$H^1_{00}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}, \tag{32}$$

where

$$\|u\|^2_{H^1(\Omega)} := \int_\Omega \left(\sum_{i=1}^{n} |\partial_i u(x)|^2 + |u(x)|^2 \right) dx. \tag{33}$$

Next the space $H^1_{0q+}(\Omega)$ of functions is defined as

$$H^1_{0q+}(\Omega) := \{u \in H^1_{00}(\Omega);\ q_+^{1/2} u \in L^2(\Omega)\}, \tag{34}$$

where q_+ is the positive part of the function q. It is known that $H^1_{0q+}(\Omega)$ is a Hilbert space under the norm $|||\cdot|||$, which is defined by

$$|||u|||^2_\Omega := \|u\|^2_{H^1(\Omega)} + \|q^{1/2}u\|^2_{L^2(\Omega)}. \tag{35}$$

Then it is quite obvious that $C_0^\infty(\Omega)$ is dense in $H^1_{0q+}(\Omega)$. Thanks to Rellich's compactness theorem [1] (1965) (cf. Theorem 3.8), it follows that weak convergence: weak-$\lim_{j\to\infty} \varphi_j = u$ in $H^1_{0q+}(\Omega)$ yields to strong convergence: $\varphi_j \to u$ strongly in $L^2(\Omega)$. On this account, one may conclude that

$$(Pu, u)_\Omega \leqslant \liminf_{j\to\infty} (P\varphi_j, \varphi_j) \leqslant \liminf_{j\to\infty} \Lambda_{R_j}(x^j; A+q). \tag{36}$$

Furthermore, the following estimate

$$(Pu_0, u_0)_{B_{R_0}(x^0)} = (Pu, u)_\Omega \leqslant \liminf_{j\to\infty} \Lambda_{R_j}(x^j; A+q) \tag{37}$$

can be derived easily from (36). Consequently, the claim $u_0 \in H^1_{0q+}(B_{R0}(x^0))$ is verified by the fact $u_0 \in H^1_{00}(B_{R0}(x^0))$, since we have $q_+^{1/2}u_0 \in L^2(B_{R0}(x^0))$. That is why in terms of the denseness

$$C_0^\infty(B_{R_0}(x^0)) \hookrightarrow H^1_{0q+}(B_{R_0}(x^0)) \tag{38}$$

and ordinary argument of convergence, it can be shown that

$$\Lambda_{R_0}(x^0; A+q) \leqslant \liminf_{j\to\infty} \Lambda_{R_j}(x^j; A+q), \tag{39}$$

which concludes that $\Lambda_R(x;\ A+q)$ is a lower semicontinuous function, and thus establishes that $\Lambda_R(x;\ A+q)$ is continuous. □

Proof of Lemma 3. Now we are going to show that if $A_m(x, \partial) := -\sum_{i,j} \partial_j a_m^{ij} \partial_i$, ($m$ =1, 2, ...) is a sequence of operators satisfying the same conditions as A and if $a_m^{ij} \to a^{ij}$ as $m \to \infty$ uniformly on compact sets, then $\Lambda_R(x;\ A_m + q)$ converges to $\Lambda_R(x;\ A+q)$ uniformly in x on compact sets. For a compact set K in $\mathbb{R}^n$ and a fixed constant $R > 0$, we may assume that

$$(P\varphi, \varphi) \geq \|\varphi\|^2 \qquad \text{and} \qquad \int_\Omega q_-(x)|\varphi(x)|^2\,dx \leqslant (P\varphi, \varphi) \tag{40}$$

for every $\varphi \in C_0^\infty(\Omega)$, taking inequalities (27) and (28) into consideration. Since $a_m^{ij}(x) \to a^{ij}(x)$ as $m \to \infty$ uniformly for $x \in \Omega$, we may deduce that there exists a sequence of positive numbers $\{\varepsilon_n\}$ with $\varepsilon_m \to 0$ (as $m \to \infty$), such that

