The L^2-representations of the Second Variations and the Łojasiewicz Inequalities for Decomposed Möbius Energies

(分解されたMöbiusエネルギーに対する第二変分の L^2 表現とŁojasiewicz不等式)

埼玉大学大学院理工学研究科博士後期課程
理工学専攻数理電子情報コース

指導教員：長澤壯之

16DM004 郡司克徳

平成31年3月20日

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Katsunori Gunji

Abstract

Knot energies, one of which is the Möbius energy, are constructed to measure the well-proportionedness of the knot. The best-proportioned knot in the given knot class may be determined by the gradient flow of the energy. Indeed, Blatt showed the global existence and convergence of the gradient flow of the Möbius energy near stationary points. The Łojasiewicz inequality played an important role in proving the results. The inequality can be proved by properties of L^2-representation of the first and second variations. On the other hand, Ishizeki and Nagasawa showed that the Möbius energy can be decomposed into parts keeping the Möbius invariance and each part has the L^2-representation of the first variation. In this thesis, we discuss the L^2-representation of the second variation for each decomposed part of the Möbius energy, and derive it explicitly. As a consequence of it and Chill's theory, the Łojasiewicz inequality is derived from the representations.

Keywords: the Möbius energy, decomposed Möbius energies, the second variation, the Łojasiewicz inequality.

2010 MSC: 53A04 (primary), 49Q10, 49J50, 26D10 (secondary).

1 Introduction

1.1 Knot energy and its minimizer

Let $\boldsymbol{f}$ be a closed curve in $\mathbb{R}^n$ parametrized by the arc-length parameter s. We define the Möbius energy $\mathcal{E}$ of $\boldsymbol{f}$, whose length is $\mathcal{L}$, by

$$\mathcal{E}(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}(\boldsymbol{f}) ds_1 ds_2,$$

where

$$\mathscr{M}(\boldsymbol{f}) = \frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2}.$$

Hereafter we use the notation

$$\Delta \boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2,\ \boldsymbol{v}_i = \boldsymbol{v}(s_i),\ \Delta s = s_1 - s_2,$$

where $s_1, s_2 \in \mathbb{R}/\mathcal{L}\mathbb{Z}, \boldsymbol{v} : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}^n$. Also $\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$ means the distance along the curve $\boldsymbol{f}$ between $\boldsymbol{f}(s_1)$ and $\boldsymbol{f}(s_2)$. Without loss of generality, we may assume $|\Delta s| \leq \frac{\mathcal{L}}{2}$. Then we can write $\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2)) = |\Delta s|$ simply. This energy is originally introcduced by O'Hara [13] under the motivation to compose energies which are well-behaved for the loop $\boldsymbol{f} \in \mathbb{R}^3$ and blow up for curves with self-intersections. Indeed, he defined as

$$\mathcal{E}_{(\alpha,p)}(\boldsymbol{f}) = \iint_{(\mathbb{R}\mathcal{L}\mathbb{Z})^2} \left(\frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^3}^{\alpha}} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^{\alpha}} \right)^p ds_1 ds_2$$

for $n = 3, \alpha, p > 0$, which is called O'Hara's energy. One of the main purposes of studying the knot energy is to consider a minimizing problem in a fixed knot type. The energy $\mathcal{E}_{\alpha,p}$ avoids the deformation with self intersection. However, knot-type changings may occure with pull-tight phenomena which are vanishments of tangles. O'Hara revealed that this problem does not occure in the case of $\alpha p > 2$ and the minimizer exists in any knot types.

Theorem 1.1 **([14])** *Let a knot K_ε be a connected sum of K and a small tangle T_ε. A difference of energy $D(\varepsilon) = \mathcal{E}_{(\alpha,p)}(K_\varepsilon) - \mathcal{E}_{(\alpha,p)}(K)$ behaves as follows in a pull-tight process $T_\varepsilon \to \{a\ point\}$: 1. $D(\varepsilon)$ blows up when $\alpha p > 2$. 2. $D(\varepsilon)$ converges a positive constant when $\alpha p = 2$. 3. $D(\varepsilon)$ vanishes when $\alpha p < 2$.*

Theorem 1.2 **([15])** *Let $n = 3$. There exists a minimizer (under rescaling) of $\mathcal{E}_{\alpha,p}$ for any knot types if and only if $\alpha p > 2$.*

The energy $\mathcal{E}$ considered in this thesis is $\mathcal{E}_{2,1}$, which is not the case of Theorem 1.2. That is, the existence of minimizers is not clear. We have only a partial answer.

Freedman-He-Wang showed that the energy $\mathcal{E}$ is invariant not only under scaling but also under Möbius tranformation.

Theorem 1.3 **([5])** *Let $\boldsymbol{f}$ be a simple closed curve in $\mathbb{R}^3$ and let T be a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$. If $T \circ \boldsymbol{f} \subset \mathbb{R}^3$, then $\mathcal{E}(T \circ \boldsymbol{f}) = \mathcal{E}$.*

Making use of the property, if the knot-type is prime, pull tight can be suitably scaled without changing energy level and a minimizer exists.

Theorem 1.4 **([5])** *Let K be an prime knot. There exists a simple closed curve $\boldsymbol{f}_K$ in $\mathbb{R}^3$ with knot-type K such that $\mathcal{E}(\boldsymbol{f}_K) \le \mathcal{E}(\boldsymbol{f})$ for any other closed curve $\boldsymbol{f}$ in $\mathbb{R}^3$ of same knot type.*

On the other hand, the existsnce of minimizers is still open for a composit knot type. Kusner-Sullivan conjectured composit knot types do not have minimizers [12]. Several mathematicians made approaches to this problem.

Blatt showed the regularity of $\boldsymbol{f}$ whose knot energy is finite.

Theorem 1.5 **([1])** *Let $\boldsymbol{f} \in C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ be an embedded regular curve parametrized by arc-length. Then $\mathcal{E}(\boldsymbol{f}) < \infty$ if and only if $\boldsymbol{f} \in H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and $\boldsymbol{f}$ is bi-Lipschitz.*

Here we say a function $\boldsymbol{f}$ is bi-Lipschitz in $\mathbb{R}/\mathcal{L}\mathbb{Z}$ when there exists a constant $b > 0$ such that, for any s_1, s_2 with $s_1 \neq s_2 \mod \mathcal{L}$,

$$b^{-1} \le \frac{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}}{|s_1 - s_2|} \le b.$$

Also we call b the bi-Lipschitz constant.

1.2 Variational formulae and decomposition of Möbius energy

We use δ to mean the first variation of a energy E, that is

$$\delta E(\boldsymbol{f})[\boldsymbol{\phi}] = \frac{d}{d\varepsilon} E(\boldsymbol{f} + \varepsilon\boldsymbol{\phi})\bigg|_{\varepsilon=0}.$$

He [7] gave the variational formulae for the Möbius energy as follows. Note that $x \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ means a general parameter.

Theorem 1.6 ([7]) *Let $\boldsymbol{f} \in C^{3,\alpha}(\mathbb{R}/\mathbb{Z})$ be a simple curve and let $0 < \alpha \leq 1$. Then for any $\boldsymbol{\phi} \in H^2(\mathbb{R}/\mathbb{Z})$, it holds that*

$$\delta\mathcal{E}(\boldsymbol{f}, \phi) = \int_{\mathbb{R}/\mathbb{Z}} (\boldsymbol{G}(\boldsymbol{f})(x_1) \cdot \boldsymbol{\phi}(x_1)) \|\boldsymbol{f}'(x_1)\|_{\mathbb{R}^n} dx_1,$$

$$\begin{aligned} &\boldsymbol{G}(\boldsymbol{f})(x_1) \\ &= 2\int_{\mathbb{R}/\mathbb{Z}} \left\{ \frac{2P_{\boldsymbol{f}'(x_1)^\perp}(\boldsymbol{f}(x_2) - \boldsymbol{f}(x_1))}{\|(\boldsymbol{f}(x_1) - \boldsymbol{f}(x_2))\|_{\mathbb{R}^3}^2} - \frac{1}{\|\boldsymbol{f}(x_1)\|_{\mathbb{R}^3}} \left(\frac{\boldsymbol{f}'(x_1)}{\|\boldsymbol{f}(x_1)\|_{\mathbb{R}^3}} \right)' \right\} \frac{\|\boldsymbol{f}'(x_2)\|_{\mathbb{R}^3}}{\|\boldsymbol{f}(x_1) - \boldsymbol{f}(x_2)\|_{\mathbb{R}^3}^2}, dx_2 \end{aligned}$$

where

$$P_{\boldsymbol{f}'(x)^\perp}(\boldsymbol{v}) = \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{f}'(x) \frac{\boldsymbol{f}'(x)}{\|\boldsymbol{f}'(x)\|_{\mathbb{R}^3}^2}.$$

Theorem 1.7 ([7]) *Let $\boldsymbol{f} \in C^\infty(\mathbb{R}/\mathbb{Z})$ be a simple curve. Then there exists a pseudo-differential operator $\tilde{L}$ whose order is less than 2 such that*

$$\delta H(\boldsymbol{f}, \boldsymbol{\phi}) = \frac{2\pi}{3} P_{\boldsymbol{f}'^\perp}(-\Delta_s)^{\frac{3}{2}} P_{\boldsymbol{f}'^\perp}\boldsymbol{\phi} + \tilde{L}\boldsymbol{\phi},$$

where Δ_s is a Laplacian with respect to arc-length parameter s.

Ishizeki-Nagasawa showed that $\mathcal{E}(\boldsymbol{f})$ is decomposed into three parts which keep the Möbius invariance.

Theorem 1.8 ([8],[9],[10]) *Let $\boldsymbol{f} \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z}) \cap W^{1,\infty}$ be bi-Lipschitz. There holds*

$$\mathcal{E}(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) + 4,$$

$$\mathcal{E}_i(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}_i(\boldsymbol{f}) ds_1 ds_2 \quad (i = 1, 2),$$

$$\mathscr{M}_1(\boldsymbol{f}) = \frac{\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},$$

$$\mathscr{M}_2(\boldsymbol{f}) = \frac{2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4} \det \begin{pmatrix} \boldsymbol{\tau}(s_1) \cdot \boldsymbol{\tau}(s_2) & \Delta\boldsymbol{f} \cdot \boldsymbol{\tau}(s_1) \\ \Delta\boldsymbol{f} \cdot \boldsymbol{\tau}(s_2) & \|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2 \end{pmatrix},$$

where $\boldsymbol{\tau} = \boldsymbol{f}' = \dfrac{d\boldsymbol{f}}{ds}$. Moreover, let

$$\boldsymbol{f} \mapsto \boldsymbol{p} = \boldsymbol{c} + \frac{r^2(\boldsymbol{f} - \boldsymbol{c})}{\|\boldsymbol{f} - \boldsymbol{c}\|_{\mathbb{R}^n}^2}$$

be the inversion with respect to sphere with center $\boldsymbol{c}$ and radius r. The following assertions hold:

1. *Each energy $\mathcal{E}_i$ is invariant under the dilation.*

2. *If $\boldsymbol{f} \in W^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then $\mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p}) + \mathcal{E}_2(\boldsymbol{p})$.*

3. *If $\boldsymbol{c} \notin \operatorname{Im} \boldsymbol{f}$ and $\mathcal{E}(\boldsymbol{f}) < \infty$, then $\mathcal{E}_1(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p})$, $\mathcal{E}_2(\boldsymbol{f}) = \mathcal{E}_2(\boldsymbol{p})$.*

4. *If $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and $\boldsymbol{c} \in \operatorname{Im} \boldsymbol{f}$, then $\mathcal{E}_1(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{p}) + 2\pi^2$, $\mathcal{E}_2(\boldsymbol{f}) = \mathcal{E}_2(\boldsymbol{p}) - 2\pi^2$.*

Furthermore, they showed that the first and second variations of $\mathcal{E}_i$ are given as (bi-)linear operators on $W^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathbb{Z})$, and the first variations have the L^2-representations in their paper [9, 11].

We use δ^2 to mean

$$\delta^2 \mathcal{E}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] = \left. \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2} \mathcal{E}_i(\boldsymbol{f} + \varepsilon_1 \boldsymbol{\phi} + \varepsilon_2 \boldsymbol{\psi}) \right|_{\varepsilon_1 = 0, \varepsilon_2 = 0}.$$

Ishizeki-Nagasawa gave the density functions of first and second variations of Möbiius energy associated with the Lebesgue measure $ds_1 ds_2$.

Theorem 1.9 ([9]) *Let $\boldsymbol{f} \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z}) \cap W^{1,\infty}$ be bi-Lipschitz. There holds*

$$\delta \mathcal{E}_i(\boldsymbol{f})[\boldsymbol{\phi}] = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}] ds_1 ds_2,$$
$$\delta^2 \mathcal{E}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] ds_1 ds_2,$$

$$\begin{aligned}
\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}] &= \frac{Q\boldsymbol{f} \cdot Q\boldsymbol{\phi}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}}^2} - \frac{2\mathscr{M}_1(\boldsymbol{f}) \Delta \boldsymbol{f} \cdot \Delta \boldsymbol{\phi}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}}^2}, \\
\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\phi}] &= \frac{\tilde{Q}_1 \boldsymbol{f} \cdot \tilde{Q}_2 \boldsymbol{\phi} + \tilde{Q}_2 \boldsymbol{f} \cdot \tilde{Q}_1 \boldsymbol{\phi}}{2\|\Delta \boldsymbol{f}\|_{\mathbb{R}}^2} - \frac{2\mathscr{M}_2(\boldsymbol{f}) \Delta \boldsymbol{f} \cdot \Delta \boldsymbol{\phi}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}}^2}, \\
\mathscr{H}_1(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] &= \frac{Q\boldsymbol{\phi} \cdot Q\boldsymbol{\psi}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{S(\boldsymbol{f}, \boldsymbol{\phi}) \cdot S(\boldsymbol{f}, \boldsymbol{\psi})}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}] \Delta \boldsymbol{f} \cdot \Delta \boldsymbol{\psi}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \\
&\quad - \frac{2\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\psi}] \Delta \boldsymbol{f} \cdot \Delta \boldsymbol{\phi}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{M}_1(\boldsymbol{f}) \Delta \boldsymbol{\phi} \cdot \Delta \boldsymbol{\psi}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}, \\
\mathscr{H}_2(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] &= \frac{\tilde{Q}_1 \boldsymbol{\phi} \cdot \tilde{Q}_2 \boldsymbol{\psi} + \tilde{Q}_2 \boldsymbol{\phi} \cdot \tilde{Q}_1 \boldsymbol{\psi}}{2\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \\
&\quad + \frac{\tilde{S}_1(\boldsymbol{f}, \boldsymbol{\phi}) \tilde{S}_2(\boldsymbol{f}, \boldsymbol{\psi}) + \tilde{S}_2(\boldsymbol{f}, \boldsymbol{\phi}) \tilde{S}_1(\boldsymbol{f}, \boldsymbol{\psi})}{2\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \\
&\quad - \frac{2\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\phi}] \Delta \boldsymbol{f} \cdot \Delta \boldsymbol{\psi}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\psi}] \Delta \boldsymbol{f} \cdot \Delta \boldsymbol{\phi}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{2\mathscr{M}_2(\boldsymbol{f}) \Delta \boldsymbol{\phi} \cdot \Delta \boldsymbol{\psi}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2},
\end{aligned}$$

where

$$Q\boldsymbol{v} = \boldsymbol{v}_1' - \boldsymbol{v}_2', \quad \tilde{Q}_i \boldsymbol{v} = (-1)^{i-1} 2\{\boldsymbol{v}_i' - (R\boldsymbol{f} \cdot \boldsymbol{\tau}_i) R\boldsymbol{v}\},$$
$$R\boldsymbol{v} = \frac{|\Delta s| \Delta \boldsymbol{v}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n} \Delta s}, \quad \hat{R}\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v}_1' + \boldsymbol{v}_2'),$$
$$S(\boldsymbol{u}, \boldsymbol{v}) = \hat{R}\boldsymbol{u} \cdot Q\boldsymbol{v} + Q\boldsymbol{u} \cdot \hat{R}\boldsymbol{v}, \quad \tilde{S}_i(\boldsymbol{u}, \boldsymbol{v}) = R\boldsymbol{u} \cdot \tilde{Q}_i \boldsymbol{v} + \tilde{Q}_i \boldsymbol{u} \cdot R\boldsymbol{v}.$$

Also we use the decomposition

$$\tilde{Q}_i \boldsymbol{\phi} = (-1)^{i-1} (\tilde{Q}_{2i} \boldsymbol{\phi} + \bar{Q}_{2i} \boldsymbol{\phi}),$$

where

$$\tilde{Q}_{2i}\boldsymbol{\phi} = -2T_i^0\boldsymbol{\phi},$$

$$\bar{Q}_{2i}\boldsymbol{\phi} = -\,2T_i^2\boldsymbol{f}\cdot\boldsymbol{\tau}_i\frac{\Delta\boldsymbol{\phi}}{\Delta s},$$

$$T_i^k\boldsymbol{\phi} = \left(\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\right)^k\frac{\Delta\boldsymbol{\phi}}{\Delta s} - \boldsymbol{\phi}_i'.$$

Let Δ_s denote the Laplacian with respect to the arc-length parameter s. Ishizeki-Nagasawa gave the L^2-representation of $\delta\mathcal{E}_i$.

Definition 1.1 *Let $\boldsymbol{\phi}\in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$. The third order pseudo-differential operators $L_i : H^3(\mathbb{R}/\mathcal{L}\mathbb{Z}) \to L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, $(i = 1, 2)$ are given by*

$$L_1\boldsymbol{\phi} = 2\pi(-\Delta_s)^{\frac{3}{2}}\boldsymbol{\phi} - 4\sum_{k\in\mathbb{Z}}\left|\frac{2\pi k}{\mathcal{L}}\right|^3 \mathrm{si}\,(|k\pi|)\langle\boldsymbol{\phi},\varphi_k\rangle_{L^2}\varphi_k + \frac{8}{\mathcal{L}}\Delta_s(\boldsymbol{\phi}-\check{\boldsymbol{\phi}}),$$

$$L_2\boldsymbol{\phi} = -\frac{4}{3}\pi(-\Delta_s)^{\frac{3}{2}}\boldsymbol{\phi} + \frac{8}{3}\sum_{k\in\mathbb{Z}}\left|\frac{2\pi k}{\mathcal{L}}\right|^3 \mathrm{si}\,(|k\pi|)\langle\boldsymbol{\phi},\varphi_k\rangle_{L^2}\varphi_k$$
$$+\frac{16}{3\mathcal{L}}\Delta_s\check{\boldsymbol{\phi}} + \frac{128}{3\mathcal{L}^3}(\boldsymbol{\phi}-\check{\boldsymbol{\phi}}),$$

where

$$\mathrm{si}\,(t) = -\int_t^\infty \frac{\sin\lambda}{\lambda}d\lambda,\ \varphi_k(s) = \frac{1}{\sqrt{\mathcal{L}}}\exp\left(\frac{2\pi iks}{\mathcal{L}}\right),\ \check{\boldsymbol{\phi}}(s) = \boldsymbol{\phi}(s+\frac{\mathcal{L}}{2}).$$

Theorem 1.10 ([11]) *Let $\boldsymbol{f}\in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ be bi-Lipschitz continuous and let $\boldsymbol{\phi}\in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Then there exist a mapping $\boldsymbol{G}_i : H^3(\mathbb{R}/\mathcal{L}\mathbb{Z}) \to L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ such that*

$$\delta\mathcal{E}_i(\boldsymbol{f})[\boldsymbol{\phi}] = \langle\boldsymbol{G}_i(\boldsymbol{f}),\boldsymbol{\phi}\rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})}$$

for each $i\in\{1,2\}$. Moreover, there exists $\boldsymbol{N}_i : H^3(\mathbb{R}/\mathcal{L}\mathbb{Z}) \to L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ such that

$$\boldsymbol{G}_i(\boldsymbol{f}) = L_i\boldsymbol{f} + \boldsymbol{N}_i(\boldsymbol{f}),$$

and for any $\alpha\in\left(0,\frac{1}{2}\right)$, there exists a constant $C_\alpha(\|\boldsymbol{f}\|_{H^{3-\alpha}})$ such that

$$\|\boldsymbol{N}_i(\boldsymbol{f})\|_{L^2} \le C_\alpha(\|\boldsymbol{f}\|_{H^{3-\alpha}})$$

for each $i\in\{1,2\}$.

We call the function $\boldsymbol{G}_i(\boldsymbol{f})$ L^2-gradient.

In this thesis, we will extend the result [11] to the existence of the L^2-representaion of the second variation, whose proof takes up a substantial part of the thesis.

Theorem 1.11 *Let $\boldsymbol{f}\in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ be a bi-Lipschitz continuous function and $\boldsymbol{\phi}\in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$, $\boldsymbol{\psi}\in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Then*

$$\delta^2\mathcal{E}_i(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \langle P_{\boldsymbol{\tau}}^{\perp}L_i\boldsymbol{\phi} - (L_i\boldsymbol{f}\cdot\boldsymbol{\tau})\boldsymbol{\phi}',\boldsymbol{\psi}\rangle + \langle\boldsymbol{N}_i(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})},$$

where

$$P_{\boldsymbol{\tau}}^{\perp}\boldsymbol{v} = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\tau})\,\boldsymbol{\tau}.$$

Furthermore, for any $\alpha \in (0, \frac{1}{2})$*,*

$$\|\boldsymbol{N}_i(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})} \le C\|\boldsymbol{f}\|_{H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})}\|\boldsymbol{\phi}\|_{H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})}$$

holds.

Theorem 1.11 will be proven in §§3–4 after the preliminaries (in §2).

1.3 The Łojasiewicz inequality

Let V, W be a pair of Banach spaces which satisfies $V \hookrightarrow W$, and let $\|\cdot\|_W$ denote a norm of W. Let E be a functional on V, and we write E' for the Fréchet derivative of E. Let $u \in V$. If there exist $\theta \in (0, \frac{1}{2}), c > 0$ and a neighborhood $U \subset V$ of u such that there holds

$$|E(v) - E(u)|^{1-\theta} \le c\|E'(v)\|_W$$

for any $v \in U$, then we say that E satisfies the Łojasiewicz inequality in the neighborhood U. We call θ the Łojasiewicz exponent.

The Łojasiewicz inequality is used for asymptotic analysis of evolution equations. For example, the following fact is known. Let V, W be a pair of Hilbert spaces satisfying $V \hookrightarrow W$. For a continuous function $\mathcal{F}$ from V to W, we consider the dynamical system

$$\frac{d}{dt}u(t) + \mathcal{F}(u(t)) = 0, \quad t \ge 0. \tag{1}$$

Now we let $\mathbb{R}_+ = [0, \infty)$. We define the ω-limit set $\omega(u)$ of $u \in C(\mathbb{R}_+, V)$ in V as

$$\omega(u) = \bigcap_{t\ge 0} \overline{\bigcup_{s \ge t}\{u(s)\}}^{\|\cdot\|_V}.$$

Chill-Haraux-Jendoubi [4] showed the following theorem.

Theorem 1.12 [4, Chill-Haraux-Jendoubi, Theorem 7.] *Let* $u \in C^1(\mathbb{R}_+, V)$ *be a solution of (1). We assume the following.*

1. *There exist a functional* $\mathcal{E} \in C^2(V, \mathbb{R})$ *and a constant* α *such that for any* $u \in V$ *satisfying* $\mathcal{F} \in V$*,*
$$\langle \mathcal{E}'(u), \mathcal{F}(u)\rangle_{V',V} \ge \alpha\|\mathcal{E}'(u)\|_{V'}\|\mathcal{F}(u)\|_W.$$
2. *Under the above condition, if* $\varphi \in V$ *satisfies* $\mathcal{E}'(\varphi) = 0$*, then* $\mathcal{F}(\varphi) = 0$ *holds.*
3. *There exist a function* $\varphi \in \omega(u)$ *and a constant* $\theta \in (0, \frac{1}{2}]$ *such that* $\mathcal{E}$ *satisfies the Łojasiewicz inequality near* φ *for the exponent* θ*.*

Then there holds $\lim_{t\to\infty} u(t) = \varphi$ *in* V*. In addition to 1-3, we assume the following condition.*

4. *There exists a constant* β *such that for any* $u \in V$ *satisfying* $\mathcal{F} \in V$*,*
$$\langle \mathcal{E}'(u), \mathcal{F}(u)\rangle_{V',V} \ge \beta\|\mathcal{E}'(u)\|_{V'}^2.$$

Then

$$\|u(t)-\varphi\|_W = \begin{cases} \mathcal{O}\left(e^{-ct}\right) & (\theta = \frac{1}{2}), \\ \mathcal{O}\left(t^{-\theta/(1-2\theta)}\right) & (\theta \in \left(0, \frac{1}{2}\right)) \end{cases}$$

holds.

In this thesis, we will show the Łojasiewicz inequality for the decomposed Möbius energies in §5, which is an application of Theorem 1.11. Blatt analyze the asymptotic behavior of the gradient flow.

Theorem 1.13 ([2]) *Let $\boldsymbol{f}_0$ be a local minimizer of $\mathcal{E}$ in $C^k(\mathbb{R}/\mathbb{Z})$. Then for any $\beta > 0$, there exists a neighborhood V of $\boldsymbol{f}_0$ in $C^{2+\beta}$ such that for all $\boldsymbol{f}_I \in V$, the heat flow*

$$\frac{\partial \boldsymbol{f}}{\partial t} = -\boldsymbol{G}(\boldsymbol{f})$$

with initial data $\boldsymbol{f}_I$ exists for all times and converges to $\boldsymbol{f}_\infty$ satisfying $\mathcal{E}(\boldsymbol{f}_\infty) = \mathcal{E}(\boldsymbol{f}_0)$ after suitable reparametrization

Here $\boldsymbol{G}$ is the L^2-representation of the first variation of $\mathcal{E}$ given by He as above. Note that we use a general parameter $x \in \mathbb{R}/\mathbb{Z}$ instead of $s \in \mathbb{R}/\mathcal{L}\mathbb{Z}$. In order to prove this theorem, he showed that $\mathcal{E}(\cdot)$ satisfies the Łojasiewicz inequality.

Theorem 1.14 ([2]) *Let a function $\boldsymbol{f}_0 \in C^\infty(\mathbb{R}/\mathbb{Z})$ be a stationary point of $\mathcal{E}_i$ for $i \in \{1,2\}$. Then there exist $\theta \in (0, \frac{1}{2}), \sigma > 0, c > 0$ such that if a function $\boldsymbol{f} \in H^3(\mathbb{R}/\mathbb{Z})$ satisfies $\|\boldsymbol{f} - \boldsymbol{f}_0\|_{H^3(\mathbb{R}/\mathbb{Z})} \leq \sigma$, the inequality*

$$\begin{aligned} |\mathcal{E}(\boldsymbol{f}) - \mathcal{E}(\boldsymbol{f}_0)|^{1-\theta} &\leq C\|\boldsymbol{G}(\boldsymbol{f})\|_{L^2(\mathbb{R}/\mathbb{Z})} \\ &= C\left(\int_{\mathbb{R}/\mathbb{Z}} \|\boldsymbol{G}(\boldsymbol{f})(x)\|_{\mathbb{R}^n}^2 \|\boldsymbol{f}'(x)\|_{\mathbb{R}^n} dx\right)^{\frac{1}{2}} \end{aligned}$$

holds for some exponent $\theta \in (0, \frac{1}{2})$ and constant $C > 0$ independent of $\boldsymbol{f}$.

We will show similar estimates for the decomposed energies.

Theorem 1.15 *Let a function $\boldsymbol{f}_0 \in C^\infty(\mathbb{R}/\mathbb{Z})$ be a stationary point of $\mathcal{E}_i$ for $i \in \{1,2\}$. Then there exist $\theta \in (0, \frac{1}{2}), \sigma > 0, c > 0$ such that if a function $\boldsymbol{f} \in H^3(\mathbb{R}/\mathbb{Z})$ satisfies $\|\boldsymbol{f} - \boldsymbol{f}_0\|_{H^3(\mathbb{R}/\mathbb{Z})} \leq \sigma$, the inequality*

$$|\mathcal{E}_i(\boldsymbol{f}) - \mathcal{E}_i(\boldsymbol{f}_0)|^{1-\theta} \leq c\|\boldsymbol{G}_i(\boldsymbol{f})\|_{L^2(\mathbb{R}/\mathbb{Z})}$$

holds.

We will show Theorem 1.15 by use of abstract theory developed by Chill [3] in §5. To apply it we need the analyticity of the first variation and the Fredholm property of the second variation. They follow from the L^2-representation of the first and second variations.

This thesis is based on [6] with the addition of details.

2 Preliminaries

Let X be a normed function space comprising of functions on $\mathbb{R}/\mathcal{L}\mathbb{Z}$. We denote the norm $\|\cdot\|_{X(\mathbb{R}/\mathcal{L}\mathbb{Z})}$ by $\|\cdot\|_X$ in §§2–5. Similarly when X is a Hilbert space, we denote the inner product $\langle\cdot,\cdot\rangle_{X(\mathbb{R}/\mathcal{L}\mathbb{Z})}$ simply by $\langle\cdot,\cdot\rangle_X$.

We write $\{\Delta s \neq 0\}$ for the set

$$\{\Delta s \neq 0\} = \{(s_1, s_2) \in (\mathbb{R}/\mathcal{L}\mathbb{Z})^2 | s_1 \neq s_2 \,(\mathrm{mod}\ \mathcal{L})\}.$$

In this section, we will see the asymptotic behavior near $\Delta s = 0$ of several functions on $\{\Delta s \neq 0\}$ which compose the integrand of the second variation of $\mathcal{E}_i$. Furthermore we will show that certain combinations of these functions become integrable and are dominated by the norms of $\boldsymbol{f}$ and $\boldsymbol{\phi}$.

Definition 2.1 *For $\boldsymbol{h} : \{\Delta s \neq 0\} \to \mathbb{R}^n$ and $\alpha > 0$, we use the notation*

$$\boldsymbol{h}(s_1, s_2) = \mathcal{O}(\Delta s)^\alpha \ \text{ as } \Delta s \to 0$$

to mean

$$\sup_{s_1 \neq s_2} \frac{\|\boldsymbol{h}(s_1, s_2)\|_{\mathbb{R}^n}}{|s_1 - s_2|^\alpha} < \infty.$$

Definition 2.2 *Let $i \in \{1, 2\}, a, b \in \mathbb{R}, \boldsymbol{u} \in C^1(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$. When $\boldsymbol{v}, \boldsymbol{w} \in L^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$, we set*

$$\boldsymbol{p}_i^{a,b}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})(s_1, s_2) := \left\{ \frac{a\tilde{Q}_{21}\boldsymbol{u} + b\tilde{Q}_{22}\boldsymbol{u}}{(\Delta s)^3} \cdot \boldsymbol{v}_i \right\} \boldsymbol{w}_i,$$

and when $v \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R})$, we set

$$\boldsymbol{q}_i^{a,b}(\boldsymbol{u}, v)(s_1, s_2) := v_i \frac{a\tilde{Q}_{21}\boldsymbol{u} + b\tilde{Q}_{22}\boldsymbol{u}}{(\Delta s)^3}.$$

Definition 2.3 *Let $\boldsymbol{f} \in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ be a bi-Lipschitz continuous function and let $\boldsymbol{\phi}$ be a function on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, and let $i, j \in \{1, 2\}$, and $k \in \mathbb{N}$. We define Δ_{ij} and $\mathscr{M}^k$ by*

$$\Delta_{ij}\boldsymbol{\phi} = \boldsymbol{\phi}_i - \boldsymbol{\phi}_j,$$

$$\mathscr{M}^k(\boldsymbol{f}) = \frac{1}{(\Delta s)^2} \left(\frac{|\Delta s|^k}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^k} - 1 \right).$$

Lemma 2.1 *Then the following assertions hold.*
(i) If $\boldsymbol{\phi} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, there holds

$$\frac{\Delta\boldsymbol{\phi}}{\Delta s} = \boldsymbol{\phi}_i' + \mathcal{O}(\Delta s),$$

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|} \left\| \frac{\Delta\boldsymbol{\phi}}{\Delta s} - \boldsymbol{\phi}_i' \right\|_{\mathbb{R}^n} \leq \|\boldsymbol{\phi}\|_{C^{1,1}},$$

$$\sup_{s_1 \neq s_2} \left\| \frac{\Delta\boldsymbol{\phi}}{\Delta s} \right\|_{\mathbb{R}^n} \leq \|\boldsymbol{\phi}\|_{C^{1,1}}$$

for each $i = 1, 2$.
(ii) Let $\alpha \in (0, \frac{1}{2})$. *If* $\boldsymbol{\phi} \in H^{2-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, *there holds*

$$\frac{\Delta \boldsymbol{\phi}}{\Delta s} = \boldsymbol{\phi}_i' + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha},$$

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}} \left\| \frac{\Delta \boldsymbol{\phi}}{\Delta s} - \boldsymbol{\phi}_i' \right\|_{\mathbb{R}^n} \leq \|\boldsymbol{\phi}\|_{H^{2-\alpha}},$$

$$\sup_{s_1 \neq s_2} \left\| \frac{\Delta \boldsymbol{\phi}}{\Delta s} \right\|_{\mathbb{R}^n} \leq \|\boldsymbol{\phi}\|_{H^{2-\alpha}}$$

for each $i = 1, 2$.

Proof (i) To prove this inequality note that

$$\begin{aligned} \frac{1}{|\Delta s|} \left\| \frac{\Delta \boldsymbol{\phi}}{\Delta s} - \boldsymbol{\phi}_i' \right\|_{\mathbb{R}^n} &\leq \left\| \frac{1}{\Delta s} \int_{s_2}^{s_1} \frac{\boldsymbol{\phi}_3' - \boldsymbol{\phi}_i'}{|\Delta s|} ds_3 \right\|_{\mathbb{R}^n} \\ &\leq C \|\boldsymbol{\phi}\|_{C^{1,1}}. \end{aligned}$$

The remaining statements follow from the inequality.
(ii) Using $H^{2-\alpha} \hookrightarrow C^{1+\frac{1}{2}-\alpha}$, we have

$$\begin{aligned} \frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}} \left\| \frac{\Delta \boldsymbol{\phi}}{\Delta s} - \boldsymbol{\phi}_i' \right\|_{\mathbb{R}^n} &\leq \left\| \frac{1}{\Delta s} \int_{s_2}^{s_1} \frac{\boldsymbol{\phi}_3' - \boldsymbol{\phi}_i'}{|\Delta s|^{\frac{1}{2}-\alpha}} ds_3 \right\|_{\mathbb{R}^n} \\ &\leq C \|\boldsymbol{\phi}\|_{H^{2-\alpha}}, \end{aligned}$$

where C is a constant dependent on α. Similary to (i), we can obtain the statements.

□

Lemma 2.2 *Then the following assertions hold.*
(i) If $\boldsymbol{\phi} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, *there holds*

$$\frac{\Delta_{ij} \boldsymbol{\phi}}{\Delta s} = \boldsymbol{\phi}_i' + \mathcal{O}(\Delta s),$$

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|} \left\| \frac{\Delta_{ij} \boldsymbol{\phi}}{\Delta s} - \boldsymbol{\phi}_i' \right\|_{\mathbb{R}^n} \leq \|\boldsymbol{\phi}\|_{C^{1,1}},$$

$$\sup_{s_1 \neq s_2} \left\| \frac{\Delta_{ij} \boldsymbol{\phi}}{\Delta s} \right\|_{\mathbb{R}^n} \leq \|\boldsymbol{\phi}\|_{C^{1,1}}$$

for each $i = 1, 2$.
(ii) Let $\alpha \in (0, \frac{1}{2})$. *If* $\boldsymbol{\phi} \in H^{2-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, *there holds*

$$\frac{\Delta_{ij} \boldsymbol{\phi}}{\Delta s} = \boldsymbol{\phi}_i' + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha},$$

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}} \left\| \frac{\Delta_{ij} \boldsymbol{\phi}}{\Delta s} - \boldsymbol{\phi}_i' \right\|_{\mathbb{R}^n} \leq \|\boldsymbol{\phi}\|_{H^{2-\alpha}},$$

$$\sup_{s_1 \neq s_2} \left\| \frac{\Delta_{ij} \boldsymbol{\phi}}{\Delta s} \right\|_{\mathbb{R}^n} \leq \|\boldsymbol{\phi}\|_{H^{2-\alpha}}$$

for each $i = 1, 2$.

Proof The assertion is obvious from Lemma 2.1. $\square$

We denote the curvature vector by $\boldsymbol{\kappa} = \boldsymbol{f}''$.

Lemma 2.3 *Let $\alpha \in (0, \frac{1}{2})$ and let $\boldsymbol{f} \in H^{3-\alpha}$ be bi-Lipschitz continuous, and let b be the bi-Lipschitz constant of $\boldsymbol{f}$. Then*

$$\mathscr{M}(\boldsymbol{f}) = \frac{\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{12} + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha},$$

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}} \left| \mathscr{M}(\boldsymbol{f}) - \frac{\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{12} \right| \leq C(b, \|\boldsymbol{f}\|_{H^{3-\alpha}})$$

for any $i \in \{1, 2\}$.

Proof Since $\boldsymbol{\kappa}$ is bounded in the norm $\|\cdot\|_{L^\infty}$, and since

$$\begin{aligned}
(\Delta s)^2 - \|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2 &= \int_{s_2}^{s_1} \int_{s_2}^{s_1} (1 - \boldsymbol{\tau}_3 \cdot \boldsymbol{\tau}_4) ds_3 ds_4 \\
&= \frac{1}{2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \|\boldsymbol{\tau}_3 - \boldsymbol{\tau}_4\|_{\mathbb{R}^n}^2 ds_3 ds_4 \\
&= \frac{1}{2} \int_{s_2}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_4} \int_{s_3}^{s_4} \boldsymbol{\kappa}_5 \cdot \boldsymbol{\kappa}_6 ds_5 ds_6 ds_3 ds_4,
\end{aligned}$$

we have

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^4} |(\Delta s)^2 - \|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2| \leq C \|\boldsymbol{f}\|_{H^{3-\alpha}}^2.$$

Moreover, we have

$$\begin{aligned}
\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^k - 1 &= \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} - 1 \right) \sum_{l=0}^{k-1} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^l \\
&= \frac{\left(\frac{|\Delta s|^2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - 1 \right)}{\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} + 1} \sum_{l=0}^{k-1} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^l \\
&= \frac{(\Delta s)^2 - \|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}{(\Delta s)^2} \frac{1}{\frac{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}{|\Delta s|} + \frac{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}{|\Delta s|^2}} \sum_{l=0}^{k-1} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^l,
\end{aligned}$$

we obtain

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^2} \left| \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^k - 1 \right| \leq C(b, \|\boldsymbol{f}\|_{H^{3-\alpha}}).$$

Using

$$\int_{s_2}^{s_1} \int_{s_2}^{s_1} \int_{s_3}^{s_4} \int_{s_3}^{s_4} ds_5 ds_6 ds_3 ds_4 = \frac{(\Delta s)^4}{6},$$

we obtain

$$\begin{aligned}
&(\Delta s)^2 - \|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2 - \frac{\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{12} (\Delta s)^4 \\
&= \frac{1}{2} \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{s_3}^{s_4} \int_{s_3}^{s_4} (\boldsymbol{\kappa}_5 \cdot \boldsymbol{\kappa}_6 - \boldsymbol{\kappa}_i \cdot \boldsymbol{\kappa}_i) ds_5 ds_6 ds_3 ds_4 \\
&= \frac{1}{2} \int_{s_1}^{s_2} \int_{s_1}^{s_2} \int_{s_3}^{s_4} \int_{s_3}^{s_4} \left\{ (\boldsymbol{\kappa}_5 - \boldsymbol{\kappa}_i) \cdot \boldsymbol{\kappa}_6 - \boldsymbol{\kappa}_i \cdot (\boldsymbol{\kappa}_6 - \boldsymbol{\kappa}_i) \right\} ds_5 ds_6 ds_3 ds_4.
\end{aligned}$$

Since $|s_5 - s_i|, |s_6 - s_i| \leq |\Delta s|$, we have

$$\begin{aligned} \|\boldsymbol{\kappa}_5 - \boldsymbol{\kappa}_i\|_{\mathbb{R}^n} &\leq \frac{\|\boldsymbol{\kappa}_5 - \boldsymbol{\kappa}_i\|_{\mathbb{R}^n}}{|s_5 - s_i|^{\frac{1}{2}-\alpha}} |\Delta s|^{\frac{1}{2}-\alpha} \\ &\leq C\|\boldsymbol{f}\|_{H^{3-\alpha}} |\Delta s|^{\frac{1}{2}-\alpha}, \end{aligned}$$

and hence it holds that

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^{4+\frac{1}{2}-\alpha}} \left| (\Delta s)^2 - \|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n} - \frac{\|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n}}{12} (\Delta s)^4 \right| \leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}}).$$

Therefore, we obtain

$$\begin{aligned} \mathscr{M}(\boldsymbol{f}) - \frac{\|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n}}{12} =& \frac{1}{\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}} - \frac{1}{|\Delta s|^2} - \frac{\|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n}}{12} \frac{|\Delta s|^2}{\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}} + \frac{\|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n}}{12} \left(\frac{|\Delta s|^2}{\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}} - 1 \right) \\ =& \frac{1}{(\Delta s)^2} \frac{1}{\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}} \left\{ (\Delta s)^2 - \|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n} - \frac{\|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n}}{12} (\Delta s)^4 \right\} \\ &+ \frac{\|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n}}{12} \left\{ \frac{|\Delta s|^2}{\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}} - 1 \right\}, \end{aligned}$$

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}} \left| \mathscr{M}(\boldsymbol{f}) - \frac{\|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n}}{12} \right| \leq C(b, \|\boldsymbol{f}\|_{H^{3-\alpha}}).$$

□

Lemma 2.4 *Let $\boldsymbol{f} \in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ be bi-Lipschitz continuous with $\alpha \in (0, \frac{1}{2})$, let b be the bi-Lipschitz constant of $\boldsymbol{f}$, and let $\boldsymbol{\phi} \in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Then, for $i, j, k, l \in \{1, 2\}$, the following assertions hold:*

$$T_i^k \boldsymbol{\phi} = \frac{(-1)^i}{2} \boldsymbol{\phi}_j'' \Delta s + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha} = \mathcal{O}(\Delta s),$$

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}} \left| \frac{T_i^k \boldsymbol{\phi}}{\Delta s} - \frac{(-1)^i}{2} \boldsymbol{\phi}_j'' \right| \leq C(b) \|\boldsymbol{\phi}\|_{H^{3-\alpha}}.$$

Furthermore there holds

$$T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}_i = -\frac{\|\boldsymbol{\kappa}_l\|^2_{\mathbb{R}^n} (\Delta s)^2}{6} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha} = \mathcal{O}(\Delta s)^2,$$

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}} \left| \frac{T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}_i}{(\Delta s)^2} + \frac{\|\boldsymbol{\kappa}_l\|^2_{\mathbb{R}^n}}{6} \right| \leq C(b) \|\boldsymbol{\phi}\|_{H^{3-\alpha}}$$

for and

$$T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}_j = \frac{\|\boldsymbol{\kappa}_l\|^2_{\mathbb{R}^n} (\Delta s)^2}{3} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha} = \mathcal{O}(\Delta s)^2,$$

$$\sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}} \left| \frac{T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}_j}{(\Delta s)^2} - \frac{\|\boldsymbol{\kappa}_l\|^2_{\mathbb{R}^n}}{3} \right| \leq C(b) \|\boldsymbol{\phi}\|_{H^{3-\alpha}}$$

for $i \neq j$.

Proof Since

$$\int_{s_2}^{s_1}\int_{s_i}^{s_3} ds_4 ds_3 = \frac{(-1)^i(\Delta s)^2}{2},$$

we have

$$\begin{aligned}
T_i^k\boldsymbol{\phi} - \frac{(-1)^i\Delta s}{2}\boldsymbol{\phi}_j'' &= \left\{\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^k - 1\right\}\frac{\Delta\boldsymbol{\phi}}{\Delta s} + \frac{\Delta\boldsymbol{\phi}}{\Delta s} - \boldsymbol{\phi}_i' - \frac{(-1)^i\Delta s}{2}\boldsymbol{\phi}_j'' \\
&= \left\{\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^k - 1\right\}\frac{\Delta\boldsymbol{\phi}}{\Delta s} + \frac{1}{\Delta s}\int_{s_2}^{s_1}\left(\boldsymbol{\phi}_3' - \boldsymbol{\phi}_i'\right) ds_3 \\
&\quad - \frac{(-1)^i\Delta s}{2}\boldsymbol{\phi}_j'' \\
&= \left\{\left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right)^k - 1\right\}\frac{\Delta\boldsymbol{\phi}}{\Delta s} + \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_i}^{s_3}\left(\boldsymbol{\phi}_4'' - \boldsymbol{\phi}_j''\right) ds_4 ds_3 \\
&= \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}.
\end{aligned}$$

Moreover, as $\boldsymbol{\tau}(s)\cdot\boldsymbol{\kappa}(s) = 0$,

$$\begin{aligned}
T_i^0\boldsymbol{f}\cdot\boldsymbol{\tau}_j &= \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_i}^{s_3}\boldsymbol{\kappa}_4\cdot\boldsymbol{\tau}_j ds_4 ds_3 \\
&= \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_i}^{s_3}\boldsymbol{\kappa}_4\cdot(\boldsymbol{\tau}_j - \boldsymbol{\tau}_4) ds_4 ds_3 \\
&= \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_i}^{s_3}\int_{s_4}^{s_j}\boldsymbol{\kappa}_4\cdot\boldsymbol{\kappa}_5 ds_5 ds_4 ds_3 = \mathcal{O}(\Delta s)^2.
\end{aligned}$$

On the other hand, we have

$$\int_{s_2}^{s_1}\int_{s_i}^{s_3}\int_{s_4}^{s_j} ds_5 ds_4 ds_3 = \begin{cases} -\dfrac{1}{6}(\Delta s)^3 & (i = j), \\ \dfrac{1}{3}(\Delta s)^3 & (i \neq j). \end{cases}$$

Consequently it holds that

$$\begin{aligned}
T_i^0\boldsymbol{f}\cdot\boldsymbol{\tau}_i + \frac{\|\boldsymbol{\kappa}_l\|_{\mathbb{R}^n}^2(\Delta s)^2}{6} &= \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_i}^{s_3}\int_{s_4}^{s_j}\boldsymbol{\kappa}_4\cdot\boldsymbol{\kappa}_5 ds_5 ds_4 ds_3 + \frac{\|\boldsymbol{\kappa}_l\|_{\mathbb{R}^n}^2(\Delta s)^2}{6} \\
&= \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_i}^{s_3}\int_{s_4}^{s_j}\left\{\boldsymbol{\kappa}_4\cdot\boldsymbol{\kappa}_5 - \|\boldsymbol{\kappa}(s_l)\|_{\mathbb{R}^n}^2\right\} ds_5 ds_4 ds_3 \\
&= \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha},
\end{aligned}$$

and if $i \neq j$, there holds

$$\begin{aligned}
T_i^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_j) - \frac{\|\boldsymbol{\kappa}_l\|_{\mathbb{R}^n}^2(\Delta s)^2}{3} &= \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_i}^{s_3}\int_{s_4}^{s_j}\boldsymbol{\kappa}_4\cdot\boldsymbol{\kappa}_5 ds_5 ds_4 ds_3 - \frac{\|\boldsymbol{\kappa}(s_l)\|_{\mathbb{R}^n}^2(\Delta s)^2}{3} \\
&= \frac{1}{\Delta s}\int_{s_2}^{s_1}\int_{s_i}^{s_3}\int_{s_4}^{s_j}\left\{\boldsymbol{\kappa}_4\cdot\boldsymbol{\kappa}_5 - \|\boldsymbol{\kappa}_l\|_{\mathbb{R}^n}^2\right\} ds_5 ds_4 ds_3 \\
&= \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned} T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}_i &= \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^k T_i^0 \boldsymbol{f} \cdot \boldsymbol{\tau}_i \\ &= (1 + \mathcal{O}(\Delta s)^2) \left(-\frac{\|\boldsymbol{\kappa}_l\|_{\mathbb{R}^n}^2 (\Delta s)^2}{6} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha} \right) \\ &= -\frac{\|\boldsymbol{\kappa}_l\|_{\mathbb{R}^n}^2 (\Delta s)^2}{6} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha}, \end{aligned}$$

and for $i \neq j$,

$$T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}_j = \frac{\|\boldsymbol{\kappa}_l\|_{\mathbb{R}^n}^2 (\Delta s)^2}{3} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha}.$$