$$(1-\varepsilon_m)\left(a^{ij}(x)\right) \leqslant \left(a_m^{ij}(x)\right) \leqslant (1+\varepsilon_m)\left(a^{ij}(x)\right) \tag{41}$$

for all $x \in \Omega$, m = 1, 2,.....Hence, we may combine (40) with (41) to get

$$(P_m\varphi, \varphi) \geq (1-2\varepsilon_m)(P\varphi, \varphi). \tag{42}$$

From this estimate, we can derive the subtler estimate

$$(1-2\varepsilon_m)\Lambda_R(x;P) \leqslant \Lambda_R(x;P_m) \leqslant (1+2\varphi_m)\Lambda_R(x;P) \tag{43}$$

for every $x \in K$ and $m = 1, 2, ...$, which implies that $\Lambda_R(x; P_m)$ converges to $\Lambda_R(x; P)$ uniformly on K. This finishes the proof. □

5. Fundamental estimate

Moreover, we can derive the following fundamental estimate: namely, under the above-mentioned conditions on P, for any $\varepsilon > 0$, there exists a positive constant $R_\varepsilon > 0$ such that

$$(P\varphi, \varphi) \geq \int_{\mathbb{R}^n} (\Lambda_R(x;P) - \varepsilon)|\varphi(x)|^2\, dx \tag{44}$$

holds for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for any $R \geq R_\varepsilon$.

The proof of (44) is easy, hence omitted. Rigorouly, we need the discussions on approximation of metrics and completeness.

6. The bottom of spectrum and essential spectrum

We begin with giving definitions of $\Lambda(P)$ and $\Sigma(P)$. As a matter of fact, the quantity $\Lambda(P)$ is defined by

$$\Lambda(P) := \inf\left\{ \frac{(P\varphi,\varphi)}{\|\varphi\|^2} :\ \varphi \in C_0^\infty(\mathbb{R}^n), \varphi \neq 0 \right\}, \tag{45}$$

while, the quantity $\Sigma(P)$ is also defined in a similar way by

$$\Sigma(P) := \sup_K\ \inf\left\{ \frac{(P\varphi,\varphi)}{\|\varphi\|^2} :\ \varphi \in C_0^\infty(\mathbb{R}^n \setminus K), \varphi \neq 0 \right\}, \tag{46}$$

where the supremum is taken over the family of compact subsets K in $\mathbb{R}^n$. $\Lambda(P)$ will be shown to be equal to the bottom of the spectrum of the self-adjoint realization of the operator P on $L^2(\mathbb{R}^n)$; on the other hand, $\Sigma(P)$ will be shown to be equal to the essential spectrum of the self-adjoint realization of P on $L^2(\mathbb{R}^n)$ when $\Sigma(P) > -\infty$. In general, both quantities may take on the value $-\infty$.

It is easy to see that $\Lambda(P) = \lim_{R\to\infty} \Lambda_R(x; P)$ for any $x \in \mathbb{R}^n$. Note that this limit exists since $\Lambda_R(x; P)$ is a decreasing function of R. Then the relationship between $\Sigma(P)$ and $\Lambda_R(x; P)$ can be given by the following lemma.

Lemma 5. *We have*

$$\Sigma(P) = \lim_{R\to\infty} \liminf_{|x|\to\infty} \Lambda_R(x;P). \tag{47}$$

Proof. Let K be a compact subset in $\mathbb{R}^n$, and $R > 0$ fixed. Clearly, $B_R(x) \subset \mathbb{R}^n \setminus K$ for $|x|$ sufficiently large. Then we can get easily

$$\inf\left\{\frac{(P\varphi,\varphi)}{\|\varphi\|^2}:\ \varphi\in C_0^\infty(\mathbb{R}^n\setminus K),\varphi\neq 0\right\}\leqslant\liminf_{|x|\to\infty}\Lambda_R(x;P). \tag{48}$$