□

We denote

$$\hat{Q}_2 \boldsymbol{\phi} = 2(\Delta s) \mathscr{M}(\boldsymbol{f}) \left\{ (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \frac{\Delta \boldsymbol{\phi}}{\Delta s} + \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \boldsymbol{\phi}_1' \right\} - 4 \Delta s \mathscr{M}^4(\boldsymbol{f}) \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \frac{\Delta \boldsymbol{\phi}}{\Delta s}.$$

Lemma 2.5 *Let $\alpha \in (0, \frac{1}{2})$ and let $\boldsymbol{f} \in H^{3-\alpha}$ be bi-Lipschitz continuous, and let $\boldsymbol{\phi} \in H^{3-\alpha}$. For each $i, j \in \{1, 2\}$, it holds that*

$$\begin{gathered} \frac{\tilde{Q}_{2i} \boldsymbol{\phi}}{\Delta s} = (-1)^{i-1} \boldsymbol{\phi}_j'' + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}, \\ \sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^{1-\alpha}} \left\| \frac{\tilde{Q}_{2i} \boldsymbol{\phi}}{\Delta s} - (-1)^{i-1} \boldsymbol{\phi}_j'' \right\|_{\mathbb{R}^n} \leq C(b) \|\boldsymbol{\phi}\|_{H^{3-\alpha}}, \\ \frac{\bar{Q}_{2i} \boldsymbol{\phi}}{(\Delta s)^2} = \frac{\|\boldsymbol{\kappa}\|_{\mathbb{R}^n}^2 \boldsymbol{\phi}_j'}{3} + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}, \\ \sup_{s_1 \neq s_2} \frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}} \left\| \frac{\bar{Q}_{2i} \boldsymbol{\phi}}{(\Delta s)^2} - \frac{\|\boldsymbol{\kappa}\|_{\mathbb{R}^n}^2 \boldsymbol{\phi}_j'}{3} \right\|_{\mathbb{R}^n} \leq C(b) \|\boldsymbol{\phi}\|_{H^{3-\alpha}}. \end{gathered}$$

Furthermore we have

$$\begin{aligned} \frac{\partial}{\partial s_1} \tilde{Q}_2 \boldsymbol{\phi} = &\left(\frac{\tilde{Q}_{21} \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \frac{\Delta \boldsymbol{\phi}}{\Delta s} + \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \frac{\tilde{Q}_{21} \boldsymbol{\phi}}{\Delta s} \\ &+ \frac{2(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left(\frac{\tilde{Q}_{21} \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_1 \right) \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \frac{\Delta \boldsymbol{\phi}}{\Delta s} + \hat{Q}_2 \boldsymbol{\phi}. \end{aligned}$$

Proof The assertions follow from Lemma 2.4 and

$$
\begin{aligned}
\frac{\partial}{\partial s_1}\tilde{Q}_2\boldsymbol{\phi} =& -2\cdot(-1)\left\{\frac{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\Delta\boldsymbol{\phi}+(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_2)\boldsymbol{\phi}_1'}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}-\frac{2(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_2)\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\right\} \\
=&\frac{2}{\Delta s}\left\{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\boldsymbol{\phi}_1'\right\} \\
&+\frac{2}{\Delta s}\left\{\frac{(\Delta s)^2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}-1\right\}\left\{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\boldsymbol{\phi}_1'\right\} \\
&-\frac{4}{\Delta s}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{\phi}}{\Delta s}-\frac{4}{\Delta s}\left\{\frac{(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)-1\right\}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{\phi}}{\Delta s} \\
=&\frac{2}{\Delta s}\left\{\left(\boldsymbol{\tau}_1-\frac{\Delta\boldsymbol{f}}{\Delta s}\right)\cdot\boldsymbol{\tau}_2\right\}\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\frac{2}{\Delta s}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\left(\boldsymbol{\phi}_1'-\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right) \\
&+\frac{2}{\Delta s}\left\{\frac{(\Delta s)^2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}-1\right\}\left\{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\boldsymbol{\phi}_1'\right\} \\
&-\frac{4(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1)}{\Delta s}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{\phi}}{\Delta s} \\
=&\frac{-2(T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}_2)}{\Delta s}\frac{\Delta\boldsymbol{\phi}}{\Delta s}-\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{2T_1^0\boldsymbol{\phi}}{\Delta s} \\
&+\frac{2}{\Delta s}\left\{\frac{(\Delta s)^2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}-1\right\}\left\{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\boldsymbol{\phi}_1'\right\} \\
&-\frac{4(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1)}{\Delta s}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{\phi}}{\Delta s} \\
=&\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\tilde{Q}_{21}\boldsymbol{\phi}}{\Delta s} \\
&+\frac{2(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\hat{Q}_2\boldsymbol{\phi}.
\end{aligned}
$$

□

Lemma 2.6 *Let $\alpha\in(0,\frac{1}{2})$ and let $\boldsymbol{f}\in H^{3-\alpha}$ be bi-Lipschitz continuous and, let $\boldsymbol{\phi}\in H^{3-\alpha}$. Then*

$$
\mathscr{M}_2(\boldsymbol{\phi}) = -\frac{\|\boldsymbol{\phi}_i''\|_{\mathbb{R}^n}^2}{2}+\mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha},
$$

$$
\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left|\mathscr{M}_2(\boldsymbol{\phi})+\frac{\|\boldsymbol{\phi}_i''\|_{\mathbb{R}^n}^2}{2}\right|\leq C(b,\|\boldsymbol{\phi}\|_{H^{3-\alpha}}).
$$

Proof The assertion is derived from Lemma 2.5 as

$$
\begin{aligned}
\mathscr{M}_2(\boldsymbol{\phi}) =& -\frac{\tilde{Q}_1\boldsymbol{\phi}\cdot\tilde{Q}_2\boldsymbol{\phi}}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} \\
=& -\frac{1}{2}\frac{\tilde{Q}_1\boldsymbol{\phi}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}+\frac{1}{2}\left\{1-\frac{(\Delta s)^2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right\}\left(\frac{\tilde{Q}_1\boldsymbol{\phi}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}\right) \\
=& -\frac{1}{2}(\boldsymbol{\phi}_i''+\mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha})\cdot(\boldsymbol{\phi}_i''+\mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha})+\frac{1}{2}\left\{1-\frac{(\Delta s)^2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right\}\left(\frac{\tilde{Q}_1\boldsymbol{\phi}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}\right) \\
=& -\frac{\|\boldsymbol{\phi}_i''\|_{\mathbb{R}^n}^2}{2}+\mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}+\mathcal{O}(\Delta s)^2 \\
=& -\frac{\|\boldsymbol{\phi}_i''\|_{\mathbb{R}^n}^2}{2}+\mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}.
\end{aligned}
$$

□

Lemma 2.7 *Let $\alpha \in (0, \frac{1}{2})$ and let $\boldsymbol{f} \in H^{3-\alpha}$ be bi-Lipschitz continuous. For all $k \in \mathbb{N}$ and $i \in \{1, 2\}$, we have*

$$\mathscr{M}^{2k}(\boldsymbol{f}) = \frac{k}{12}\|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n} + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}.$$

Proof The assertion follows from

$$\begin{aligned}
\frac{|\Delta s|^{2k}}{\|\Delta \boldsymbol{f}\|^{2k}_{\mathbb{R}^n}} - 1 =& \left(\frac{|\Delta s|^2}{\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}} - 1 \right) \sum_{l=0}^{k-1} \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^{2\cdot l} \\
=& \left(\frac{|\Delta s|^2}{\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}} - 1 \right) \sum_{l=0}^{k-1} \left(1 + \mathcal{O}(\Delta s)^2 \right)^l \\
=& \left\{ (\Delta s)^2 \frac{1}{12} \|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha} \right\} (k + \mathcal{O}(\Delta s)^2) \\
=& (\Delta s)^2 \frac{k}{12} \|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n} + \mathcal{O}(\Delta s)^{\frac{5}{2}-\alpha}.
\end{aligned}$$

□

Lemma 2.8 *Let $\alpha \in (0, \frac{1}{2})$ and let $\boldsymbol{f} \in H^{3-\alpha}$ be bi-Lipschitz continuous. Then it holds that*

$$\frac{\partial}{\partial s_1} \mathscr{M}(\boldsymbol{f}) = \frac{-2(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}_1)}{(\Delta s)^3} - 2\frac{\mathscr{M}^4(\boldsymbol{f})}{\Delta s}.$$

Proof Since s is arc-length parameter, we have

$$\begin{aligned}
\frac{\partial}{\partial s_1} \mathscr{M}(\boldsymbol{f}) =& \frac{\partial}{\partial s_1} \left\{ \frac{1}{\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}} - \frac{1}{(\Delta s)^2} \right\} \\
=& \frac{-2\Delta \boldsymbol{f} \cdot \boldsymbol{\tau}_1}{\|\Delta \boldsymbol{f}\|^4_{\mathbb{R}^n}} + \frac{2}{(\Delta s)^3} \\
=& \frac{2(\Delta s)^4}{\|\Delta \boldsymbol{f}\|^4_{\mathbb{R}^n}} \left\{ -\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_1 + 1 \right\} \frac{1}{(\Delta s)^3} - 2 \left\{ \frac{(\Delta s)^4}{\|\Delta \boldsymbol{f}\|^4_{\mathbb{R}^n}} - 1 \right\} \frac{1}{(\Delta s)^3} \\
=& \frac{2(\Delta s)^4}{\|\Delta \boldsymbol{f}\|^4_{\mathbb{R}^n}} \left\{ \left(-\frac{\Delta \boldsymbol{f}}{\Delta s} + \boldsymbol{\tau}_1 \right) \cdot \boldsymbol{\tau}_1 \right\} \frac{1}{(\Delta s)^3} - 2\frac{\mathscr{M}^4(\boldsymbol{f})}{\Delta s} \\
=& -2\frac{T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}_1}{(\Delta s)^3} - 2\frac{\mathscr{M}^4(\boldsymbol{f})}{\Delta s}.
\end{aligned}$$

□

Lemma 2.9 *Let $\boldsymbol{\phi} \in H^3$ and $\boldsymbol{\psi} \in C^2$. Then it holds that*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\Delta \boldsymbol{\phi}''}{(\Delta s)^2} \cdot \boldsymbol{\psi}_1 ds_1 ds_2 = -\frac{1}{2} \langle L_1 \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{L^2}.$$

Proof According to [11, Proposition 3.1],

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\Delta \boldsymbol{\phi}'}{(\Delta s)^2} \cdot \Delta \boldsymbol{\psi}' ds_1 ds_2 = \langle L_1 \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{L^2}$$

holds. Now we have

$$
\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon} \frac{\Delta\boldsymbol{\phi}'}{(\Delta s)^2}\cdot\Delta\boldsymbol{\psi}' ds_1 ds_2 \\
=& \iint_{|\Delta s|\geq\varepsilon} \frac{\Delta\boldsymbol{\phi}'}{(\Delta s)^2}\cdot\boldsymbol{\psi}_1' ds_1 ds_2 - \iint_{|\Delta s|\geq\varepsilon} \frac{\Delta\boldsymbol{\phi}'}{(\Delta s)^2}\cdot\boldsymbol{\psi}_2' ds_1 ds_2 \\
=& \iint_{|\Delta s|\geq\varepsilon} \frac{\Delta\boldsymbol{\phi}'}{(\Delta s)^2}\cdot\boldsymbol{\psi}_1' ds_1 ds_2 - \iint_{|\Delta s|\geq\varepsilon} \frac{-\Delta\boldsymbol{\phi}'}{(\Delta s)^2}\cdot\boldsymbol{\psi}_1' ds_1 ds_2 \\
=& 2\iint_{|\Delta s|\geq\varepsilon} \frac{\Delta\boldsymbol{\phi}'}{(\Delta s)^2}\cdot\frac{\partial}{\partial s_1}(\Delta\boldsymbol{\psi}) ds_1 ds_2 \\
=& \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ 2\left(\frac{\boldsymbol{\phi}'(s-\varepsilon)-\boldsymbol{\phi}'(s)}{\varepsilon}\right)\cdot\left(\frac{\boldsymbol{\psi}(s-\varepsilon)-\boldsymbol{\psi}(s)}{\varepsilon}\right)\right. \\
&\left. -2\left(\frac{\boldsymbol{\phi}'(s+\varepsilon)-\boldsymbol{\phi}'(s)}{-\varepsilon}\right)\cdot\left(\frac{\boldsymbol{\psi}(s+\varepsilon)-\boldsymbol{\psi}(s)}{-\varepsilon}\right)\right\} ds \\
&-2\iint_{|\Delta s|\geq\varepsilon}\left(\frac{\partial}{\partial s_1}\frac{\Delta\boldsymbol{\phi}'}{(\Delta s)^2}\right)\Delta\boldsymbol{\psi} ds_1 ds_2 \\
=& \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(-2\boldsymbol{\phi}''(s)\cdot\boldsymbol{\psi}'(s)+\mathcal{O}(\varepsilon)+2\boldsymbol{\phi}''(s)\cdot\boldsymbol{\psi}'(s)+\mathcal{O}(\varepsilon)\right) ds \\
&-2\iint_{|\Delta s|\geq\varepsilon}\left(\frac{\partial}{\partial s_1}\frac{\Delta\boldsymbol{\phi}'}{(\Delta s)^2}\right)\cdot\Delta\boldsymbol{\psi} ds_1 ds_2 \\
=& -2\iint_{|\Delta s|\geq\varepsilon}\left\{\frac{\partial}{\partial s_1}\frac{\Delta\boldsymbol{\phi}'}{(\Delta s)^2}\right\}\cdot\Delta\boldsymbol{\psi} ds_1 ds_2 + \mathcal{O}(\varepsilon) \\
=& -2\iint_{|\Delta s|\geq\varepsilon}\left\{\frac{\boldsymbol{\phi}_1''}{(\Delta s)^2} - 2\frac{\Delta\boldsymbol{\phi}'}{(\Delta s)^3}\right\}\cdot\Delta\boldsymbol{\psi} ds_1 ds_2 + \mathcal{O}(\varepsilon) \\
=& -2\iint_{|\Delta s|\geq\varepsilon}\left\{\frac{\Delta\boldsymbol{\phi}''}{(\Delta s)^2} + 0\right\}\cdot\boldsymbol{\psi}_1 ds_1 ds_2 + \mathcal{O}(\varepsilon).
\end{aligned}
$$

Therefore we obtain

$$
\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}\frac{\Delta\boldsymbol{\phi}''}{(\Delta s)^2}\cdot\boldsymbol{\psi}_1 ds_1 ds_2 = -\frac{1}{2}\langle L_1\boldsymbol{\phi},\boldsymbol{\psi}\rangle_{L^2}.
$$

□

Lemma 2.10 *For any $\boldsymbol{\phi},\boldsymbol{\psi}\in H^3$, it holds that*

$$
\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\frac{\tilde{Q}_{21}\boldsymbol{\phi}+\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\boldsymbol{\psi}_1 ds_1 ds_2 = \frac{1}{4}\langle L_2\boldsymbol{\phi},\boldsymbol{\psi}\rangle_{L^2}.
$$

Furthermore, the following assertions hold:

$$
\begin{aligned}
&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\left(\boldsymbol{p}_1^{a,b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})(s_1,s_2)-\boldsymbol{p}_2^{a,b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})(s_2,s_1)\right)\cdot\boldsymbol{\psi}_1 ds_1 ds_2 \\
=&\frac{a+b}{4}\langle(\boldsymbol{v}\cdot L_2\boldsymbol{u})\boldsymbol{w},\boldsymbol{\psi}\rangle_{L^2},
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\left(\boldsymbol{q}_1^{a,b}(\boldsymbol{u},v)(s_1,s_2)-\boldsymbol{q}_2^{a,b}(\boldsymbol{u},v)(s_2,s_1)\right)\cdot\boldsymbol{\psi}_1 ds_1 ds_2 \\
=&\frac{a+b}{4}\langle L_2\boldsymbol{u},v\boldsymbol{\psi}\rangle_{L^2}
\end{aligned}
$$

Proof Using result of [11, section 2], we have

$$\begin{aligned}\langle L_2\boldsymbol{\phi},\boldsymbol{\psi}\rangle_{L^2} &= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\tilde{Q}_{21}\boldsymbol{\phi}\cdot\tilde{Q}_{22}\boldsymbol{\psi}+\tilde{Q}_{22}\boldsymbol{\phi}\cdot\tilde{Q}_{21}\boldsymbol{\psi}}{2(\Delta s)^2}ds_1ds_2\\ &= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\tilde{Q}_{21}\boldsymbol{\phi}\cdot\tilde{Q}_{22}\boldsymbol{\psi}}{2(\Delta s)^2}ds_1ds_2 + \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\tilde{Q}_{22}\boldsymbol{\phi}\cdot\tilde{Q}_{21}\boldsymbol{\psi}}{2(\Delta s)^2}ds_1ds_2\\ &= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\tilde{Q}_{21}\boldsymbol{\phi}\cdot\tilde{Q}_{22}\boldsymbol{\psi}}{(\Delta s)^2}ds_1ds_2.\end{aligned}$$

It follows from the definition of $\tilde{Q}_{22}$ that

$$\iint_{|\Delta s|\geq\varepsilon} \frac{\tilde{Q}_{21}\boldsymbol{\phi}\cdot\tilde{Q}_{22}\boldsymbol{\psi}}{(\Delta s)^2}ds_1ds_2 = 2\iint_{|\Delta s|\geq\varepsilon} \frac{\tilde{Q}_{21}\boldsymbol{\phi}\cdot\boldsymbol{\psi}_2'}{(\Delta s)^2}ds_1ds_2 - 2\iint_{|\Delta s|\geq\varepsilon} \frac{\tilde{Q}_{21}\boldsymbol{\phi}\cdot\Delta\boldsymbol{\psi}}{(\Delta s)^3}ds_1ds_2.$$

We perform the integration by parts on the first intrgration of the right hand side. To do this, we calculate the derivative as follows:

$$\begin{aligned}\frac{\partial}{\partial s_2}\frac{\tilde{Q}_{21}\boldsymbol{\phi}}{(\Delta s)^2} &= \frac{\frac{\partial}{\partial s_2}\tilde{Q}_{21}\boldsymbol{\phi}}{(\Delta s)^2} + \frac{2\tilde{Q}_{21}\boldsymbol{\phi}}{(\Delta s)^3}\\ &= \frac{2\frac{\partial}{\partial s_2}\left(\boldsymbol{\phi}'(s_1) - \frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)}{(\Delta s)^2} + \frac{\tilde{Q}_{21}\boldsymbol{\phi}}{(\Delta s)^3}\\ &= \frac{2\left(\frac{\boldsymbol{\phi}'(s_2)}{\Delta s} - \frac{\Delta\boldsymbol{\phi}}{(\Delta s)^2}\right)}{(\Delta s)^2} + \frac{2\tilde{Q}_{21}\boldsymbol{\phi}}{(\Delta s)^3}\\ &= \frac{2\tilde{Q}_{21}\boldsymbol{\phi}+\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}.\end{aligned}$$

Hence we have

$$
\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon}\frac{\tilde{Q}_{21}\boldsymbol{\phi}\cdot\boldsymbol{\psi}_2'}{(\Delta s)^2}ds_1ds_2\\
=&\iint_{|\Delta s|\geq\varepsilon}\frac{\tilde{Q}_{21}\boldsymbol{\phi}}{(\Delta s)^2}\cdot\frac{\partial}{\partial s_2}(-\Delta\boldsymbol{\psi})ds_1ds_2\\
=&-\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left(\left[\frac{\tilde{Q}_{21}\boldsymbol{\phi}}{(\Delta s)^2}\cdot\Delta\boldsymbol{\psi}\right]_{s_2=s_1+\varepsilon}^{s_2=s_1+\mathcal{L}/2}+\left[\frac{\tilde{Q}_{21}\boldsymbol{\phi}}{(\Delta s)^2}\cdot\Delta\boldsymbol{\psi}\right]_{s_2=s_1-\mathcal{L}/2}^{s_2=s_1-\varepsilon}\right)ds_1\\
&+\iint_{|\Delta s|\geq\varepsilon}\frac{\partial}{\partial s_2}\left(\frac{\tilde{Q}_{21}\boldsymbol{\phi}}{(\Delta s)^2}\right)\cdot\Delta\boldsymbol{\psi}ds_1ds_2\\
=&-\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{\frac{\tilde{Q}_{21}\boldsymbol{\phi}(s_1,s_1+\mathcal{L}/2)}{(\mathcal{L}/2)^2}\cdot(\boldsymbol{\psi}(s_1)-\boldsymbol{\psi}(s_1+\mathcal{L}/2))-\frac{\tilde{Q}_{21}\boldsymbol{\phi}(s_1,s_1+\varepsilon)}{\varepsilon^2}\cdot(\boldsymbol{\psi}(s_1)-\boldsymbol{\psi}(s_1+\varepsilon))\right.\\
&\left.+\frac{\tilde{Q}_{21}\boldsymbol{\phi}(s_1,s_1-\varepsilon)}{\varepsilon^2}\cdot(\boldsymbol{\psi}(s_1)-\boldsymbol{\psi}(s_1-\varepsilon))-\frac{\tilde{Q}_{21}\boldsymbol{\phi}(s_1,s_1-\mathcal{L}/2)}{(\mathcal{L}/2)^2}\cdot(\boldsymbol{\psi}(s_1)-\boldsymbol{\psi}(s_1-\mathcal{L}/2))\right\}ds_1\\
&+\iint_{|\Delta s|\geq\varepsilon}\frac{2\tilde{Q}_{21}\boldsymbol{\phi}+\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\Delta\boldsymbol{\psi}ds_1ds_2\\
=&-\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{-(\boldsymbol{\phi}_1''+\mathcal{O}(\varepsilon)^{\frac12-\alpha})\cdot(\boldsymbol{\psi}_1'+\mathcal{O}(\varepsilon))+(\boldsymbol{\phi}_1''+\mathcal{O}(\varepsilon)^{\frac12-\alpha})\cdot(\boldsymbol{\psi}_1'+\mathcal{O}(\varepsilon))\right\}ds_1\\
&+\iint_{|\Delta s|\geq\varepsilon}\frac{2\tilde{Q}_{21}\boldsymbol{\phi}+\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\Delta\boldsymbol{\psi}ds_1ds_2\\
=&\mathcal{O}(\varepsilon)^{\frac12-\alpha}+\iint_{|\Delta s|\geq\varepsilon}\frac{2\tilde{Q}_{21}\boldsymbol{\phi}+\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\Delta\boldsymbol{\psi}ds_1ds_2.
\end{aligned}
$$