Moreover, by the definition of $\Sigma(P)$, it follows immediately that

$$\Sigma(P)\leqslant\lim_{R\to\infty}\liminf_{|x|\to\infty}\Lambda_R(x;P). \tag{49}$$

Here we may apply the fundamental estimate result stated in the previous section §5, to obtain

$$(P\varphi,\varphi)\geq\int_{\mathbb{R}^n}(\Lambda_R(x;P)-\varepsilon)|\varphi(x)|^2\,dx \tag{50}$$

for all $R\geq R_\varepsilon$ and $\varphi\in C_0^\infty(\mathbb{R}^n)$. On the other hand, a simple calculation with (50) leads to the estimate

$$\Sigma(P)\geq\lim_{R\to\infty}\liminf_{|x|\to\infty}\Lambda_R(x;P)-2\varepsilon. \tag{51}$$

Hence, we can easily deduce from (49) and (51) the statement in Lemma 5. □

Now let us introduce a new function $K(\omega)=K(\omega,P)$ which, roughly speaking, approximates the lower bound of the quadratic form $(P\varphi,\varphi)$. Let $S^{n-1}=\{\omega\in\mathbb{R}^n:|\omega|=1\}$. For $\omega\in S^{n-1}$, $0<\varepsilon<\pi$, and $N>0$, we define

$$\Gamma_\omega^{\varepsilon,N}:=\{x\in\mathbb{R}^n:\ \langle x,\omega\rangle>|x|\cos\varepsilon,|x|>N\}, \tag{52}$$

$$\Sigma^{\varepsilon,N}(\omega):=\inf\left\{\frac{(P\varphi,\varphi)}{\|\varphi\|^2}:\ \varphi\in C_0^\infty(\Gamma_\omega^{\varepsilon,N}),\varphi\neq 0\right\}, \tag{53}$$

$$K(\omega)\equiv K(\omega,P):=\lim_{\varepsilon\to\infty}\lim_{N\to\infty}\Sigma^{\varepsilon,N}(\omega), \tag{54}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$. Note that the limit in (54) exists and $K(\omega)$ may take the value $+\infty$. The following is the main assertion in this section.

Proposition 6. (i) *$K(\omega)$ is a lower semicontinuous function of ω on S^{n-1}.*
(ii) *The following holds*:

$$\Sigma(P)=\min\{K(\omega):\ \omega\in S^{n-1}\}. \tag{55}$$

Proof. (i) Let $\{\omega_j\}\subset S^{n-1}$ satisfying $\omega_j\to\omega$ as $j\to\infty$. Fix a number L such that $L<K(\omega)$. Then from the definition of $K(\omega)$, it follows immediately that there exist $\varepsilon\in(0,\pi)$ and $N>0$ such that $\Sigma^{\varepsilon,N}(\omega)>L$. Hence, we obtain

$$L<\Sigma^{\varepsilon,N}(\omega)\leqslant\Sigma^{\varepsilon/2,N}(\omega_j)\leqslant K(\omega_j) \tag{56}$$

for any $j\geq j_0$ ($\exists$ some j_0). Consequently, it proves to be that

$$\liminf_{j\to\infty}K(\omega_j)\geq K(\omega), \tag{57}$$

which completes the proof of lower semicontinuity of $K(\omega)$. □

(ii) We can rewrite

$$\Sigma(P) = \sup_{K:\text{compact}} \inf\left\{\frac{(P\varphi,\varphi)}{\|\varphi\|^2} :\ \varphi \in C_0^\infty(\mathbb{R}^n \setminus K), \varphi \neq 0\right\}$$
$$\leqslant \inf\left\{\frac{(P\varphi,\varphi)}{\|\varphi\|^2} :\ \varphi \in C_0^\infty(\mathbb{R}^n \setminus K_0) :\ \varphi \neq 0\right\} + \delta \tag{58}$$

for fixed $\delta > 0$ and some compact set K_0. On the other hand, for $R_0 > 0$ satisfying $K_0 \subset B_{R0}(0)$, we have