We have used Lemmas 2.5 and 2.1 at the fourth equality. Therefore we have

$$
\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon}\frac{\tilde{Q}_{21}\boldsymbol{\phi}\cdot\tilde{Q}_{22}\boldsymbol{\psi}}{(\Delta s)^2}ds_1ds_2\\
=&2\iint_{|\Delta s|\geq\varepsilon}\frac{2\tilde{Q}_{21}\boldsymbol{\phi}+\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\Delta\boldsymbol{\psi}ds_1ds_2-2\iint_{|\Delta s|\geq\varepsilon}\frac{\tilde{Q}_{21}\boldsymbol{\phi}\cdot\Delta\boldsymbol{\psi}}{(\Delta s)^3}ds_1ds_2\\
&+\mathcal{O}(\varepsilon)^{\frac12-\alpha}\\
=&\iint_{|\Delta s|\geq\varepsilon}\frac{2\tilde{Q}_{21}\boldsymbol{\phi}+2\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\Delta\boldsymbol{\psi}ds_1ds_2+\mathcal{O}(\varepsilon)^{\frac12-\alpha}
\end{aligned}
$$

for any $\alpha\in(0,\frac12)$. Consequently it holds that

$$
\begin{aligned}
\langle L_2\boldsymbol{\phi},\boldsymbol{\psi}\rangle_{L^2}=&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\frac{2\tilde{Q}_{21}\boldsymbol{\phi}+2\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\Delta\boldsymbol{\psi}ds_1ds_2\\
=&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\frac{2\tilde{Q}_{21}\boldsymbol{\phi}+2\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\boldsymbol{\psi}_1ds_1ds_2-\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\frac{2\tilde{Q}_{21}\boldsymbol{\phi}+2\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\boldsymbol{\psi}_2ds_1ds_2\\
=&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\frac{2\tilde{Q}_{21}\boldsymbol{\phi}+2\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\boldsymbol{\psi}_1ds_1ds_2-\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\frac{2\tilde{Q}_{22}\boldsymbol{\phi}+2\tilde{Q}_{21}\boldsymbol{\phi}}{-(\Delta s)^3}\cdot\boldsymbol{\psi}_1ds_1ds_2\\
=&4\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\frac{\tilde{Q}_{21}\boldsymbol{\phi}+\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\boldsymbol{\psi}_1ds_1ds_2.
\end{aligned}
$$

Using this relation, we obtain

$$
\begin{aligned}
&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \Big(\boldsymbol{p}_1^{a,b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})(s_1,s_2) - \boldsymbol{p}_2^{a,b}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})(s_2,s_1)\Big) \cdot \boldsymbol{\psi}_1 ds_1 ds_2 \\
=&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left\{ \frac{a\tilde{Q}_{21}\boldsymbol{u} + b\tilde{Q}_{22}\boldsymbol{u}}{(\Delta s)^3} - \frac{a\tilde{Q}_{22}\boldsymbol{u} + b\tilde{Q}_{21}\boldsymbol{u}}{-(\Delta s)^3} \right\} \cdot \boldsymbol{v}_1(\boldsymbol{w}_1\cdot\boldsymbol{\psi}_1) ds_1 ds_2 \\
=&(a+b)\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\tilde{Q}_{21}\boldsymbol{u} + \tilde{Q}_{22}\boldsymbol{u}}{(\Delta s)^3} \cdot \{(\boldsymbol{w}_1\cdot\boldsymbol{\psi}_1)\boldsymbol{v}_1\}\, ds_1 ds_2 \\
=&\frac{a+b}{4}\langle L_2\boldsymbol{u}, (\boldsymbol{w}\cdot\boldsymbol{\psi})\boldsymbol{v}\rangle_{L^2} \\
=&\frac{a+b}{4}\langle (\boldsymbol{v}\cdot L_2\boldsymbol{u})\boldsymbol{w}, \boldsymbol{\psi}\rangle_{L^2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \Big(\boldsymbol{q}_1^{a,b}(\boldsymbol{u},v)(s_1,s_2) - \boldsymbol{q}_2^{a,b}(\boldsymbol{u},v)(s_2,s_1)\Big) \cdot \boldsymbol{\psi}_1 ds_1 ds_2 \\
=&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} v_1 \left\{ \frac{a\tilde{Q}_{21}\boldsymbol{u} + b\tilde{Q}_{22}\boldsymbol{u}}{(\Delta s)^3} - \frac{a\tilde{Q}_{22}\boldsymbol{u} + b\tilde{Q}_{21}\boldsymbol{u}}{-(\Delta s)^3} \right\} \cdot \boldsymbol{\psi}_1 ds_1 ds_2 \\
=&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} (a+b)v_1 \frac{\tilde{Q}_{21}\boldsymbol{u} + \tilde{Q}_{22}\boldsymbol{u}}{(\Delta s)^3} \cdot \boldsymbol{\psi}_1 ds_1 ds_2 \\
=&\frac{a+b}{4}\langle L_2\boldsymbol{u}, v\boldsymbol{\psi}\rangle_{L^2}.
\end{aligned}
$$

□

Let $\alpha > 0$ and $u \in L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2, \mathbb{R})$. When there exists a limit $\tilde{u}$ such that

$$\tilde{u}(s) = \lim_{s\to r} u(r,s),$$

we define $E(u,\alpha,i)$ and $E(u,\alpha)$ by

$$E(u,\alpha,i) = \sup_{s_1\neq s_2} \frac{|u(s_1,s_2) - \tilde{u}_i|}{|\Delta s|^\alpha} + \|u\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)},$$

$$E(u,\alpha) = \sup\{E(u,\alpha,1), E(u,\alpha,2)\}.$$

Lemma 2.11 *Let $\alpha,\beta \in (0,1)$ and $i \in \{1,2\}$ and that $u,v \in L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2,\mathbb{R})$. Assume that there exists $\lim_{s\to r} u(r,s)$, $\lim_{s\to r} v(r,s)$ and that u,v satisfy*

$$E(u,\alpha,i) < \infty,\ E(v,\beta,i) < \infty.$$

Then there exists a function $N \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ such that the following assertions hold.
(i) For any $\psi \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$,

$$\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon} \frac{u(s_1,s_2)v(s_1,s_2)}{\Delta s}\psi_i ds_1 ds_2 = \langle N,\psi\rangle_{L^2}.$$

(ii) There exists a positive constant $C = C(\alpha,\beta)$ such that

$$\|N\|_{L^2} \leq CE(u,\alpha,i)E(v,\beta,i).$$

Proof Without loss of generality, we may assume that $i = 1$. Put $\tilde{u}(r) = \lim_{s \to r} u(r,s)$ and $\tilde{v}(r) = \lim_{s \to r} v(r,s)$. Let $\psi \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and $\varepsilon > 0$. Noting that $\int_{s_1-\mathcal{L}/2+\varepsilon}^{s_1+\mathcal{L}/2-\varepsilon} \frac{1}{\Delta s} ds_2 = 0$, we have

$$\begin{aligned}
&\iint_{|\Delta s| \geq \varepsilon} \frac{u(s_1,s_2)v(s_1,s_2)}{\Delta s} \psi_1 ds_1 ds_2 \\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\int_{s_1-\mathcal{L}/2+\varepsilon}^{s_1+\mathcal{L}/2-\varepsilon} \frac{u(s_1,s_2)v(s_1,s_2)}{\Delta s} ds_2 - \int_{s_1-\mathcal{L}/2+\varepsilon}^{s_1+\mathcal{L}/2-\varepsilon} \frac{\tilde{u}_1 \tilde{v}_1}{\Delta s} ds_2 \right) \psi_1 ds_1 \\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\int_{s_1-\mathcal{L}/2+\varepsilon}^{s_1+\mathcal{L}/2-\varepsilon} \frac{u(s_1,s_2)v(s_1,s_2) - \tilde{u}_1 \tilde{v}_1}{\Delta s} ds_2 \right) \psi_1 ds_1,
\end{aligned}$$

and we obtain the estimate

$$\begin{aligned}
&\sup_{s_1 \in \mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s_1-\mathcal{L}/2+\varepsilon}^{s_1+\mathcal{L}/2-\varepsilon} \left| \frac{u(s_1,s_2)v(s_1,s_2) - \tilde{u}_1 \tilde{v}_1}{\Delta s} \right| ds_2 \\
&= \sup_{s_1 \in \mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s_1-\mathcal{L}/2+\varepsilon}^{s_1+\mathcal{L}/2-\varepsilon} \left| \left(\frac{u(s_1,s_2) - \tilde{u}_1}{\Delta s} \right) v(s_1,s_2) + \tilde{u}_1 \left(\frac{v(s_1,s_2) - \tilde{v}_1}{\Delta s} \right) \right| ds_2 \\
&\leq \sup_{s_1 \in \mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{s_1-\mathcal{L}/2+\varepsilon}^{s_1+\mathcal{L}/2-\varepsilon} \left(\frac{1}{|\Delta s|^{1-\alpha}} \frac{|u(s_1,s_2) - \tilde{u}_1|}{|\Delta s|^\alpha} |v(s_1,s_2)| + \frac{1}{|\Delta s|^{1-\beta}} |\tilde{u}_1| \frac{|v(s_1,s_2) - \tilde{v}_1|}{|\Delta s|^\beta} \right) ds_2 \\
&\leq \sup_{s_1 \in \mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\int_{s_1-\mathcal{L}/2+\varepsilon}^{s_1+\mathcal{L}/2-\varepsilon} \frac{ds_2}{|\Delta s|^{1-\alpha}} + \int_{s_1-\mathcal{L}/2+\varepsilon}^{s_1+\mathcal{L}/2-\varepsilon} \frac{ds_2}{|\Delta s|^{1-\beta}} \right) E(u,\alpha,1)E(v,\beta,1) \\
&\leq CE(u,\alpha,1)E(v,\beta,1).
\end{aligned}$$

for any $\varepsilon > 0$. Lebesgue's convergence theorem implies that

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left| \frac{u(s_1,s_2)v(s_1,s_2) - \tilde{u}_1 \tilde{v}_1}{\Delta s} \right| ds_2 \right) |\psi_1| ds_1 < \infty.$$

Hence Fubini's theorem implies that there exists a function

$$N(s_1) := \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{u(s_1,s_2)v(s_1,s_2) - \tilde{u}(s_1)\tilde{v}(s_1)}{\Delta s} ds_2.$$

Since ψ is arbitrary, $N \in L^2$ holds. From the above calculation N satisfies (ii) and

$$\lim_{\varepsilon \downarrow 0} \iint_{|\Delta s| \geq \varepsilon} \frac{u(s_1,s_2)v(s_1,s_2)}{\Delta s} \psi_1 ds_1 ds_2 = \langle N, \psi \rangle_{L^2}.$$

□

Corollary 2.1 *Let $\alpha, \beta > 0$ and $i \in \{0,1\}$ and $u, v : (\mathbb{R}/\mathcal{L}\mathbb{Z})^2 \to \mathbb{R}$. Assume that there exist $\lim_{s \to r} u(r,s)$, $\lim_{s \to r} v(r,s)$ and u, v satisfy*

$$E(u,\alpha) < \infty, \; E(v,\beta) < \infty.$$

Then there exists a function $N \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ such that the following assertions hold.
(i) For any $\psi \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$,

$$\lim_{\varepsilon \downarrow 0} \iint_{|\Delta s| \geq \varepsilon} \frac{u(s_1,s_2)v(s_1,s_2)}{\Delta s} \Delta\psi ds_1 ds_2 = \langle N, \psi \rangle_{L^2}.$$

(ii) There exists a positive constant $C = C(\alpha, \beta)$ such that

$$\|N\|_{L^2} \leq CE(u, \alpha)E(v, \beta).$$

Proof Put $\tilde{u}(r) = \lim_{s \to r} u(r, s)$ and $\tilde{v}(r) = \lim_{s \to r} v(r, s)$. Let $\psi \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R})$. Using Lemma 2.11, there exist $N_1, N_2 \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R})$ such that

$$\lim_{\varepsilon \downarrow 0} \iint_{|\Delta s| \geq \varepsilon} \frac{u(s_1, s_2)v(s_1, s_2)}{\Delta s} \psi_1 ds_1 ds_2 = \langle N_1, \psi \rangle_{L^2},$$

$$\lim_{\varepsilon \downarrow 0} \iint_{|\Delta s| \geq \varepsilon} \frac{u(s_1, s_2)v(s_1, s_2)}{\Delta s} \psi_2 ds_1 ds_2 = \langle N_2, \psi \rangle_{L^2},$$

and

$$\|N_i\|_{L^2} \leq CE(u, \alpha)E(v, \beta).$$

Letting $N = N_1 - N_2$, we obtain

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \iint_{|\Delta s| \geq \varepsilon} \frac{u(s_1, s_2)v(s_1, s_2)}{\Delta s} \Delta \psi ds_1 ds_2 &= \langle N_1, \psi \rangle_{L^2} - \langle N_2, \psi \rangle_{L^2} \\ &= \langle N, \psi \rangle_{L^2}, \end{aligned}$$

and

$$\|N\|_{L^2} \leq CE(u, \alpha)E(v, \beta).$$

□

When ϕ is a scalar function, we define $\tilde{Q}_{ij}\phi$ similarly to $\tilde{Q}_{ij}\boldsymbol{\phi}$

Lemma 2.12 *Let $\alpha \in (0, \frac{1}{2})$, $\phi \in H^3$ and $u : \{\Delta s \neq 0\} \to \mathbb{R}$. Assume that there exist a number $\beta > 0$ and functions $v, w^1, w^2 \in L^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R})$ such that*

$$u(s_1, s_2) = v(s_i) + \Delta s w^i(s_i) + \mathcal{O}(\Delta s)^{1+\beta},$$

for any $i \in \{1, 2\}$. If the function $h_j : (\mathbb{R}/\mathcal{L}\mathbb{Z})^2 \to \mathbb{R}$ is given by

$$h_j(s_1, s_2) = \frac{\tilde{Q}_{2j}\phi}{(\Delta s)^3} u(s_1, s_2),$$

for $j \in \{1, 2\}$. Then there exists an L^2 function N_j such that

$$\lim_{\varepsilon \downarrow 0} \iint_{|\Delta s| \geq \varepsilon} (h_j(s_1, s_2)\Delta \psi) ds_1 ds_2 = \frac{1}{4} \langle u_1 L_2 \phi + N_j, \psi \rangle_{L^2},$$

for any $\psi \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R})$ and any $j \in \{1, 2\}$. Moreover, the estimate

$$\|N_j\|_{L^2} \leq C\|\phi\|_{H^{3-\alpha}} E\left(\frac{u - v}{\Delta s}, \beta\right)$$

holds.

Proof Firstly, we show the case $j = 1$. We decompose h_1 as

$$h_1(s_1, s_2) = \frac{\tilde{Q}_{21}\phi}{(\Delta s)^3} v_i + \frac{1}{\Delta s}\left(\frac{\tilde{Q}_{21}\phi}{\Delta s}\frac{u(s_1, s_2) - v_i}{\Delta s}\right),$$

and the formula holds for each $i \in \{1, 2\}$ from the assumption. Now we have

$$\begin{aligned}
\iint_{|\Delta s|\geq\varepsilon} h_1(s_1, s_2)\Delta\psi ds_1 ds_2 &= \iint_{|\Delta s|\geq\varepsilon}\left\{\frac{\tilde{Q}_{21}\phi}{(\Delta s)^3}v_1 + \frac{1}{\Delta s}\frac{\tilde{Q}_{21}\phi}{\Delta s}\frac{u(s_1, s_2) - v_1}{\Delta s}\right\}\psi_1 ds_1 ds_2 \\
&\quad - \iint_{|\Delta s|\geq\varepsilon}\left\{\frac{\tilde{Q}_{21}\phi}{(\Delta s)^3}v_2 + \frac{1}{\Delta s}\frac{\tilde{Q}_{21}\phi}{\Delta s}\frac{u(s_1, s_2) - v_2}{\Delta s}\right\}\psi_2 ds_1 ds_2 \\
&= \iint_{|\Delta s|\geq\varepsilon}\left\{\frac{\tilde{Q}_{21}\phi + \tilde{Q}_{22}\phi}{(\Delta s)^3}v_1 + \frac{1}{\Delta s}\frac{\tilde{Q}_{21}\phi}{\Delta s}\frac{u(s_1, s_2) - v_1}{\Delta s}\right. \\
&\quad \left. + \frac{1}{\Delta s}\frac{\tilde{Q}_{22}\phi}{\Delta s}\frac{u(s_2, s_1) - v_1}{\Delta s}\right\}\psi_1 ds_1 ds_2. \qquad (2.1)
\end{aligned}$$

From the assumption, we have

$$\frac{u(s_1, s_2) - v_1}{\Delta s} = w_1^1 + \mathcal{O}(\Delta s)^\beta, \ \frac{u(s_2, s_1) - v_1}{\Delta s} = w_1^2 + \mathcal{O}(\Delta s)^\beta,$$

$$\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^\beta}\left|\frac{u(s_1, s_2) - v_1}{\Delta s} - w_1^1\right| \leq E\left(\frac{u - v}{\Delta s}, \beta\right) < \infty$$

and

$$\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^\beta}\left|\frac{u(s_2, s_2) - v_1}{\Delta s} - w_1^2\right| \leq E\left(\frac{u - v}{\Delta s}, \beta\right) < \infty.$$

Using Lemma 2.10 for the first term of right-hand side of (2.1) and using Corollary 2.1 for the remaining term, we can conclude that there exists an L^2 function N_1 such that

$$\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}(h_1(s_1, s_2)\Delta\psi)ds_1 ds_2 = \frac{1}{4}\langle vL_2\phi + N_1, \psi\rangle_{L^2},$$

$$\|N_1\|_{L^2} \leq C\|\phi\|_{H^{3-\alpha}}E\left(\frac{u - v}{\Delta s}, \beta\right).$$

Similarly, we can show that there exists an L^2 function N_2 such that

$$\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}(h_2(s_1, s_2)\Delta\psi)ds_1 ds_2 = \frac{1}{4}\langle vL_2\phi + N_2, \psi\rangle_{L^2},$$

$$\|N_2\|_{L^2} \leq C\|\phi\|_{H^{3-\alpha}}E\left(\frac{u - v}{\Delta s}, \beta\right).$$

□

Lemma 2.13 *Let $\alpha > 0$ and $\boldsymbol{\zeta} \in C^1(\{s_1 \neq s_2\})$, $\boldsymbol{\psi} \in C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})$. If there exists a bounded function $\xi : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}$ such that*

$$\boldsymbol{\zeta}(s_1, s_2)\cdot\Delta\boldsymbol{\psi} = \xi_1 + \mathcal{O}(\Delta s)^\alpha,$$

then

$$\iint_{|\Delta s|\geq\varepsilon} \boldsymbol{\zeta}(s_1,s_2)\cdot\boldsymbol{\psi}_1' ds_1 ds_2$$
$$= -\iint_{|\Delta s|\geq\varepsilon} \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2)\cdot\Delta\boldsymbol{\psi} ds_1 ds_2 + \mathcal{O}(\varepsilon^\alpha)$$

as $\varepsilon \to +0$.

Proof Integrating by parts, we have

$$\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon} \boldsymbol{\zeta}(s_1,s_2)\cdot\boldsymbol{\psi}_1' ds_1 ds_2\\
&= \iint_{|\Delta s|\geq\varepsilon} \boldsymbol{\zeta}(s_1,s_2)\cdot\frac{\partial}{\partial s_1}(\Delta\boldsymbol{\psi})\, ds_1 ds_2\\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left([\boldsymbol{\zeta}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}]_{s_2-\mathcal{L}/2}^{s_1=s_2-\varepsilon} + [\boldsymbol{\zeta}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}]_{s_2+\varepsilon}^{s_2+\mathcal{L}/2}\right) ds_2\\
&\quad - \iint_{|\Delta s|\geq\varepsilon} \frac{\partial}{\partial s_1}(\boldsymbol{\zeta}(s_1,s_2))\cdot\Delta\boldsymbol{\psi} ds_1 ds_2\\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \{\boldsymbol{\zeta}(s-\varepsilon,s)\cdot(\boldsymbol{\psi}(s-\varepsilon)-\boldsymbol{\psi}(s)) - \boldsymbol{\zeta}(s-\mathcal{L}/2,s)\cdot(\boldsymbol{\psi}(s-\mathcal{L}/2)-\boldsymbol{\psi}(s))\\
&\quad +\boldsymbol{\zeta}(s+\mathcal{L}/2,s)\cdot(\boldsymbol{\psi}(s+\mathcal{L}/2)-\boldsymbol{\psi}(s)) - \boldsymbol{\zeta}(s+\varepsilon,s)\cdot(\boldsymbol{\psi}(s+\varepsilon)-\boldsymbol{\psi}(s))\}\, ds\\
&\quad - \iint_{|\Delta s|\geq\varepsilon} \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2)\cdot\Delta\boldsymbol{\psi} ds_1 ds_2\\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \{\boldsymbol{\zeta}(s-\varepsilon,s)\cdot(\boldsymbol{\psi}(s-\varepsilon)-\boldsymbol{\psi}(s)) - \boldsymbol{\zeta}(s+\varepsilon,s)\cdot(\boldsymbol{\psi}(s+\varepsilon)-\boldsymbol{\psi}(s))\}\, ds\\
&\quad - \iint_{|\Delta s|\geq\varepsilon} \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2)\cdot\Delta\boldsymbol{\psi} ds_1 ds_2\\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\xi(s)+\mathcal{O}(\varepsilon)^\alpha - \xi(s) + \mathcal{O}(\varepsilon)^\alpha) ds - \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2)\cdot\Delta\boldsymbol{\psi} ds_1 ds_2\\
&= \mathcal{O}(\varepsilon^\alpha) - \iint_{|\Delta s|\geq\varepsilon} \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2)\cdot\Delta\boldsymbol{\psi} ds_1 ds_2
\end{aligned}$$

as $\varepsilon \to 0$.

□

Lemma 2.14 *Let* $\alpha > 0$ *and* $\boldsymbol{\zeta} \in C^1(\{s_1 \neq s_2\})$, $\boldsymbol{\psi} \in C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})$. *If*

$$\boldsymbol{\zeta}(s_1,s_2) = -\boldsymbol{\zeta}(s_2,s_1),$$

and if there exists a bounded function $\xi : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}$ *such that*

$$\boldsymbol{\zeta}(s_1,s_2)\cdot\Delta\boldsymbol{\psi} = \xi_1 + \mathcal{O}(\Delta s)^\alpha,$$

then

$$\iint_{|\Delta s|\geq\varepsilon} \boldsymbol{\zeta}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}' ds_1 ds_2$$
$$= -\iint_{|\Delta s|\geq\varepsilon} 2\left\{\left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2) - \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_2,s_1)\right\}\cdot\boldsymbol{\psi}_1 ds_1 ds_2 + \mathcal{O}(\varepsilon^\alpha)$$

as $\varepsilon \to +0$.

Proof Using Lemma 2.13 we have

$$
\begin{aligned}
&-\iint_{|\Delta s|\geq\varepsilon} \boldsymbol{\zeta}(s_1,s_2)\cdot\boldsymbol{\psi}_2' ds_1 ds_2 \\
&= \iint_{|\Delta s|\geq\varepsilon} \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_2,s_1)\cdot\Delta\boldsymbol{\psi} ds_1 ds_2 + \mathcal{O}(\varepsilon^\alpha)
\end{aligned}
$$

as $\varepsilon \to +0$. Therefore it follows that

$$
\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon} \boldsymbol{\zeta}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}' ds_1 ds_2 \\
&= -\iint_{|\Delta s|\geq\varepsilon} \left\{\left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2) - \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_2,s_1)\right\}\cdot\Delta\boldsymbol{\psi} ds_1 ds_2 + \mathcal{O}(\varepsilon^\alpha).
\end{aligned}
$$

Since

$$
\begin{aligned}
&-\iint_{|\Delta s|\geq\varepsilon} \left\{\left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2) - \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_2,s_1)\right\}\cdot\boldsymbol{\psi}_2 ds_1 ds_2 \\
&= \iint_{|\Delta s|\geq\varepsilon} \left\{\left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2) - \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_2,s_1)\right\}\cdot\boldsymbol{\psi}_1 ds_1 ds_2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon} \left\{\left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2) - \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_2,s_1)\right\}\cdot\Delta\boldsymbol{\psi} ds_1 ds_2 \\
&= \iint_{|\Delta s|\geq\varepsilon} 2\left\{\left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_1,s_2) - \left(\frac{\partial}{\partial s_1}\boldsymbol{\zeta}\right)(s_2,s_1)\right\}\cdot\boldsymbol{\psi}_1 ds_1 ds_2 + \mathcal{O}(\varepsilon^\alpha).
\end{aligned}
$$

□

A stated in §1, Ishizeki-Nagasawa gave us the L^2-representation of the first variation $\delta\mathcal{E}_i$ in [11]. To show Theorem 1.11, we need the L^2-representation not only of $\delta\mathcal{E}_i$ but also of $\delta^2\mathcal{E}_i$. To do this, we decompose $\mathscr{H}_1(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]$ into the following five parts

$$
\mathscr{H}_1(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \sum_{i=1}^{5} H_{1i}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}],
$$

$$
\begin{aligned}
H_{11}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] &= \frac{Q\boldsymbol{\phi}\cdot Q\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}, \\
H_{12}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] &= -\frac{S(\boldsymbol{f},\boldsymbol{\phi})\cdot S(\boldsymbol{f},\boldsymbol{\psi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}, \\
H_{13}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] &= -\frac{2\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}, \\
H_{14}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] &= -\frac{2\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}, \\
H_{15}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] &= -\frac{2\mathscr{M}_1(\boldsymbol{f})\Delta\boldsymbol{\phi}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},
\end{aligned}
$$

and $\mathscr{H}_2(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]$ into the following five parts

$$
\mathscr{H}_2(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \sum_{i=1}^{5} H_{2i}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}],
$$

$$H_{21}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \frac{\tilde{Q}_1\boldsymbol{\phi}\cdot\tilde{Q}_2\boldsymbol{\psi}+\tilde{Q}_2\boldsymbol{\phi}\cdot\tilde{Q}_1\boldsymbol{\psi}}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},$$
$$H_{22}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \frac{\tilde{S}_1(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_2(\boldsymbol{f},\boldsymbol{\psi})+\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_1(\boldsymbol{f},\boldsymbol{\psi})}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},$$
$$H_{23}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = -\frac{2\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},$$
$$H_{24}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = -\frac{2\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},$$
$$H_{25}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = -\frac{2\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{\phi}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}.$$

For Banach spaces X and Y, we write $\mathscr{B}(X,Y)$ for the set of all bounded linear operators from X to Y. Also, we set

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{ij}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = K_{ij}.$$

We would like to identify a mapping $\boldsymbol{H}_{ij} : H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \to \mathscr{B}(H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z}), L^2(\mathbb{R}/\mathcal{L}\mathbb{Z}))$ which satisfies

$$K_{ij} = \langle \boldsymbol{H}_{ij}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})}$$

under appropriate conditions. Such a mapping will be called the L^2-representation of K_{ij}. We will show the existence and estimates of the L^2-representation of K_{1j} in §3 and of K_{2j} in §4 respectively.

3 The L^2-representation of the second variation of $\mathcal{E}_1$

3.1 The L^2-representation of K_{11}

We use the decomposition

$$H_{11}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = H_{111}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] + H_{112}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}],$$
$$H_{111}[\boldsymbol{\phi},\boldsymbol{\psi}] = \frac{Q\boldsymbol{\phi}\cdot Q\boldsymbol{\psi}}{(\Delta s)^2} = \frac{\Delta\boldsymbol{\phi}'\cdot\Delta\boldsymbol{\psi}'}{(\Delta s)^2},$$
$$H_{112}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \mathscr{M}(\boldsymbol{f})Q\boldsymbol{\phi}\cdot Q\boldsymbol{\psi} = \mathscr{M}(\boldsymbol{f})\Delta\boldsymbol{\phi}'\cdot\Delta\boldsymbol{\psi}',$$

and give the L^2-representation of each part.

Proposition 3.1 *If $\boldsymbol{\phi},\boldsymbol{\psi} \in H^3$, then*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{111}[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = \langle L_1\boldsymbol{\phi},\boldsymbol{\psi}\rangle_{L^2}.$$

The proof is the same as [11, Proposition 3.1].

Lemma 3.1 *Let $\alpha \in (0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{112} : H^{3-\alpha} \to \mathscr{B}(H^{3-\alpha}, L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f} \in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi} \in H^3$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{112}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = \langle \boldsymbol{N}_{112}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover $\boldsymbol{N}_{112}(\boldsymbol{f})[\boldsymbol{\phi}]$ *satisfies*

$$||\boldsymbol{N}_{112}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \le C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}.$$

Proof Let us set

$$\boldsymbol{h}_{112}(s_1,s_2) = \left\{\frac{1}{\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}} - \frac{1}{(\Delta s)^2}\right\}\Delta\boldsymbol{\phi}'.$$

Lemma 2.3 implies that

$$\boldsymbol{h}_{112}(s_1,s_2) = \mathscr{M}(\boldsymbol{f})\frac{\Delta\boldsymbol{\phi}'}{\Delta s}\Delta s = \mathcal{O}(\Delta s) \quad (\Delta s \to 0).$$

We can calculate the partial derivative of $\boldsymbol{h}_{112}$ and Lemma 2.4 implies that

$$\begin{aligned}\frac{\partial}{\partial s_1}\left(\boldsymbol{h}_{112}(s_1,s_2)\right) &= \frac{\partial}{\partial s_1}(\mathscr{M}(\boldsymbol{f}))\Delta\boldsymbol{\phi}' + \mathscr{M}(\boldsymbol{f})\frac{\partial}{\partial s_1}(\boldsymbol{\phi}_1'),\\ &= -\frac{2}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1)\Delta\boldsymbol{\phi}' + \mathscr{M}(\boldsymbol{f})\boldsymbol{\phi}_1''\\ &= \mathcal{O}(1).\end{aligned}$$

Since it holds that

$$\begin{aligned}\boldsymbol{h}_{112}(s_2,s_1) &= \left\{\frac{1}{\|-\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}} - \frac{1}{(-\Delta s)^2}\right\}(-\Delta\boldsymbol{\phi}')\\ &= -\boldsymbol{h}_{112}(s_1,s_2),\end{aligned}$$

using Lemmas 2.14, we obtain

$$\begin{aligned}&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{112}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2\\ =&\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\ge\varepsilon}\boldsymbol{h}_{112}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}'ds_1ds_2\\ =&-\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} 2\left\{\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{112}\right)(s_1,s_2) - \left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{112}\right)(s_2,s_1)\right\}\cdot\boldsymbol{\psi}_1 ds_1ds_2.\end{aligned}$$

Letting

$$u(s_1,s_2) = \frac{T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1}{\Delta s},\ \boldsymbol{v}(s_1,s_2) = \frac{\Delta\boldsymbol{\phi}'}{\Delta s},$$

we have

$$\lim_{s_2\to s_1} u(s_1,s_2) = 0,\ \sup_{s_2\ne s_1}\left|\frac{u(s_1,s_2)-0}{\Delta s}\right| \le C\|\boldsymbol{f}\|^2_{C^2},$$

$$\lim_{s_2\to s_1}\boldsymbol{v}(s_1,s_2) = \boldsymbol{\phi}_1'',\ \sup_{s_2\ne s_1}\left|\frac{\boldsymbol{v}(s_1,s_2)-\boldsymbol{\phi}''(s_1)}{|\Delta s|^{\frac{1}{2}-\alpha}}\right| \le C\|\boldsymbol{\phi}\|_{H^{3-\alpha}}$$

from Lemma 2.4 and 2.1. Using Lemma 2.11 for the first termof $\frac{\partial}{\partial s_1}\boldsymbol{h}_{112}$, we can conclude that there exists a mapping $\boldsymbol{N}_{112} : H^{3-\alpha} \to \mathscr{B}(H^{3-\alpha}, L^2)$ such that

$$\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\ge\varepsilon} H_{112}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = \langle \boldsymbol{N}_{112}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2},$$

and

$$||\boldsymbol{N}_{112}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \le C(||\boldsymbol{f}||_{H^{3-\alpha}})||\boldsymbol{\phi}||_{H^{3-\alpha}}.$$

Using Young's inequality, we have

$$\left| \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{112}(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] ds_1 ds_2 \right| \leq ||\boldsymbol{N}_{112}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \|\boldsymbol{\psi}\|_{L^2}$$

for any $\boldsymbol{\psi} \in H^3$. □

3.2 The L^2-representation of K_{12}

$H_{12}(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]$ can be written as

$$\begin{aligned} H_{12}(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] &= \frac{-S(\boldsymbol{f}, \boldsymbol{\phi}) S(\boldsymbol{f}, \boldsymbol{\psi})}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \\ &= -\frac{S(\boldsymbol{f}, \boldsymbol{\phi}) \hat{R} \boldsymbol{f}}{|\Delta s|^2} \cdot \Delta \boldsymbol{\psi}' - \mathscr{M}(\boldsymbol{f}) S(\boldsymbol{f}, \boldsymbol{\phi}) \hat{R} \boldsymbol{f} \cdot \Delta \boldsymbol{\psi}' - \frac{S(\boldsymbol{f}, \boldsymbol{\phi}) \Delta \boldsymbol{\tau}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \cdot \hat{R} \boldsymbol{\psi}'. \end{aligned}$$

Now we use the functions

$$\begin{aligned} \boldsymbol{h}_{121}(s_1, s_2) &= -\frac{S(\boldsymbol{f}, \boldsymbol{\phi}) \hat{R} \boldsymbol{f}}{|\Delta s|^2}, \\ \boldsymbol{h}_{122}(s_1, s_2) &= -\mathscr{M}(\boldsymbol{f}) S(\boldsymbol{f}, \boldsymbol{\phi}) \hat{R} \boldsymbol{f}, \\ \boldsymbol{h}_{123}(s_1, s_2) &= -\frac{S(\boldsymbol{f}, \boldsymbol{\phi}) \Delta \boldsymbol{\tau}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}. \end{aligned}$$

For $\boldsymbol{\psi} \in C^1$ we define H_{12k} by

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{12k}(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] ds_1 ds_2 = \lim_{\varepsilon \downarrow 0} \iint_{|\Delta s| \geq \varepsilon} h_{12k}(s_1, s_2) \cdot \Delta \boldsymbol{\psi}' ds_1 ds_2$$

for $k = 1, 2$ and

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{123}(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] ds_1 ds_2 = \lim_{\varepsilon \downarrow 0} \iint_{|\Delta s| \geq \varepsilon} h_{123}(s_1, s_2) \cdot \hat{R} \boldsymbol{\psi}' ds_1 ds_2.$$

Lemma 3.2 *Let $\alpha \in (0, \frac{1}{2})$. There exists a mapping $\boldsymbol{N}_{121} : H^{3-\alpha} \to \mathscr{B}(H^{3-\alpha}, L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f} \in H^3$ and functions $\boldsymbol{\phi}, \boldsymbol{\psi} \in H^3$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{121}(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] ds_1 ds_2 = -\langle (L_1 \boldsymbol{f} \cdot \boldsymbol{\phi}' + \boldsymbol{\tau} \cdot L_1 \boldsymbol{\phi}) \boldsymbol{\tau}, \boldsymbol{\psi} \rangle_{L^2} + \langle \boldsymbol{N}_{121}(\boldsymbol{f})[\boldsymbol{\phi}], \boldsymbol{\psi} \rangle_{L^2}.$$

Moreover $\boldsymbol{N}_{121}(\boldsymbol{f})[\boldsymbol{\phi}]$ satisfies

$$||\boldsymbol{N}_{121}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right) ||\boldsymbol{\phi}||_{H^{3-\alpha}}.$$

Proof We have

$$\begin{aligned} \boldsymbol{h}_{121}(s_1, s_2) \cdot \Delta \boldsymbol{\psi} &= \frac{-(\hat{R} \boldsymbol{f} \cdot Q \boldsymbol{\phi} + Q \boldsymbol{f} \cdot \hat{R} \boldsymbol{\phi}) \left(\hat{R} \boldsymbol{f} \cdot \Delta \boldsymbol{\psi} \right)}{(\Delta s)^2} \\ &= -(\boldsymbol{\tau}(s_2) \cdot \boldsymbol{\phi}''(s_2) + \boldsymbol{\kappa}(s_2) \cdot \boldsymbol{\phi}'(s_2))(\boldsymbol{\tau}(s_2) \cdot \boldsymbol{\psi}'(s_2)) + \mathcal{O}(\Delta s)^{\frac{1}{2}} \end{aligned}$$

as $s_1 \to s_2$, and

$$
\begin{aligned}
&\frac{\partial}{\partial s_1}\boldsymbol{h}_{121}(s_1,s_2)\\
&=-\frac{\frac{\partial}{\partial s_1}S(\boldsymbol{f},\boldsymbol{\phi})\hat{R}\boldsymbol{f}}{(\Delta s)^2}-\frac{S(\boldsymbol{f},\boldsymbol{\phi})\frac{1}{2}\boldsymbol{\kappa}_1}{(\Delta s)^2}+\frac{2S(\boldsymbol{f},\boldsymbol{\phi})\hat{R}\boldsymbol{f}}{(\Delta s)^3}\\
&=-\frac{\left(\frac{1}{2}\boldsymbol{\kappa}_1\cdot Q\boldsymbol{\phi}+\hat{R}\boldsymbol{f}\cdot\boldsymbol{\phi}_1''+\boldsymbol{\kappa}_1\cdot\hat{R}\boldsymbol{\phi}+\frac{1}{2}Q\boldsymbol{f}\cdot\boldsymbol{\phi}_1''\right)\hat{R}\boldsymbol{f}}{(\Delta s)^2}-\frac{S(\boldsymbol{f},\boldsymbol{\phi})\frac{1}{2}\boldsymbol{\kappa}_1}{(\Delta s)^2}+\frac{2S(\boldsymbol{f},\boldsymbol{\phi})\hat{R}\boldsymbol{f}}{(\Delta s)^3},
\end{aligned}
$$

and

$$
\boldsymbol{h}_{121}(s_2,s_1)=-\frac{\{\hat{R}\boldsymbol{f}\cdot(-Q\boldsymbol{\phi})+(-Q\boldsymbol{f})\cdot\hat{R}\boldsymbol{\phi}\}\hat{R}\boldsymbol{f}}{|\Delta s|^2}=-\boldsymbol{h}_{121}(s_1,s_2).
$$

Using Lemma 2.14, we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{121}(s_1,s_2)\Delta\boldsymbol{\psi}_1' ds_1ds_2\\
&=-\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}2\left\{\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{121}\right)(s_1,s_2)-\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{121}\right)(s_2,s_1)\right\}\cdot\boldsymbol{\psi}_1 ds_1ds_2\\
&=\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}\left\{\frac{2S(\boldsymbol{\tau},\boldsymbol{\phi})\hat{R}\boldsymbol{f}}{(\Delta s)^2}+\frac{2S(\boldsymbol{f},\boldsymbol{\phi}')\hat{R}\boldsymbol{f}}{(\Delta s)^2}+\frac{2S(\boldsymbol{f},\boldsymbol{\phi})\hat{R}\boldsymbol{\tau}}{(\Delta s)^2}\right\}\cdot\boldsymbol{\psi}_1 ds_1ds_2\\
&=\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{1211}(s_1,s_2)\cdot\boldsymbol{\psi}_1 ds_1ds_2,
\end{aligned}
$$

where

$$
\boldsymbol{h}_{1211}(s_1,s_2)=\frac{2S(\boldsymbol{\tau},\boldsymbol{\phi})\hat{R}\boldsymbol{f}}{(\Delta s)^2}+\frac{2S(\boldsymbol{f},\boldsymbol{\phi}')\hat{R}\boldsymbol{f}}{(\Delta s)^2}+\frac{2S(\boldsymbol{f},\boldsymbol{\phi})\hat{R}\boldsymbol{\tau}}{(\Delta s)^2}.
$$

Now we give estimates of $S(\boldsymbol{v},\boldsymbol{w})\hat{R}\boldsymbol{w}$ for $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in C^2$. Using Lemma 2.1, we have

$$
\begin{aligned}
&\sup_{s_1\neq s_2}\frac{1}{|\Delta s|}\left\|\frac{S(\boldsymbol{u},\boldsymbol{v})}{\Delta s}-\left(\frac{\Delta\boldsymbol{u}'}{\Delta s}\cdot\boldsymbol{v}_1'+\boldsymbol{u}_1'\cdot\frac{\Delta\boldsymbol{v}'}{\Delta s}\right)\right\|_{\mathbb{R}^n}\\
&=\sup_{s_1\neq s_2}\frac{1}{|\Delta s|}\left\|\frac{\Delta\boldsymbol{u}'}{\Delta s}\cdot\frac{\boldsymbol{v}_2'-\boldsymbol{v}_1'}{2}+\frac{\boldsymbol{u}_2'-\boldsymbol{u}_1'}{2}\cdot\frac{\Delta\boldsymbol{v}'}{\Delta s}\right\|_{\mathbb{R}^n}\\
&\leq 2\|\boldsymbol{u}\|_{C^2}\|\boldsymbol{v}\|_{C^2},
\end{aligned}
$$

and

$$
\sup_{s_1\neq s_2}\frac{1}{|\Delta s|}\left\|\hat{R}\boldsymbol{u}-\boldsymbol{u}_1'\right\|_{\mathbb{R}^n}=\left\|\frac{\boldsymbol{u}_2'-\boldsymbol{u}_1'}{2\Delta s}\right\|_{\mathbb{R}^n}\leq\|\boldsymbol{u}\|_{C^2}.
$$

Hence

$$
\sup_{s_1\neq s_2}\frac{1}{|\Delta s|}\left\|\frac{S(\boldsymbol{u},\boldsymbol{v})\hat{R}\boldsymbol{w}}{\Delta s}-\left(\frac{\Delta\boldsymbol{u}'}{\Delta s}\cdot\boldsymbol{v}_1'+\boldsymbol{u}_1'\cdot\frac{\Delta\boldsymbol{v}'}{\Delta s}\right)\boldsymbol{w}_1'\right\|_{\mathbb{R}^n}\leq 2\|\boldsymbol{u}\|_{C^2}\|\boldsymbol{v}\|_{C^2}\|\boldsymbol{w}\|_{C^2}
$$

holds. Overall we have

$$
\begin{aligned}
\boldsymbol{h}_{1211}=&2\frac{1}{\Delta s}\left(\frac{\Delta\boldsymbol{\kappa}}{\Delta s}\cdot\boldsymbol{\phi}_1'+\boldsymbol{\kappa}_1'\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\boldsymbol{\tau}_1+2\frac{1}{\Delta s}\left(\frac{\Delta\boldsymbol{\tau}}{\Delta s}\cdot\boldsymbol{\phi}_1''+\boldsymbol{\tau}_1\cdot\frac{\Delta\boldsymbol{\phi}''}{\Delta s}\right)\boldsymbol{\tau}_1\\
&+2\frac{1}{\Delta s}\left(\frac{\Delta\boldsymbol{\tau}}{\Delta s}\cdot\boldsymbol{\phi}_1'+\boldsymbol{\tau}_1\cdot\frac{\Delta\boldsymbol{\phi}'}{\Delta s}\right)\boldsymbol{\tau}_1+\mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}\\
=&2\frac{1}{\Delta s}\left(\frac{\Delta\boldsymbol{\kappa}}{\Delta s}\cdot\boldsymbol{\phi}_1'+\boldsymbol{\tau}_1\cdot\frac{\Delta\boldsymbol{\phi}''}{\Delta s}\right)\boldsymbol{\tau}_1+\mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha},
\end{aligned}
\tag{2.2}
$$

and

$$\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left\|\boldsymbol{h}_{1211}-2\frac{1}{\Delta s}\left(\frac{\Delta\boldsymbol{\kappa}}{\Delta s}\cdot\boldsymbol{\phi}_1'+\boldsymbol{\tau}_1\cdot\frac{\Delta\boldsymbol{\phi}''}{\Delta s}\right)\boldsymbol{\tau}_1\right\|_{\mathbb{R}^n}\leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}\|\boldsymbol{f}\|_{C^2}.$$

Using Lemma 2.9 for the first term of (2.2), and Lemma 2.11 for remaining term of $\boldsymbol{h}_{1211}$, we can conclude that $\boldsymbol{h}_{1211}$ is integrable on $\mathbb{R}/\mathcal{L}\mathbb{Z}$ and there exists a mapping $\boldsymbol{N}_{121}: H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\boldsymbol{h}_{121}(s_1,s_2)\cdot\boldsymbol{\psi}_1 ds_1ds_2=-\langle(L_1\boldsymbol{f}\cdot\boldsymbol{\phi}'+\boldsymbol{\tau}\cdot L_1\boldsymbol{\phi})\boldsymbol{\tau},\boldsymbol{\psi}\rangle_{L^2}+\langle\boldsymbol{N}_{121}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}$$

and

$$||(L_1\boldsymbol{f}\cdot\boldsymbol{\phi}+\boldsymbol{\tau}\cdot L_1\boldsymbol{\phi})\boldsymbol{\tau}||_{L^2}\leq(C||\boldsymbol{f}||_{H^3})\,||\boldsymbol{\phi}||_{H^3},\quad ||\boldsymbol{N}_{121}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2}\leq(C||\boldsymbol{f}||_{H^3})\,||\boldsymbol{\phi}||_{H^3}.$$

□

Lemma 3.3 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{122}: H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^{3-\alpha}$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}H_{122}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2=\langle\boldsymbol{N}_{122}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover $\boldsymbol{N}_{122}(\boldsymbol{f})[\boldsymbol{\phi}]$ satisfies

$$||\boldsymbol{N}_{122}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2}\leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}.$$

Proof Since we have

$$\boldsymbol{h}_{122}(s_1,s_2)\cdot\boldsymbol{\psi}(s_2)=-\mathscr{M}(\boldsymbol{f})S(\boldsymbol{f},\boldsymbol{\phi})\hat{R}\boldsymbol{f}\cdot\boldsymbol{\psi}(s_2)=\mathcal{O}(\Delta s),$$

and

$$\boldsymbol{h}_{122}(s_2,s_1)=-\mathscr{M}(-S(\boldsymbol{f},\boldsymbol{\phi}))\hat{R}\boldsymbol{f}=-\boldsymbol{h}_{122}(s_1,s_2),$$

using Lemma 2.14

$$\begin{aligned}&\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{122}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}'ds_1ds_2\\&=-2\iint_{|\Delta s|\geq\varepsilon}\left\{\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{122}\right)(s_1,s_2)-\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{122}\right)(s_2,s_1)\right\}\cdot\boldsymbol{\psi}_1ds_1ds_2.+\mathcal{O}(\varepsilon)\end{aligned}$$

Moreover we have

$$\begin{aligned}\frac{\partial}{\partial s_1}\left(2\boldsymbol{h}_{122}(s_1,s_2)\right)&=\frac{\partial}{\partial s_1}\left(-2\mathscr{M}(\boldsymbol{f})S(\boldsymbol{f},\boldsymbol{\phi})\hat{R}\boldsymbol{f}\right)\\&=-2\frac{\partial}{\partial s_1}\left(\mathscr{M}(\boldsymbol{f})\right)S(\boldsymbol{f},\boldsymbol{\phi})\hat{R}\boldsymbol{f}-2\mathscr{M}(\boldsymbol{f})\frac{\partial}{\partial s_1}\left(S(\boldsymbol{f},\boldsymbol{\phi})\right)\hat{R}\boldsymbol{f}\\&\quad-2\mathscr{M}(\boldsymbol{f})S(\boldsymbol{f},\boldsymbol{\phi})\frac{\partial}{\partial s_1}\left(\hat{R}\boldsymbol{f}\right)\\&=\frac{4}{(\Delta s)^3}T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1S(\boldsymbol{f},\boldsymbol{\phi})\hat{R}\boldsymbol{f}\\&\quad-\mathscr{M}(\boldsymbol{f})\left(\boldsymbol{\kappa}_1\cdot Q\boldsymbol{\phi}+2\hat{R}\boldsymbol{f}\cdot\boldsymbol{\phi}_1''+2\boldsymbol{\kappa}_1\cdot\hat{R}\boldsymbol{\phi}+Q\boldsymbol{f}\cdot\boldsymbol{\phi}_1''\right)\hat{R}\boldsymbol{f}\\&\quad-\mathscr{M}(\boldsymbol{f})S(\boldsymbol{f},\boldsymbol{\phi})\boldsymbol{\kappa}_1\\&=\mathcal{O}(1)+\mathcal{O}(\Delta s)+\mathcal{O}(\Delta s)\\&=\mathcal{O}(1).\end{aligned}$$