$$\Sigma(P) \leqslant \inf\left\{\frac{(P\varphi,\varphi)}{\|\varphi\|^2} :\ \varphi \in C_0^\infty(\Gamma_\omega^{\varepsilon,N}), \varphi \neq 0\right\} + \delta$$
$$= \Sigma^{\varepsilon,N}(\omega) + \delta \tag{59}$$

for any $\omega \in S^{n-1}$, $0 < \varepsilon < \pi$, and $N > R_0$, which implies that

$$\Sigma(P) \leqslant \lim_{\varepsilon\to 0}\lim_{N\to\infty} \Sigma^{\varepsilon,N}(\omega) + \delta = K(\omega) + \delta \tag{60}$$

for any $\omega \in S^{n-1}$. Likewise, applying Lemma 5 we can deduce that

$$\Sigma(P) \geq \liminf_{|x|\to\infty} \Lambda_{R_1}(x;P) - \frac{\delta}{2} \geq \Lambda_{R_1}(x_m;P) - \delta \tag{61}$$

for $m = 1, 2, ..., \exists R_1 > 0, \delta > 0, x_m \in \mathbb{R}^n$ $(m = 1, 2, ...)$ with $|x_m| \to \infty$ (as $j \to \infty$). Hence

$$\Sigma(P) \geq \min\{K(\omega) :\ \omega \in S^{n-1}\} - \delta \tag{62}$$

holds. Finally, we can deduce that $\Sigma(P) = \min\{K(\omega): \omega \in S^{n-1}\}$. □

7. The proof of the main theorem

Since $K(\omega) - g(\omega)$ is positive and lower semicontinuous, we note together with Lemma 5 that there exists $R > 0$ such that

$$\int (\Lambda_R(x;P) - \delta)|\varphi(x)|^2\, dx \leqslant \int \left(|\nabla_A\varphi(x)|^2 + q(x)|\varphi(x)|^2\right)\, dx \tag{63}$$

holds for every $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then, for fixed $\omega_0 \in S^{n-1}$, $0 < \varepsilon_0 < \pi/2$, $N_0 > 0$ and some neighborhood $U_0 = U(\omega_0) \subset S^{n-1}$ satisfying

$$g(\omega) < g(\omega_0) + \delta, \qquad \Gamma_\omega^{\varepsilon_0/2,N_0} \subset \Gamma_{\omega_0}^{\varepsilon_0,N_0} \tag{64}$$

as far as ω is taken from U_0. Therefore, the compactness of S^{n-1} and a covering argument allow us to conclude that

$$\Sigma^{\varepsilon,N}(\omega) > g(\omega) + \delta \tag{65}$$

for every $\omega \in S^{n-1}$ and some $\varepsilon > 0$, $N > 0$. If we set

$$C := \max\{N + R, R/\sin\varepsilon\}, \tag{66}$$

then we have an inclusion

$$B_R(x) \subset \Gamma^{\varepsilon,N}_{x/|x|} \tag{67}$$

as far as $|x| > C$, and hence it follows that

$$\begin{aligned}
&\Lambda_R(x;P) \\
&\geq \inf\left\{\frac{(P\varphi,\varphi)}{\|\varphi\|^2} \;:\; \varphi \in C_0^\infty(\Gamma^{\varepsilon,N}_{x/|x|}), \varphi \neq 0\right\} \\
&= \Sigma^{\varepsilon,N}\left(\frac{x}{|x|}\right) \\
&\geq g\left(\frac{x}{|x|}\right) + \delta,
\end{aligned} \tag{68}$$

which proves that

$$\Lambda_R(x;P) - \delta \geq g\left(\frac{x}{|x|}\right) \tag{69}$$

for $|x| > C$. This completes the proof. □

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Isamu Dôku
Department of Mathematics
Faculty of Education, Saitama University Saitama, 338–8570 Japan
e-mail: idoku@mail.saitama-u.ac.jp

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