We used Lemmas 2.4, 2.1, 2.3 for the second line from the last. Therefore $\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{122}\right)(s_1,s_2)$ and $\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{122}\right)(s_2,s_1)$ are bounded in $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$. The assertions hold since Fubini's theorem and Hölder's inequality imply that the function

$$s_1 \mapsto \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ \left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{122}\right)(s_1,s_2) - \left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{122}\right)(s_2,s_1) \right\} ds_2$$

belongs to L^2. □

Lemma 3.4 *Let $\alpha \in (0, \frac{1}{2})$. For any bi-Lipschitz continuous function $\boldsymbol{f} \in H^{3-\alpha}$ and functions $\boldsymbol{\phi}, \boldsymbol{\psi} \in H^{3-\alpha}$, there exists a mapping $\boldsymbol{N}_{123} : H^{3-\alpha} \to \mathscr{B}(H^{3-\alpha}, L^2)$ such that*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{123}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] ds_1 ds_2 = \langle \boldsymbol{N}_{123}(\boldsymbol{f})[\boldsymbol{\phi}], \boldsymbol{\psi}\rangle_{L^2}.$$

Moreover $\boldsymbol{N}_{123}(\boldsymbol{f})[\boldsymbol{\phi}]$ satisfies

$$||\boldsymbol{N}_{123}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \le C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}.$$

Proof Using the notation

$$A\boldsymbol{\psi} = \frac{\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2}{2},$$

we have

$$\begin{aligned}
&\iint_{|\Delta s|\ge\varepsilon} \boldsymbol{h}_{123}(s_1,s_2)\cdot \hat{R}\boldsymbol{\psi} ds_1 ds_2 \\
&= \iint_{|\Delta s|\ge\varepsilon} (\boldsymbol{h}_{123}(s_1,s_2) + \boldsymbol{h}_{123}(s_2,s_1))\cdot \frac{1}{2}\boldsymbol{\psi}_1' ds_1 ds_2 \\
&= \iint_{|\Delta s|\ge\varepsilon} 2\boldsymbol{h}_{123}(s_1,s_2)\cdot \frac{\partial}{\partial s_1}(A\boldsymbol{\psi}) ds_1 ds_2 \\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \{\boldsymbol{h}_{123}(s-\varepsilon,s)\cdot(\boldsymbol{\psi}(s-\varepsilon)+\boldsymbol{\psi}(s)) - \boldsymbol{h}_{123}(s+\varepsilon,s)\cdot(\boldsymbol{\psi}(s+\varepsilon)+\boldsymbol{\psi}(s))\} ds \\
&\quad - \iint_{|\Delta s|\ge\varepsilon} 2\frac{\partial}{\partial s_1}(\boldsymbol{h}_{123}(s_1,s_2))\cdot A\boldsymbol{\psi} ds_1 ds_2.
\end{aligned}$$

Since Lemma 2.1 implies that

$$\begin{aligned}
\boldsymbol{h}_{123}(s_1,s_2) &= -\frac{(\Delta s)^2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}\left(\hat{R}\boldsymbol{f}\cdot\frac{Q\boldsymbol{\phi}}{\Delta s} + \frac{Q\boldsymbol{f}}{\Delta s}\cdot\hat{R}\boldsymbol{\phi}\right)\frac{\Delta\boldsymbol{\tau}}{\Delta s} \\
&= -(\boldsymbol{\tau}(s_2)\cdot\boldsymbol{\phi}''(s_2) + \boldsymbol{\kappa}(s_2)\cdot\boldsymbol{\phi}'(s_2))\boldsymbol{\kappa}(s_2) + \mathcal{O}(\Delta s)^{\frac{1}{2}},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \Big\{\boldsymbol{h}_{123}(s-\varepsilon,s)\cdot(\boldsymbol{\psi}(s-\varepsilon)+\boldsymbol{\psi}(s)) - \boldsymbol{h}_{123}(s+\varepsilon,s)\cdot(\boldsymbol{\psi}(s+\varepsilon)+\boldsymbol{\psi}(s))\Big\} ds \\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \Big\{-(\boldsymbol{\tau}(s)\cdot\boldsymbol{\phi}''(s) + \boldsymbol{\kappa}(s)\cdot\boldsymbol{\phi}'(s))\boldsymbol{\kappa}(s)\cdot(\boldsymbol{\psi}(s-\varepsilon)+\boldsymbol{\psi}(s)) + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \\
&\quad + (\boldsymbol{\tau}(s)\cdot\boldsymbol{\phi}''(s) + \boldsymbol{\kappa}(s)\cdot\boldsymbol{\phi}'(s))\{\boldsymbol{\kappa}(s)\cdot(\boldsymbol{\psi}(s+\varepsilon)+\boldsymbol{\psi}(s))\} + \mathcal{O}(\varepsilon)^{\frac{1}{2}}\Big\} ds \\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{\tau}(s)\cdot\boldsymbol{\phi}''(s) + \boldsymbol{\kappa}(s)\cdot\boldsymbol{\phi}'(s))(-\boldsymbol{\psi}(s-\varepsilon)+\boldsymbol{\psi}(s+\varepsilon)) ds + \mathcal{O}(\varepsilon^{\frac{1}{2}}) \\
&= \mathcal{O}(\varepsilon^{\frac{1}{2}}).
\end{aligned}$$

On the other hand, since

$$
\begin{aligned}
&\frac{\partial}{\partial s_1}\left(\boldsymbol{h}_{123}(s_1,s_2)\right)\\
&= -\frac{\frac{\partial}{\partial s_1}\left(S(\boldsymbol{f},\boldsymbol{\phi})\right)\Delta\boldsymbol{\tau}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{S(\boldsymbol{f},\boldsymbol{\phi})\frac{\partial}{\partial s_1}\left(\Delta\boldsymbol{\tau}\right)}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \left(S(\boldsymbol{f},\boldsymbol{\phi})\frac{\partial}{\partial s_1}\left(\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^{-2}\right)\Delta\boldsymbol{\tau}\right)\\
&= -\frac{\left(\frac{1}{2}\boldsymbol{\kappa}_1\cdot Q\boldsymbol{\phi} + \hat{R}\boldsymbol{f}\cdot\boldsymbol{\phi}_1'' + \boldsymbol{\kappa}_1\cdot\hat{R}\boldsymbol{\phi} + \frac{1}{2}Q\boldsymbol{f}\cdot\boldsymbol{\phi}_1''\right)\Delta\boldsymbol{\tau}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{S(\boldsymbol{f},\boldsymbol{\phi})\boldsymbol{\kappa}_1}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\\
&\quad + \frac{2\left(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1\right)S(\boldsymbol{f},\boldsymbol{\phi})\Delta\boldsymbol{\tau}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon}\left\{-2\frac{\partial}{\partial s_1}\left(\boldsymbol{h}_{123}(s_1,s_2)\right)\cdot A\boldsymbol{\psi}\right\}ds_1ds_2\\
&= -\iint_{|\Delta s|\geq\varepsilon}\left\{\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{123}\right)(s_1,s_2) + \left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{123}\right)(s_2,s_1)\right\}\cdot\boldsymbol{\psi}_1 ds_1ds_2\\
&= -\iint_{|\Delta s|\geq\varepsilon}\left\{\frac{\left(S(\boldsymbol{\tau},\boldsymbol{\phi}) + S(\boldsymbol{f},\boldsymbol{\phi}')\right)\cdot Q\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{S(\boldsymbol{f},\boldsymbol{\phi})\cdot\Delta\boldsymbol{\kappa}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right.\\
&\quad\left. + \frac{2(\Delta\boldsymbol{f}\cdot Q\boldsymbol{f})S(\boldsymbol{f},\boldsymbol{\phi})Q\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\right\}\cdot\boldsymbol{\psi}_1 ds_1ds_2
\end{aligned}
$$

Now the integrand is absolutely integrable on $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$. Therefore the assertion follows from Fubini's theorem. □

3.3 The L^2-representation of K_{13}

Lemma 3.5 *Let $\alpha \in (0, \frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{13} : H^{3-\alpha} \to \mathscr{B}(H^{3-\alpha}, L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f} \in H^{3-\alpha}$ and functions $\boldsymbol{\phi}, \boldsymbol{\psi} \in H^{3-\alpha}$, we have*

$$
\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{13}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = \langle \boldsymbol{N}_{13}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.
$$

Moreover $\boldsymbol{N}_{13}(\boldsymbol{f})[\boldsymbol{\phi}]$ satisfies

$$
||\boldsymbol{N}_{13}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}.
$$

Proof Assume that $\boldsymbol{\psi} \in H^3$. Define

$$
\boldsymbol{h}_{13}(s_1,s_2) = \frac{2\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}.
$$

Then we have

$$
\iint_{|\Delta s|\geq\varepsilon} H_{13}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = \iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{13}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2.
$$

Using Lemmas 2.3 and 2.1, we can show the estimate

$$
\begin{aligned}
\mathscr{M}(\boldsymbol{f}) &= \frac{(\Delta s)^2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\frac{\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2}{(\Delta s)^2}\\
&= \|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2 + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha},
\end{aligned}
$$

thus we have

$$\begin{aligned}\boldsymbol{h}_{13}(s_1,s_2) &= \frac{2Q\boldsymbol{f}\cdot Q\boldsymbol{\phi} - 4\mathscr{M}_1(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\Delta\boldsymbol{f} \\ &= \frac{2}{\Delta s}\left(\frac{Q\boldsymbol{f}}{\Delta s}\cdot\frac{Q\boldsymbol{\phi}}{\Delta s} - 2\mathscr{M}_1(\boldsymbol{f})\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{f}}{\Delta s} + \mathcal{O}(\Delta s)^5 \\ &= \frac{2}{\Delta s}(\boldsymbol{\kappa}_i\cdot\boldsymbol{\phi}_i'' - 2\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i)\boldsymbol{\tau}_i + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha},\end{aligned}$$

and

$$\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left\|\boldsymbol{h}_{13}(s_1,s_2) - \frac{1}{\Delta s}(\boldsymbol{\kappa}_i\cdot\boldsymbol{\phi}_i'' - 2\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i)\boldsymbol{\tau}_i\right\|_{\mathbb{R}^n} \leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}.$$

Corollary 2.1 implies that there exists a mapping $\boldsymbol{N}_{13} : H^{3-\alpha} \to \mathscr{B}(H^{3-\alpha}, L^2)$ such that

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{13}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = \langle\boldsymbol{N}_{13}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2},$$

and

$$\|\boldsymbol{N}_{13}(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^2} \leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}.$$

□

3.4 The L^2-representation of K_{14}

Lemma 3.6 *Let $\alpha \in (0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{14} : H^{3-\alpha} \to \mathscr{B}(H^{3-\alpha}, L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f} \in H^3$ and functions $\boldsymbol{\phi},\boldsymbol{\psi} \in H^{3-\alpha}$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{14}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = \langle 2(\boldsymbol{\tau}\cdot\boldsymbol{\phi}')L_1\boldsymbol{f} + \boldsymbol{N}_{14}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover

$$||\boldsymbol{N}_{14}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}$$

holds.

Proof We decompose H_{14} as

$$\begin{aligned}H_{14}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] &= -\frac{2\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} \\ &= \frac{-2\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\right)\left(Q\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}'\right)}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4} + \frac{2\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\right)\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}\right)}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6} \\ &= \boldsymbol{h}_{141}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}' + \boldsymbol{h}_{142}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}.\end{aligned}$$

We have

$$\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon} \boldsymbol{h}_{141}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}' ds_1 ds_2 \\
&= \iint_{|\Delta s|\geq\varepsilon} 2\boldsymbol{h}_{141}(s_1,s_2)\cdot\boldsymbol{\psi}_1' ds_1 ds_2 \\
&= \iint_{|\Delta s|\geq\varepsilon} 2\boldsymbol{h}_{141}(s_1,s_2)\cdot\frac{\partial}{\partial s_1}\left(\Delta\boldsymbol{\psi}'\right) ds_1 ds_2 \\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \Big\{ 2\boldsymbol{h}_{141}(s-\varepsilon,s)\cdot(\boldsymbol{\psi}(s-\varepsilon)-\boldsymbol{\psi}(s)) - 2\boldsymbol{h}_{141}(s+\varepsilon,s)\cdot(\boldsymbol{\psi}(s+\varepsilon)-\boldsymbol{\psi}(s)) \Big\} ds \\
&\qquad - \iint_{|\Delta s|\geq\varepsilon} \frac{\partial}{\partial s_1}\left(2\boldsymbol{h}_{141}(s_1,s_2)\right)\cdot\Delta\boldsymbol{\psi} ds_1 ds_2.
\end{aligned}$$

Since

$$\begin{aligned}
&2\boldsymbol{h}_{141}(s-\varepsilon,s)\cdot(\boldsymbol{\psi}(s-\varepsilon)-\boldsymbol{\psi}(s)) - 2\boldsymbol{h}_{141}(s+\varepsilon,s)\cdot(\boldsymbol{\psi}(s+\varepsilon)-\boldsymbol{\psi}(s)) \\
&= -4\left(\boldsymbol{\tau}(s)\cdot\boldsymbol{\phi}'(s)\right)\left(\boldsymbol{\kappa}(s)\cdot\boldsymbol{\psi}'(s)\right) + \mathcal{O}(\varepsilon)^{\frac{1}{2}-\alpha} + 4\left(\boldsymbol{\tau}(s)\cdot\boldsymbol{\phi}'(s)\right)\left(\boldsymbol{\kappa}(s)\cdot\boldsymbol{\psi}'(s)\right) + \mathcal{O}(\varepsilon)^{\frac{1}{2}-\alpha} \\
&= \mathcal{O}(\varepsilon)^{\frac{1}{2}-\alpha},
\end{aligned}$$

the integral on the boundary of the interval converges to 0 as $\varepsilon\to+0$. Since

$$\begin{aligned}
\frac{\partial}{\partial s_1}\left(2\boldsymbol{h}_{141}(s_1,s_2)\right) = {}& \frac{-4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\Big\{ \left(\boldsymbol{\tau}_1\cdot\Delta\boldsymbol{\phi}\right)Q\boldsymbol{f} + \left(\Delta\boldsymbol{f}\cdot\boldsymbol{\phi}_1'\right)Q\boldsymbol{f} + \left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\right)\boldsymbol{\kappa}_1 \Big\} \\
&+ \frac{16\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\right)\left(\Delta\boldsymbol{f}\cdot\boldsymbol{f}_1\right)Q\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^6},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon} \frac{\partial}{\partial s_1}\left(2\boldsymbol{h}_{141}(s_1,s_2)\right)\cdot\Delta\boldsymbol{\psi} ds_1 ds_2 \\
&= \iint_{|\Delta s|\geq\varepsilon} \left\{ \left(\frac{\partial}{\partial s_1}2\boldsymbol{h}_{141}\right)(s_1,s_2) - \left(\frac{\partial}{\partial s_1}2\boldsymbol{h}_{141}\right)(s_2,s_1) \right\}\cdot\boldsymbol{\psi}_1 ds_1 ds_2 \\
&= \iint_{|\Delta s|\geq\varepsilon} \left\{ -\frac{4\left(\Delta\boldsymbol{\tau}\cdot\Delta\boldsymbol{\phi}\right)Q\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4} - \frac{4\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}'\right)Q\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4} - \frac{4\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\right)\Delta\boldsymbol{\kappa}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4} \right. \\
&\qquad \left. + \frac{16\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\right)\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{f}\right)Q\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^6} \right\}\cdot\boldsymbol{\psi}_1 ds_1 ds_2.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{(\Delta s)^2} \\
&= \left\{ \left(\frac{\Delta\boldsymbol{f}}{(\Delta s)^2} - \frac{\boldsymbol{\tau}_1}{\Delta s}\right)\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s} \right\}\frac{\Delta\boldsymbol{\kappa}}{\Delta s} + \left\{ \boldsymbol{\tau}_1\cdot\left(\frac{\Delta\boldsymbol{\phi}}{(\Delta s)^2} - \frac{\boldsymbol{\phi}_1'}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{\Delta s} \right\} \\
&\quad + \boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'\frac{\Delta\boldsymbol{\kappa}}{(\Delta s)^2} \\
&= \left(\frac{T_1^0\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{\Delta s} + \left(\boldsymbol{\tau}_1\cdot\frac{T_1^0\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{\Delta s} + \boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'\frac{\Delta\boldsymbol{\kappa}}{(\Delta s)^2},
\end{aligned}$$

in conjunction with Lemma 2.1, it follows that

$$\begin{aligned}
&\left(\frac{\partial}{\partial s_1}2\boldsymbol{h}_{141}\right)(s_1,s_2)-\left(\frac{\partial}{\partial s_1}2\boldsymbol{h}_{141}\right)(s_2,s_1)\\
&=\frac{-4(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\Big\{\left(\frac{\Delta\boldsymbol{\tau}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\tau}}{(\Delta s)^2}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}'}{\Delta s}\right)\frac{\Delta\boldsymbol{\tau}}{(\Delta s)^2}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{(\Delta s)^2}\Big\}\\
&\quad+\frac{16(\Delta s)^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\tau}}{\Delta s}\right)\frac{\Delta\boldsymbol{\tau}}{(\Delta s)^2}\\
&=-\frac{4\left(\boldsymbol{\kappa}_1\cdot\boldsymbol{\phi}_1'\right)\boldsymbol{\kappa}_1}{\Delta s}+\mathcal{O}(\Delta s)^{-\frac{1}{2}-\alpha}-\frac{4\left(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1''\right)\boldsymbol{\kappa}_1}{\Delta s}+\mathcal{O}(\Delta s)^{-\frac{1}{2}-\alpha}\\
&\quad-4\left(\frac{T_1^0\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{(\Delta s)^2}-4\left(\boldsymbol{\tau}_1\cdot\frac{T_1^0\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{\Delta s}-4\left(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'\right)\frac{\Delta\boldsymbol{\kappa}}{(\Delta s)^2}\\
&\quad+\frac{16(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1')(\boldsymbol{\tau}_1\cdot\boldsymbol{\kappa}_1)\boldsymbol{\kappa}_1}{\Delta s}+\mathcal{O}(\Delta s)^{-\frac{1}{2}-\alpha}\\
&=-\frac{4\left(\boldsymbol{\kappa}_1\cdot\boldsymbol{\phi}_1'\right)\boldsymbol{\kappa}_1}{\Delta s}-\frac{4\left(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1''\right)\boldsymbol{\kappa}_1}{\Delta s}\\
&\quad+\frac{16(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1')(\boldsymbol{\tau}_1\cdot\boldsymbol{\kappa}_1)\boldsymbol{\kappa}_1}{\Delta s}\\
&\quad-4\left(\frac{T_1^0\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{\Delta s}-4\left(\boldsymbol{\tau}_1\cdot\frac{T_1^0\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{\Delta s}-4\left(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'\right)\frac{\Delta\boldsymbol{\kappa}}{(\Delta s)^2}+\mathcal{O}(\Delta s)^{-\frac{1}{2}-\alpha}.
\end{aligned}$$

Now, the function

$$-4\left(\frac{T_1^0\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{\Delta s}-4\left(\boldsymbol{\tau}_1\cdot\frac{T_1^0\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{\Delta s}+\mathcal{O}(\Delta s)^{-\frac{1}{2}-\alpha}$$

is square-integrable in $\mathbb{R}/\mathcal{L}\mathbb{Z}$ and we have

$$\lim_{\varepsilon\downarrow 0}\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\frac{\Delta\boldsymbol{\kappa}}{(\Delta s)^2}\cdot\boldsymbol{\psi}_1 ds_1ds_2=-\frac{1}{2}\langle L_1\boldsymbol{f},\boldsymbol{\psi}\rangle_{L^2}$$

from Lemma 2.9. Using Fubini's theorem and Lemma 2.11, we have

$$\begin{aligned}
&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}H_{141}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2\\
&=2\langle(\boldsymbol{\tau}\cdot\boldsymbol{\phi}')L_1\boldsymbol{f},\boldsymbol{\psi}\rangle_{L^2}\\
&\quad+\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\Bigg[\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\Bigg\{-\frac{4\left(\Delta\boldsymbol{\tau}\cdot\Delta\boldsymbol{\phi}\right)Q\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}-\frac{4\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}'\right)Q\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}-\frac{4\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\right)\Delta\boldsymbol{\kappa}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\\
&\quad+\frac{16\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\right)\left(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{f}\right)Q\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^6}+\frac{4\left(\boldsymbol{\kappa}_1\cdot\boldsymbol{\phi}_1'\right)\boldsymbol{f}_1}{\Delta s}+\frac{4\left(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1''\right)\boldsymbol{f}_1}{\Delta s}\\
&\quad-\frac{16(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1')(\boldsymbol{\tau}_1\cdot\boldsymbol{\kappa}_1)\boldsymbol{\kappa}_1}{\Delta s}-4\left(\frac{T_1^0\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{\Delta s}\\
&\quad-4\left(\boldsymbol{\tau}_1\cdot\frac{T_1^0\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{\kappa}}{\Delta s}\Bigg\}ds_1\Bigg]\cdot\boldsymbol{\psi}_1 ds_1.
\end{aligned}$$

Next we calculate the integral of $\boldsymbol{h}_{142}(s_1,s_2)$. We have

$$\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{142}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2=\iint_{|\Delta s|\geq\varepsilon}2\boldsymbol{h}_{142}(s_1,s_2)\cdot\boldsymbol{\psi}_1 ds_1ds_2,$$

where

$$\boldsymbol{h}_{142}(s_1,s_2) = \frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}$$
$$= \frac{2(\Delta s)^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2}{(\Delta s)^2}\frac{\Delta\boldsymbol{f}}{\Delta s}\frac{1}{\Delta s}.$$

Using Lemma 2.1, it holds that

$$\boldsymbol{h}_{142}(s_1,s_2) = 2(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1')\|\boldsymbol{\kappa}_1\|_{\mathbb{R}^n}^2\frac{\boldsymbol{\tau}_1}{\Delta s} + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha-1}.$$

Therefore the function

$$\boldsymbol{h}_{142}(s_1,s_2) - 2(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1')\|\boldsymbol{\kappa}_1\|_{\mathbb{R}^n}^2\frac{\boldsymbol{\tau}_1}{\Delta s}$$

is integrable is with respect to s_2. Noting that

$$\int_{|\Delta s|\geq\varepsilon}\frac{1}{\Delta s}ds_2 = 0,$$

we have

$$\begin{aligned}
&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{142}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2\\
&= \lim_{\varepsilon\to+0}\iint_{|\Delta s|\geq\varepsilon}\left\{2\boldsymbol{h}_{142}(s_1,s_2) - 4(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1')\|\boldsymbol{\kappa}_1\|_{\mathbb{R}^n}^2\frac{\boldsymbol{\tau}_1}{\Delta s}\right\}\cdot\boldsymbol{\psi}_1ds_2ds_1\\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left[\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{\frac{2\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6} - 4\left(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'\right)\|\Delta\boldsymbol{\kappa}\|_{\mathbb{R}^n}^2\frac{\Delta\boldsymbol{\tau}}{\Delta s}\right\}ds_2\right]\cdot\boldsymbol{\psi}_1ds_1\\
&= \langle N_{14}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})},
\end{aligned}$$

where

$$N_{14}(\boldsymbol{f})[\boldsymbol{\phi}] = \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{\frac{2\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6} - 4\left(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'\right)\|\Delta\boldsymbol{\kappa}\|_{\mathbb{R}^n}^2\frac{\Delta\boldsymbol{\tau}}{\Delta s}\right\}ds_2.$$

Moreover, Lemma 2.11 implies that

$$||\boldsymbol{N}_{14}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}.$$

□

3.5 The L^2-representaion of K_{15}

Lemma 3.7 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{15}: H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi}\in H^{3-\alpha},\boldsymbol{\psi}\in L^2$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{15}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = \langle\boldsymbol{N}_{15}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover $\boldsymbol{N}_{15}(\boldsymbol{f})[\boldsymbol{\phi}]$ satisfies

$$||\boldsymbol{N}_{15}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}.$$

Proof If we define

$$\boldsymbol{h}_{15}(s_1,s_2) = -\frac{2\mathscr{M}_1(\boldsymbol{f})\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},$$

then we have

$$\iint_{|\Delta s|\geq\varepsilon} H_{15}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = \iint_{|\Delta s|\geq\varepsilon} 2\boldsymbol{h}_{15}(s_1,s_2)\cdot\boldsymbol{\psi}_1 ds_1ds_2.$$

Now, it holds that

$$\begin{aligned}\frac{\mathscr{M}_1(\boldsymbol{f})\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^2} &= \frac{1}{2\Delta s}\frac{(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\left|\frac{\Delta\boldsymbol{\tau}}{\Delta s}\right|_{\mathbb{R}^n}^2\frac{\Delta\boldsymbol{\phi}}{\Delta s}\\ &= \frac{1}{2\Delta s}\left|\frac{\Delta\boldsymbol{\tau}}{\Delta s}\right|_{\mathbb{R}^n}^2\frac{\Delta\boldsymbol{\phi}}{\Delta s} + \frac{1}{2}\mathscr{M}^4(\boldsymbol{f})\left|\frac{\Delta\boldsymbol{\tau}}{\Delta s}\right|_{\mathbb{R}^n}^2\Delta\boldsymbol{\phi},\end{aligned}$$

and Lemma 2.1 implies that

$$\sup_{s_1\neq s_2}\frac{1}{2|\Delta s|^{\frac{1}{2}-\alpha}}\left\|\left\|\frac{\Delta\boldsymbol{\tau}}{\Delta s}\right\|_{\mathbb{R}^n}^2\frac{\Delta\boldsymbol{\phi}}{\Delta s} - \|\boldsymbol{\kappa}_1\|_{\mathbb{R}^n}^2\boldsymbol{\phi}_1'\right\|_{\mathbb{R}^n}.$$

Using Lemma 2.11 for the first term and Fubini's theorem for second term, we obtain the assertions.

□

3.6 The L^2-representation of the second variation of $\mathcal{E}_1$

We obtain the L^2-representation of $\delta^2\mathcal{E}_1$ from the results obtained above in Proposition 3.1 through to Lemma 3.7.

Theorem 3.1 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_1 : H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^3$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^3$, we have*

$$\delta^2\mathcal{E}_1(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \langle P_{\boldsymbol{\tau}}^{\perp}L_1\boldsymbol{\phi} - (L_1\boldsymbol{f}\cdot\boldsymbol{\phi}')\boldsymbol{\tau} + 2(\boldsymbol{\tau}\cdot\boldsymbol{\phi}')L_1\boldsymbol{f} + \boldsymbol{N}_1(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2},$$

where

$$\begin{aligned}
\boldsymbol{N}_1(\boldsymbol{f})[\boldsymbol{\phi}] = & \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left\{ \frac{2}{(\Delta s)^3} T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1) \Delta \boldsymbol{\phi}' - \mathscr{M}(\boldsymbol{f}) \boldsymbol{\phi}''(s_1) \right\} ds_2 \\
& + \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ \left(2 \frac{Q\boldsymbol{f}}{\Delta s} \cdot \frac{Q\boldsymbol{\phi}}{\Delta s} \right) \hat{R}\boldsymbol{f} + \left(2\boldsymbol{\tau}(s_1) \cdot \frac{Q\boldsymbol{\phi}'}{\Delta s} \right) \frac{Q\boldsymbol{f}}{\Delta s} \right. \\
& \left. - \left(2 \frac{Q\boldsymbol{\tau}}{\Delta s} \cdot \frac{Q\boldsymbol{\phi}}{\Delta s} \right) \hat{R}\boldsymbol{f} - \left(2 \frac{Q\boldsymbol{\tau}}{\Delta s} \cdot \boldsymbol{\phi}'(s_1) \right) \frac{Q\boldsymbol{f}}{\Delta s} \right\} ds_2 \\
& + \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{2}{\Delta s} \left\{ \left(\hat{R}\boldsymbol{\tau} \cdot \frac{Q\boldsymbol{\phi}}{(\Delta s)^2} \right) \hat{R}\boldsymbol{f} + \left(\frac{Q\boldsymbol{f}}{(\Delta s)^2} \cdot \hat{R}\boldsymbol{\phi}' \right) \hat{R}\boldsymbol{f} + \left(\hat{R}\boldsymbol{f} \cdot \frac{Q\boldsymbol{\phi}}{(\Delta s)^2} \right) \hat{R}\boldsymbol{\tau} \right. \right. \\
& \left. + \left(\frac{Q\boldsymbol{f}}{(\Delta s)^2} \cdot \hat{R}\boldsymbol{\phi}' \right) \hat{R}\boldsymbol{\tau} \right\} - \frac{4\boldsymbol{\kappa}(s_1) \cdot \boldsymbol{\phi}''(s_1) \boldsymbol{\tau}(s_1)}{\Delta s} - \frac{2\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\phi}''(s_1) \boldsymbol{\kappa}(s_1)}{\Delta s} \\
& \left. - \frac{2\boldsymbol{\kappa}(s_1) \cdot \boldsymbol{\phi}''(s_1) \boldsymbol{\kappa}(s_1)}{\Delta s} \right] ds_2 \\
& + \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ \frac{4}{(\Delta s)^3} \left(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1) \right) S(\boldsymbol{f}, \boldsymbol{\phi}) \hat{R}\boldsymbol{f} \right. \\
& - \mathscr{M}(\boldsymbol{f}) \left(\boldsymbol{\kappa}(s_1) \cdot Q\boldsymbol{\phi} + 2\hat{R}\boldsymbol{f} \cdot \boldsymbol{\phi}''(s_1) + 2\boldsymbol{\kappa}(s_1) \cdot \hat{R}\boldsymbol{f} + Q\boldsymbol{f} \cdot \boldsymbol{\phi}''(s_1) \right) \hat{R}\boldsymbol{f} \\
& \left. - \mathscr{M}(\boldsymbol{f}) S(\boldsymbol{f}, \boldsymbol{\phi}) \cdot \boldsymbol{\kappa}(s_1) \right\} ds_2 \\
& + \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ \frac{(S(\boldsymbol{\tau}, \boldsymbol{\phi}) + S(\boldsymbol{f}, \boldsymbol{\phi}')) \cdot Q\boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - \frac{S(\boldsymbol{f}, \boldsymbol{\phi}) \cdot \Delta \boldsymbol{\kappa}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \right. \\
& \left. + \frac{2 \left(S(\boldsymbol{f}, \boldsymbol{\phi}) \cdot Q\boldsymbol{f} \right) \left(\Delta \boldsymbol{f} \cdot Q\boldsymbol{f} \right)}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \right\} ds_2 \\
& + \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[4 \frac{\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}]}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} - \left\{ \boldsymbol{\kappa}(s_1) \cdot \boldsymbol{\phi}''(s_1) + \|\boldsymbol{\kappa}(s_1)\|_{\mathbb{R}^n}^2 (\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\phi}(s_1)) \right\} \frac{\boldsymbol{\tau}(s_1)}{\Delta s} \right] ds_2 \\
& + \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ - \frac{4 \left(\Delta \boldsymbol{\tau} \cdot \Delta \boldsymbol{\phi} \right) Q\boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}}^4} - \frac{4 \left(\Delta \boldsymbol{f} \cdot \Delta \boldsymbol{\phi}' \right) Q\boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}}^4} - \frac{4 \left(\Delta \boldsymbol{f} \cdot \Delta \boldsymbol{\phi} \right) \Delta \boldsymbol{\kappa}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}}^4} \right. \\
& + \frac{16 \left(\Delta \boldsymbol{f} \cdot \Delta \boldsymbol{\phi} \right) \left(\Delta \boldsymbol{f} \cdot \Delta \boldsymbol{f} \right) Q\boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}}^6} + \frac{4 \left(\boldsymbol{\kappa}(s_1) \cdot \boldsymbol{\phi}'(s_1) \right) \boldsymbol{f}(s_1)}{\Delta s} + \frac{4 \left(\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\phi}''(s_1) \right) \boldsymbol{f}(s_1)}{\Delta s} \\
& - \frac{16 (\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\phi}'(s_1)) (\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\kappa}(s_1)) \boldsymbol{\kappa}(s_1)}{\Delta s} \\
& - \left(4 \frac{T_1^0 \boldsymbol{f}}{\Delta s} \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s} \right) \frac{\Delta \boldsymbol{\kappa}}{\Delta s} - \left(4\boldsymbol{\tau}(s_1) \cdot \frac{T_1^0 \boldsymbol{\phi}}{\Delta s} \right) \frac{\Delta \boldsymbol{\kappa}}{\Delta s} \\
& \left. + \frac{\left(2\Delta \boldsymbol{f} \cdot \Delta \boldsymbol{\phi} \right) \|\Delta \boldsymbol{\tau}\|_{\mathbb{R}^n}^2 \Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6} - 4 \left(\boldsymbol{\tau}(s_1) \cdot \boldsymbol{\phi}'(s_1) \right) \|\Delta \boldsymbol{\kappa}\|_{\mathbb{R}^n}^2 \frac{\Delta \boldsymbol{\tau}}{\Delta s} \right\} ds_2 \\
& + \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left\{ \frac{4\mathscr{M}(\boldsymbol{f}) T_1^2(\boldsymbol{f}, \boldsymbol{\phi})}{\Delta s} + \frac{4}{\Delta s} \left(\mathscr{M}(\boldsymbol{f}) \frac{1}{2} \|\Delta \boldsymbol{\kappa}\|_{\mathbb{R}^n}^2 \right) \boldsymbol{\phi}'(s_1) \right\} ds_2 .
\end{aligned}$$

Moreover, $\boldsymbol{N}_1$ *satisfies*

$$||\boldsymbol{N}_1(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \leq C \left(||\boldsymbol{f}||_{H^{3-\alpha}} \right) ||\boldsymbol{\phi}||_{H^{3-\alpha}}.$$

Furthermore, we can extend the domain of the linear form

$$\boldsymbol{\psi} \mapsto \delta^2 \mathcal{E}_1(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}] = \langle P_{\boldsymbol{\tau}}^{\perp} L_1 \boldsymbol{\phi} - L_1(\boldsymbol{f} \cdot \boldsymbol{\tau}) \boldsymbol{\phi}' + \boldsymbol{N}_1(\boldsymbol{f})[\boldsymbol{\phi}], \boldsymbol{\psi} \rangle_{L^2}$$

to $\boldsymbol{\psi} \in L^2$.

4 The L^2-representation of the second variation of $\mathcal{E}_2$

4.1 The L^2-representation of K_{21}

We decompose the integrand H_{21} of K_{21} as

$$H_{21}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = H_{211}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] + H_{212}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}],$$

$$H_{211}[\boldsymbol{\phi},\boldsymbol{\psi}] = \frac{\tilde{Q}_1\boldsymbol{\phi}\cdot\tilde{Q}_2\boldsymbol{\psi}+\tilde{Q}_2\boldsymbol{\phi}\cdot\tilde{Q}_1\boldsymbol{\psi}}{2(\Delta s)^2},$$

$$H_{212}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \mathscr{M}(\boldsymbol{f})\frac{\tilde{Q}_1\boldsymbol{\phi}\cdot\tilde{Q}_2\boldsymbol{\psi}+\tilde{Q}_2\boldsymbol{\phi}\cdot\tilde{Q}_1\boldsymbol{\psi}}{2},$$

and give the L^2-representation of each part.

Proposition 4.1 *Let $\boldsymbol{\phi},\boldsymbol{\psi}\in H^3$. Then we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{211}[\boldsymbol{\phi},\boldsymbol{\psi}] = \langle L_2\boldsymbol{\phi},\boldsymbol{\psi}\rangle_{L^2}.$$

The proof is the same as [8, Proposition 3.1], so we omit the details.

Proposition 4.2 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{212}: H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^{3-\alpha}$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} H_{212}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 = \langle \boldsymbol{N}_{212}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover $\boldsymbol{N}_{212}(\boldsymbol{f})[\boldsymbol{\phi}]$ satisfies

$$||\boldsymbol{N}_{212}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2}\le C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}.$$

Proof Let $\varepsilon>0$. Interchanging the variables s_1 and s_2, we have

$$\begin{aligned}
&\iint_{|\Delta s|\ge\varepsilon} H_{212}[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2\\
&=\iint_{|\Delta s|\ge\varepsilon}\mathscr{M}(\boldsymbol{f})\frac{\tilde{Q}_1\boldsymbol{\phi}\cdot\tilde{Q}_2\boldsymbol{\psi}+\tilde{Q}_2\boldsymbol{\phi}\cdot\tilde{Q}_1\boldsymbol{\psi}}{2}ds_1ds_2\\
&=\iint_{|\Delta s|\ge\varepsilon}\mathscr{M}(\boldsymbol{f})\frac{\tilde{Q}_1\boldsymbol{\phi}\cdot\tilde{Q}_2\boldsymbol{\psi}}{2}ds_1ds_2+\iint_{|\Delta s|\ge\varepsilon}\mathscr{M}(\boldsymbol{f})\frac{\tilde{Q}_2\boldsymbol{\phi}\cdot\tilde{Q}_1\boldsymbol{\psi}}{2}ds_1ds_2\\
&=\iint_{|\Delta s|\ge\varepsilon}\mathscr{M}(\boldsymbol{f})\tilde{Q}_2\boldsymbol{\phi}\cdot\tilde{Q}_1\boldsymbol{\psi}ds_1ds_2\\
&=\iint_{|\Delta s|\ge\varepsilon}2\mathscr{M}(\boldsymbol{f})\tilde{Q}_2\boldsymbol{\phi}\cdot\left\{\boldsymbol{\psi}_1'-\frac{(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\Delta\boldsymbol{\psi}\right\}ds_1ds_2\\
&=\iint_{|\Delta s|\ge\varepsilon}2\mathscr{M}(\boldsymbol{f})\tilde{Q}_2\boldsymbol{\phi}\cdot\boldsymbol{\psi}_1'ds_1ds_2-\iint_{|\Delta s|\ge\varepsilon}2\mathscr{M}(\boldsymbol{f})\frac{(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\tilde{Q}_2\boldsymbol{\phi}\cdot\Delta\boldsymbol{\psi}ds_1ds_2\\
&=\iint_{|\Delta s|\ge\varepsilon}\boldsymbol{h}_{2121}(s_1,s_2)\cdot\boldsymbol{\psi}_1'ds_1ds_2+\iint_{|\Delta s|\ge\varepsilon}\boldsymbol{h}_{2122}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2,
\end{aligned}$$

where

$$\boldsymbol{h}_{2121}(s_1,s_2)=2\mathscr{M}(\boldsymbol{f})\tilde{Q}_2\boldsymbol{\phi},$$

$$\boldsymbol{h}_{2122}(s_1,s_2)=\frac{-2\mathscr{M}(\boldsymbol{f})(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)\tilde{Q}_2\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}.$$

First of all, we estimate $\boldsymbol{h}_{2121}(s_1,s_2)$. Since it holds that

$$\boldsymbol{h}_{2121}(s_1,s_2)=\mathcal{O}(\Delta s),$$

using Lemma 2.13, we have

$$\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{2121}(s_1,s_2)\cdot\boldsymbol{\psi}_1' ds_1ds_2\\
=&\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\{\boldsymbol{h}_{2121}(s-\varepsilon,s)\cdot(\boldsymbol{\psi}(s-\varepsilon)-\boldsymbol{\psi}(s))-\boldsymbol{h}_{2121}(s+\varepsilon,s)\cdot(\boldsymbol{\psi}(s+\varepsilon)-\boldsymbol{\psi}(s))\}\,ds\\
&-\iint_{|s_1-s_2|\geq\varepsilon}\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{2121}\right)(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2.
\end{aligned}$$

Since $\boldsymbol{h}_{2121}$ is bounded in $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$ and since $\boldsymbol{\psi}(s+\varepsilon)-\boldsymbol{\psi}(s)\to 0$ uniformly as $\varepsilon\to 0$,

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\{\boldsymbol{h}_{2121}(s-\varepsilon,s)\cdot(\boldsymbol{\psi}(s-\varepsilon)-\boldsymbol{\psi}(s))-\boldsymbol{h}_{2121}(s+\varepsilon,s)\cdot(\boldsymbol{\psi}(s+\varepsilon)-\boldsymbol{\psi}(s))\}\,ds\to 0$$

as $\varepsilon\to 0$. Lemmas 2.3–2.5 and 2.8 imply that

$$\begin{aligned}
&\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{2121}\right)(s_1,s_2)\\
=&2\left(\frac{\partial}{\partial s_1}\mathscr{M}(\boldsymbol{f})\right)\tilde{Q}_2\boldsymbol{\phi}+2\mathscr{M}(\boldsymbol{f})\left(\frac{\partial}{\partial s_1}\tilde{Q}_2\boldsymbol{\phi}\right)\\
=&2\frac{-2(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1)}{(\Delta s)^3}\tilde{Q}_2\boldsymbol{\phi}+2\mathscr{M}(\boldsymbol{f})\left[\frac{-2(T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}_2)}{\Delta s}\frac{\Delta\boldsymbol{\phi}}{\Delta s}-\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{2T_1^0\boldsymbol{\phi}}{\Delta s}\right.\\
&+\frac{2}{\Delta s}\left\{\frac{(\Delta s)^2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}-1\right\}\left\{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\boldsymbol{\phi}_1'\right\}\\
&\left.-\frac{4(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1)}{\Delta s}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right]\\
=&\frac{-4(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1)}{(\Delta s)^2}\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}+4\mathscr{M}(\boldsymbol{f})\left[\frac{-(T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}_2)}{\Delta s}\frac{\Delta\boldsymbol{\phi}}{\Delta s}-\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{T_1^0\boldsymbol{\phi}}{\Delta s}\right.\\
&+\frac{1}{\Delta s}\left\{\frac{(\Delta s)^2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}-1\right\}\left\{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\boldsymbol{\phi}_1'\right\}\\
&\left.-\frac{4(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1)}{\Delta s}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right]\\
=&\mathcal{O}(1).
\end{aligned}$$

Secondly, Lemmas 2.3 and 2.5 imply that $\boldsymbol{h}_{2122}$ is bounded. From Corollary 2.1, there exists a mapping $\boldsymbol{N}_{212}:H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^{3-\alpha}$, we have the representation

$$\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{212}(s_1,s_2)\cdot\boldsymbol{\psi}_1' ds_1ds_2=\langle\boldsymbol{N}_{212}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2},$$

and the estimate

$$\|\boldsymbol{N}_{212}(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^2}\leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}.$$

□

4.2 The L^2-representation of K_{22}

We can calculate K_{22} as

$$\begin{aligned}
K_{22} =& \lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon} H_{22}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2\\
=& \lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon} \frac{\tilde{S}_1(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_2(\boldsymbol{f},\boldsymbol{\psi})+\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_1(\boldsymbol{f},\boldsymbol{\psi})}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}ds_1ds_2\\
=& \lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon} \frac{\tilde{S}_1(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_2(\boldsymbol{f},\boldsymbol{\psi})}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}ds_1ds_2+\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon} \frac{\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_1(\boldsymbol{f},\boldsymbol{\psi})}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}ds_1ds_2\\
=& \lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon} \frac{\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_1(\boldsymbol{f},\boldsymbol{\psi})}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}ds_1ds_2+\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon} \frac{\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_1(\boldsymbol{f},\boldsymbol{\psi})}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}ds_1ds_2\\
=& \lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon} \frac{\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_1(\boldsymbol{f},\boldsymbol{\psi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}ds_1ds_2.
\end{aligned}$$

We decompose the integrand on the last line into two parts as

$$\begin{aligned}
&\frac{\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_1(\boldsymbol{f},\boldsymbol{\psi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\\
=&\frac{\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})(R\boldsymbol{f}\cdot\tilde{Q}_1\boldsymbol{\psi}+\tilde{Q}_1\boldsymbol{f}\cdot R\boldsymbol{\psi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\\
=&\frac{\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\left[\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot 2\left\{\boldsymbol{\psi}_1'-\frac{(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right\}+\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\left(\tilde{Q}_1\boldsymbol{f}\cdot\frac{\Delta\boldsymbol{\psi}}{\Delta s}\right)\right]\\
=&\frac{\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\left\{-\frac{2(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2\Delta s}+\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\right\}\cdot\Delta\boldsymbol{\psi}+\left(\frac{2|\Delta s|\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^3}\frac{\Delta\boldsymbol{f}}{\Delta s}\right)\cdot\boldsymbol{\psi}_1'\\
=&\boldsymbol{h}_{221}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}+\boldsymbol{h}_{222}(s_1,s_2)\cdot\boldsymbol{\psi}_1',
\end{aligned}$$

where

$$\boldsymbol{h}_{221}(s_1,s_2)=\frac{\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\left\{-\frac{2(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2\Delta s}+\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\right\},$$

$$\boldsymbol{h}_{222}(s_1,s_2)=\frac{2|\Delta s|\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^3}\frac{\Delta\boldsymbol{f}}{\Delta s}.$$

Lemma 4.1 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{221}:H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^{3-\alpha}$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}(\boldsymbol{h}_{221}(s_1,s_2)\cdot\Delta\boldsymbol{\psi})ds_1ds_2=\frac{1}{2}\langle(\boldsymbol{\tau}\cdot L_2\boldsymbol{\phi}+\boldsymbol{\phi}'\cdot L_2\boldsymbol{f})\boldsymbol{\tau},\boldsymbol{\psi}\rangle_{L^2}+\langle\boldsymbol{N}_{221}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover

$$||\boldsymbol{N}_{221}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2}\leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}$$

holds.

Proof The integrand $\boldsymbol{h}_{221}$ can be rewritten as

$$\begin{aligned}
\boldsymbol{h}_{221} =&\frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})\left\{-\frac{2(\Delta \boldsymbol{f}\cdot\boldsymbol{\tau}_1)\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2\Delta s}+\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\right\}\\
=&\frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}(R\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot R\boldsymbol{\phi})\left\{-\frac{2(\Delta \boldsymbol{f}\cdot\boldsymbol{\tau}_1)\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2\Delta s}+\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\right\}\\
=&\frac{1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}\frac{(\Delta s)^2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left\{-\frac{2(\Delta \boldsymbol{f}\cdot\boldsymbol{\tau}_1)\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2\Delta s}+\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\right\}\\
=&-\frac{2(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left\{\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{(\Delta s)^3}+\frac{\tilde{Q}_2\boldsymbol{f}}{(\Delta s)^3}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right\}\frac{\Delta \boldsymbol{f}}{\Delta s}\\
&+\frac{(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left\{\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{(\Delta s)^2}+\frac{\tilde{Q}_2\boldsymbol{f}}{(\Delta s)^2}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right\}\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\\
=&\frac{2(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left\{\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}+\frac{\tilde{Q}_{22}\boldsymbol{f}}{(\Delta s)^3}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right\}\frac{\Delta \boldsymbol{f}}{\Delta s}\\
&+\frac{2(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left\{\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\bar{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}+\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^3}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right\}\frac{\Delta \boldsymbol{f}}{\Delta s}\\
&+\frac{1}{\Delta s}\frac{(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left\{\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}+\frac{\tilde{Q}_2\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right\}\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}.
\end{aligned}$$

Now we put

$$\boldsymbol{h}_{221}=\boldsymbol{h}_{2211}+\boldsymbol{h}_{2212},$$

where

$$\begin{aligned}
\boldsymbol{h}_{2211} =&\frac{2(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left\{\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}+\frac{\tilde{Q}_{22}\boldsymbol{f}}{(\Delta s)^3}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right\}\frac{\Delta \boldsymbol{f}}{\Delta s},\\
\boldsymbol{h}_{2212} =&\frac{1}{\Delta s}\frac{2(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left\{\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\bar{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^2}+\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right\}\frac{\Delta \boldsymbol{f}}{\Delta s}\\
&+\frac{1}{\Delta s}\frac{(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left\{\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}+\frac{\tilde{Q}_2\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right\}\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}.
\end{aligned}$$

Now we verify that $\boldsymbol{h}_{2211}$ satisfy the assumptions of Lemma 2.12. We have

$$\begin{aligned}
\boldsymbol{h}_{2211} =&\left\{\left(\frac{\tilde{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^3}\cdot\frac{\Delta \boldsymbol{f}}{\Delta s}\right)\frac{2(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\right\}\frac{\Delta \boldsymbol{f}}{\Delta s}\\
&+\left\{\left(\frac{\tilde{Q}_{22}\boldsymbol{f}}{(\Delta s)^3}\cdot\frac{\Delta \boldsymbol{\phi}}{\Delta s}\right)\frac{2(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\right\}\frac{\Delta \boldsymbol{f}}{\Delta s},
\end{aligned}$$

and Lemma 2.4 implies that

$$\begin{aligned}
\frac{\Delta \boldsymbol{f}}{\Delta s} =&\boldsymbol{\tau}_i+\Delta s\frac{(-1)^i}{2}\boldsymbol{\phi}_i''+\mathcal{O}^{\frac{3}{2}+\alpha},\\
\frac{2(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\frac{\Delta \boldsymbol{f}}{\Delta s} =&(1+\mathcal{O}^2)(\boldsymbol{\tau}_i+\Delta s\frac{(-1)^i}{2}\boldsymbol{\phi}_i''+\mathcal{O}(\Delta s)^{\frac{3}{2}+\alpha})\cdot\boldsymbol{\tau}_1\\
&\times(\boldsymbol{\tau}_i+\Delta s\frac{(-1)^i}{2}\boldsymbol{\phi}_i''+\mathcal{O}(\Delta s)^{\frac{3}{2}+\alpha})\\
=&(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_1)\boldsymbol{\tau}_i+\Delta s(-1)^i\boldsymbol{\phi}_i''+\mathcal{O}(\Delta s)^{\frac{3}{2}+\alpha}.
\end{aligned}$$

Moreover, Lemmas 2.1 and 2.5

$$\begin{aligned}&\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left|(\Delta s)\boldsymbol{h}_{2212}-2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_1)\left\{(-1)^{i-1}\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i''+(-1)^{i-1}\boldsymbol{\kappa}_i\cdot\boldsymbol{\phi}_i'\right\}\boldsymbol{\tau}_i\right.\\&\left.-\left(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i''+\boldsymbol{\kappa}_i\cdot\boldsymbol{\phi}_i'\right)\boldsymbol{\kappa}_i\right|\\&\leq C\|\boldsymbol{f}\|_{H^{3-\alpha}}^2\|\boldsymbol{\phi}\|_{H^{3-\alpha}}.\end{aligned}$$

From the above, we can apply Lemma 2.12 to $\boldsymbol{h}_{2211}$ and Lemmas 2.5, Corollary 2.1 to $\boldsymbol{h}_{2212}$ and thus we can find $\boldsymbol{N}_{2211}(\boldsymbol{f})[\boldsymbol{\phi}]$, $\boldsymbol{N}_{2212}(\boldsymbol{f})[\boldsymbol{\phi}]$ such that

$$\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{2211}\cdot\Delta\boldsymbol{\psi}ds_1ds_2=\frac{1}{2}\langle(\boldsymbol{\tau}\cdot L_2\boldsymbol{\phi}+\boldsymbol{\phi}'\cdot L_2\boldsymbol{f})\boldsymbol{\tau},\boldsymbol{\psi}\rangle_{L^2}+\langle\boldsymbol{N}_{2211}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2},$$

$$\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{2212}\cdot\Delta\boldsymbol{\psi}ds_1ds_2=\langle\boldsymbol{N}_{2212}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover it holds that

$$\begin{aligned}\|\boldsymbol{N}_{2211}(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^2}&\leq C\|\boldsymbol{f}\|_{H^{3-\alpha}}\|\boldsymbol{\phi}\|_{H^{3-\alpha}},\\\|\boldsymbol{N}_{2212}(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^2}&\leq C\|\boldsymbol{f}\|_{H^{3-\alpha}}\|\boldsymbol{\phi}\|_{H^{3-\alpha}}.\end{aligned}$$

The assertions follow from the fact that $\boldsymbol{h}_{221}=\boldsymbol{h}_{2211}+\boldsymbol{h}_{2212}$. □

Lemma 4.2 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{222}:H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^{3-\alpha}$, we have*

$$\begin{aligned}\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\boldsymbol{h}_{222}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2=&-\langle\frac{3}{2}(\boldsymbol{\tau}\cdot L_2\boldsymbol{\phi})\boldsymbol{\tau}+3(\boldsymbol{\tau}\cdot L_2\boldsymbol{f})(\boldsymbol{\phi}'\cdot\boldsymbol{\tau})\boldsymbol{\tau}+\frac{3}{2}(\boldsymbol{\phi}'\cdot L_2\boldsymbol{f})\boldsymbol{\tau},\boldsymbol{\psi}\rangle_{L^2}\\&+\langle\boldsymbol{N}_{222}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle.\end{aligned}$$

Moreover

$$||\boldsymbol{N}_{222}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2}\leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}$$

holds.

Proof We compute $\boldsymbol{h}_{222}\cdot\Delta\boldsymbol{\psi}$ as follows:

$$\begin{aligned}\boldsymbol{h}_{222}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}=&\frac{2|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^3}(R\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot R\boldsymbol{\phi})\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\Delta\boldsymbol{\psi}\right)\\=&\frac{2|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^3}\left\{\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\tilde{Q}_2\boldsymbol{\phi}\right)+\frac{|\Delta s|}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}}\left(\tilde{Q}_2\boldsymbol{f}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\right\}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\Delta\boldsymbol{\psi}\right)\\=&\frac{2(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}+\frac{\tilde{Q}_2\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\psi}}{\Delta s}\right)\\=&2(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1''+\boldsymbol{\kappa}_1\cdot\boldsymbol{\phi}_1')(\boldsymbol{\tau}_1\cdot\boldsymbol{\psi}_1')+\mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}.\end{aligned}$$

From Lemma 2.13, we have

$$\begin{aligned}&\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{222}(s_1,s_2)\cdot\boldsymbol{\psi}_1'ds_1ds_2\\&=-\iint_{|\Delta s|\geq\varepsilon}\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{222}\right)(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2+\mathcal{O}(\varepsilon)^{\frac{1}{2}-\alpha}.\end{aligned}$$

By using Lemma 2.5, the derivative of $\boldsymbol{h}_{222}$ may be calculated as follows.

$$
\begin{aligned}
\frac{\partial}{\partial s_1}\boldsymbol{h}_{222}(s_1,s_2) =& \frac{2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\boldsymbol{\tau}_1\cdot\tilde{Q}_2\boldsymbol{\phi}+\Delta\boldsymbol{f}\cdot\frac{\partial\tilde{Q}_2\boldsymbol{\phi}}{\partial s_1}+\frac{\partial\tilde{Q}_2\boldsymbol{f}}{\partial s_1}\cdot\Delta\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot\boldsymbol{\phi}_1'\right)\Delta\boldsymbol{f} \\
&+\frac{2(\Delta\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\boldsymbol{\tau}_1}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}-\frac{8(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)(\Delta\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6} \\
=&\frac{2(\boldsymbol{\tau}_1\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot\boldsymbol{\phi}_1')\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4} \\
&+\frac{2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\Delta\boldsymbol{f}\cdot\frac{\partial\tilde{Q}_2\boldsymbol{\phi}}{\partial s_1}\right)\Delta\boldsymbol{f}+\frac{2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\partial\tilde{Q}_2\boldsymbol{f}}{\partial s_1}\cdot\Delta\boldsymbol{\phi}\right)\Delta\boldsymbol{f} \\
&+\frac{2(\Delta\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\boldsymbol{\tau}_1}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}-\frac{8(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)(\Delta\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6} \\
=&-\frac{2(\boldsymbol{\tau}_1\cdot\tilde{Q}_{22}\boldsymbol{\phi}+\tilde{Q}_{22}\boldsymbol{f}\cdot\boldsymbol{\phi}_1')\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}-\frac{2(\boldsymbol{\tau}_1\cdot\bar{Q}_{22}\boldsymbol{\phi}+\bar{Q}_{22}\boldsymbol{f}\cdot\boldsymbol{\phi}_1')\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4} \\
&+\frac{2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left\{\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\left(\frac{\Delta\boldsymbol{\phi}}{\Delta s}\cdot\Delta\boldsymbol{f}\right)+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\left(\frac{\tilde{Q}_{21}\boldsymbol{\phi}}{\Delta s}\cdot\Delta\boldsymbol{f}\right)\right. \\
&\left.+\frac{2(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\left(\frac{\Delta\boldsymbol{\phi}}{\Delta s}\cdot\Delta\boldsymbol{f}\right)+\hat{Q}_2\boldsymbol{\phi}\cdot\Delta\boldsymbol{f}\right\}\Delta\boldsymbol{f} \\
&+\frac{2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left\{\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\Delta\boldsymbol{\phi}\right)+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\Delta\boldsymbol{\phi}\right)\right. \\
&\left.+\frac{2(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\Delta\boldsymbol{\phi}\right)+\hat{Q}_2\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}\right\}\Delta\boldsymbol{f} \\
&-\frac{2(\Delta\boldsymbol{f}\cdot\tilde{Q}_{22}\boldsymbol{\phi}+\tilde{Q}_{22}\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\boldsymbol{\tau}_1}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}+\frac{8(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)(\Delta\boldsymbol{f}\cdot\tilde{Q}_{22}\boldsymbol{\phi}+\tilde{Q}_{22}\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6} \\
&-\frac{2(\Delta\boldsymbol{f}\cdot\bar{Q}_{22}\boldsymbol{\phi}+\bar{Q}_{22}\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\boldsymbol{\tau}_1}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}+\frac{8(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)(\Delta\boldsymbol{f}\cdot\bar{Q}_{22}\boldsymbol{\phi}+\bar{Q}_{22}\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6} \\
=&\boldsymbol{h}_{2221}(s_1,s_2)+\boldsymbol{h}_{2222}(s_1,s_2),
\end{aligned}
$$

where $\boldsymbol{h}_{2221}$ consists of terms of order $\mathcal{O}(\Delta s)^{-2}$, and we let $\boldsymbol{h}_{2222}$ be the remaining terms. That is,

$$
\begin{aligned}
\boldsymbol{h}_{2221} =& -\frac{2(\boldsymbol{\tau}_1 \cdot \tilde{Q}_{22}\boldsymbol{\phi} + \tilde{Q}_{22}\boldsymbol{f} \cdot \boldsymbol{\phi}_1')\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \\
&+ \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left\{ \left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \left(\frac{\Delta \boldsymbol{\phi}}{\Delta s} \cdot \Delta \boldsymbol{f} \right) + \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \left(\frac{\tilde{Q}_{21}\boldsymbol{\phi}}{\Delta s} \cdot \Delta \boldsymbol{f} \right) \right. \\
&\left. + \frac{2(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_1 \right) \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \left(\frac{\Delta \boldsymbol{\phi}}{\Delta s} \cdot \Delta \boldsymbol{f} \right) \right\} \Delta \boldsymbol{f} \\
&+ \frac{2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left\{ \left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \Delta \boldsymbol{\phi} \right) + \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s} \cdot \Delta \boldsymbol{\phi} \right) \right. \\
&\left. + \frac{2(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_1 \right) \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \Delta \boldsymbol{\phi} \right) \right\} \Delta \boldsymbol{f} \\
&- \frac{2(\Delta \boldsymbol{f} \cdot \tilde{Q}_{22}\boldsymbol{\phi} + \tilde{Q}_{22}\boldsymbol{f} \cdot \Delta \boldsymbol{\phi})\boldsymbol{\tau}_1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} + \frac{8(\Delta \boldsymbol{f} \cdot \boldsymbol{\tau}_1)(\Delta \boldsymbol{f} \cdot \tilde{Q}_{22}\boldsymbol{\phi} + \tilde{Q}_{22}\boldsymbol{f} \cdot \Delta \boldsymbol{\phi})\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6} \\
=& -\frac{2}{(\Delta s)^2} \frac{(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left(\boldsymbol{\tau}_1 \cdot \frac{\tilde{Q}_{22}\boldsymbol{\phi}}{\Delta s} + \frac{\tilde{Q}_{22}\boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\phi}_1' \right) \frac{\Delta \boldsymbol{f}}{\Delta s} \\
&+ \frac{2}{(\Delta s)^2} \frac{(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left\{ 2 \left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \left(\frac{\Delta \boldsymbol{\phi}}{\Delta s} \cdot \frac{\Delta \boldsymbol{f}}{\Delta s} \right) \right. \\
&+ \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \left(\frac{\tilde{Q}_{21}\boldsymbol{\phi}}{\Delta s} \cdot \frac{\Delta \boldsymbol{f}}{\Delta s} + \frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s} \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s} \right) \\
&\left. + \frac{4(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_1 \right) \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2 \right) \left(\frac{\Delta \boldsymbol{\phi}}{\Delta s} \cdot \frac{\Delta \boldsymbol{f}}{\Delta s} \right) \right\} \frac{\Delta \boldsymbol{f}}{\Delta s} \\
&- \frac{2}{(\Delta s)^2} \frac{(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \frac{\tilde{Q}_{22}\boldsymbol{\phi}}{\Delta s} + \frac{\tilde{Q}_{22}\boldsymbol{f}}{\Delta s} \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s} \right) \boldsymbol{\tau}_1 \\
&+ \frac{8}{(\Delta s)^2} \frac{(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6} \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_1 \right) \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \frac{\tilde{Q}_{22}\boldsymbol{\phi}}{\Delta s} + \frac{\tilde{Q}_{22}\boldsymbol{f}}{\Delta s} \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s} \right) \frac{\Delta \boldsymbol{f}}{\Delta s}, \\
\boldsymbol{h}_{2222} =& -\frac{2(\boldsymbol{\tau}_1 \cdot \bar{Q}_{22}\boldsymbol{\phi} + \bar{Q}_{22}\boldsymbol{f} \cdot \boldsymbol{\phi}_1')\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} + \frac{(2\hat{Q}_2\boldsymbol{\phi} \cdot \Delta \boldsymbol{f})\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} + \frac{(2\hat{Q}_2\boldsymbol{f} \cdot \Delta \boldsymbol{\phi})\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \\
&- \frac{2(\Delta \boldsymbol{f} \cdot \bar{Q}_{22}\boldsymbol{\phi} + \bar{Q}_{22}\boldsymbol{f} \cdot \Delta \boldsymbol{\phi})\boldsymbol{\tau}_1}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} + \frac{8(\Delta \boldsymbol{f} \cdot \boldsymbol{\tau}_1)(\Delta \boldsymbol{f} \cdot \bar{Q}_{22}\boldsymbol{\phi} + \bar{Q}_{22}\boldsymbol{f} \cdot \Delta \boldsymbol{\phi})\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6} \\
=& \frac{2}{\Delta s} \frac{(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left\{ -\boldsymbol{\tau}_1 \cdot \frac{\bar{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^2} - \frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2} \cdot \boldsymbol{\phi}_1' + \frac{\hat{Q}_2\boldsymbol{\phi}}{\Delta s} \cdot \frac{\Delta \boldsymbol{f}}{\Delta s} + \frac{\hat{Q}_2\boldsymbol{f}}{\Delta s} \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s} \right\} \frac{\Delta \boldsymbol{f}}{\Delta s} \\
&- \frac{2}{\Delta s} \frac{(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4} \left\{ \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \frac{\bar{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^2} + \frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2} \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s} \right\} \boldsymbol{\tau}_1 \\
&+ \frac{8}{\Delta s} \frac{(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6} \left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_1 \right) \left\{ \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \frac{\bar{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^2} + \frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2} \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s} \right\} \frac{\Delta \boldsymbol{f}}{\Delta s}.
\end{aligned}
$$

Firstly, we consider the integrability of $\boldsymbol{h}_{2221}$. We have

$$\frac{(\Delta s)^{2n}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^{2n}} = 1 + \mathcal{O}(\Delta s)^2$$

for $n \in \mathbb{N}$. From Lemma 2.2,

$$\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 + (\Delta s)\boldsymbol{\kappa}_2 + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}, \qquad \boldsymbol{\phi}_1' = \boldsymbol{\phi}_2' + (\Delta s)\boldsymbol{\phi}_2'' + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}.$$

hold since $\boldsymbol{\tau}, \boldsymbol{\phi} \in H^{2-\alpha}$. Lemma 2.4 implies

$$\frac{\Delta \boldsymbol{f}}{\Delta s} = \boldsymbol{\tau}_i + (\Delta s)\frac{(-1)^i}{2} + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}.$$

Hence, each term of

$$-\frac{2}{(\Delta s)^2}\frac{(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\boldsymbol{\tau}_1 \cdot \frac{\tilde{Q}_{22}\boldsymbol{\phi}}{\Delta s} + \frac{\tilde{Q}_{22}\boldsymbol{f}}{\Delta s}\cdot \boldsymbol{\phi}_1'\right)\frac{\Delta \boldsymbol{f}}{\Delta s}$$

satisfies the assertions of Lemma 2.12. Secondly, using Lemmas 2.1, 2.4, we have

$$\begin{aligned}\frac{\Delta \boldsymbol{\phi}}{\Delta s}\cdot\frac{\Delta \boldsymbol{f}}{\Delta s} =& \{\boldsymbol{\phi}_i' + (\Delta s)\frac{(-1)^i}{2}\boldsymbol{\phi}_i'' + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\}\cdot\{\boldsymbol{\tau}_i + (\Delta s)\frac{(-1)^i}{2}\boldsymbol{\kappa}_i + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\}\\ =& \boldsymbol{\phi}_i'\cdot\boldsymbol{\tau}_i + (\Delta s)\frac{(-1)^i}{2}(\boldsymbol{\phi}_i'\cdot\boldsymbol{\kappa}_i + \boldsymbol{\phi}_i''\cdot\boldsymbol{\tau}_i) + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha},\end{aligned}$$

$$\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2 = 1 + \mathcal{O}(\Delta s)^2.$$

Thus each term of remaining part of $\boldsymbol{h}_{2221}$ satisfies the assertions of Lemma 2.12. Using Lemma 2.12 for $\boldsymbol{h}_{2221}$, we obtain that there exists an L^2 function $\boldsymbol{N}_{2221}(\boldsymbol{f})[\boldsymbol{\phi}]$ such that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}(-\boldsymbol{h}_{2221})\cdot\Delta\boldsymbol{\psi}\,ds_1ds_2 =& \langle\frac{1}{2}(\boldsymbol{\tau}\cdot L_2\boldsymbol{\phi} + \boldsymbol{\phi}'\cdot L_2\boldsymbol{f})\boldsymbol{\tau} - (\boldsymbol{\tau}\cdot L_2\boldsymbol{f})(\boldsymbol{\phi}'\cdot\boldsymbol{\tau})\boldsymbol{\tau}\\ &- \frac{1}{2}(\boldsymbol{\tau}\cdot L_2\boldsymbol{\phi} + \boldsymbol{\phi}'\cdot L_2\boldsymbol{f})\boldsymbol{\tau} - 2(\boldsymbol{\tau}\cdot L_2\boldsymbol{f})(\boldsymbol{\phi}'\cdot\boldsymbol{\tau})\boldsymbol{\tau}\\ &+ \frac{1}{2}(\boldsymbol{\tau}\cdot L_2\boldsymbol{\phi} + \boldsymbol{\phi}'\cdot L_2\boldsymbol{f})\boldsymbol{\tau} - 2(\boldsymbol{\tau}\cdot L_2\boldsymbol{\phi} + \boldsymbol{\phi}'\cdot L_2\boldsymbol{f})\boldsymbol{\tau}\\ &+ \boldsymbol{N}_{2221}(\boldsymbol{f})[\boldsymbol{\phi}], \boldsymbol{\psi}\rangle_{L^2}\\ =& \langle -\frac{3}{2}(\boldsymbol{\tau}\cdot L_2\boldsymbol{\phi})\boldsymbol{\tau} - 3(\boldsymbol{\tau}\cdot L_2\boldsymbol{f})(\boldsymbol{\phi}'\cdot\boldsymbol{\tau})\boldsymbol{\tau} - \frac{3}{2}(\boldsymbol{\phi}'\cdot L_2\boldsymbol{f})\boldsymbol{\tau}, \boldsymbol{\psi}\rangle_{L^2}\\ &+ \langle\boldsymbol{N}_{2221}(\boldsymbol{f})[\boldsymbol{\phi}], \boldsymbol{\psi}\rangle_{L^2},\end{aligned}$$

and there exists a constant $C(\|\boldsymbol{f}\|_{H^{3-\alpha}})$ such that the estimate

$$\|\boldsymbol{N}_{2221}(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^2} \leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}$$

holds.

Next, we consider the integrability of $\boldsymbol{h}_{2222}$. Using Lemmas 2.1, 2.3, 2.7 we have

$$\begin{aligned}\frac{\hat{Q}_2\boldsymbol{\phi}}{\Delta s} =& 2\mathscr{M}(\boldsymbol{f})\left\{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\frac{\Delta\boldsymbol{\phi}}{\Delta s} + \left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\boldsymbol{\phi}_1'\right\} - 4\mathscr{M}^4(\boldsymbol{f})\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{\phi}}{\Delta s}\\ =& \left(\frac{\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{12} + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}\right)\{(1+\mathcal{O}(\Delta s))(\boldsymbol{\phi}_i' + \mathcal{O}(\Delta s)) + (1+\mathcal{O}(\Delta s)^2)(\boldsymbol{\phi}_i' + \mathcal{O}(\Delta s))\}\\ &- 4\left(\frac{2\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{12} + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}\right)(1+\mathcal{O}(\Delta s)^2)(\boldsymbol{\phi}_i' + \mathcal{O}(\Delta s))\\ =& \frac{\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{6}(\boldsymbol{\phi}_i' + \boldsymbol{\phi}_i') - \frac{2\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{3}\boldsymbol{\phi}_i' + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}\\ =& -\frac{\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{3}\boldsymbol{\phi}_i' + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}.\end{aligned}$$

In addition to Lemma 2.5, the following estimates hold:

$$\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left\|\frac{2(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left\{-\boldsymbol{\tau}_1\cdot\frac{\bar{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^2}-\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2}\cdot\boldsymbol{\phi}_1'+\frac{\hat{Q}_2\boldsymbol{\phi}}{\Delta s}\cdot\frac{\Delta \boldsymbol{f}}{\Delta s}+\frac{\hat{Q}_2\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta \boldsymbol{\phi}}{\Delta s}\right\}\frac{\Delta \boldsymbol{f}}{\Delta s}\right.$$
$$\left.+\frac{8\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{3}(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i)\boldsymbol{\tau}_i\right\|_{\mathbb{R}^n}$$
$$\leq C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{H^{3-\alpha}},$$

$$\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left\|-\frac{2(\Delta s)^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left\{\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\bar{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^2}+\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2}\cdot\frac{\Delta \boldsymbol{\phi}}{\Delta s}\right\}\boldsymbol{\tau}_1+\frac{4\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{3}(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i)\boldsymbol{\tau}_i\right\|_{\mathbb{R}^n}$$
$$\leq C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{H^{3-\alpha}},$$

$$\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left\|\frac{8(\Delta s)^6}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left\{\frac{\Delta \boldsymbol{f}}{\Delta s}\cdot\frac{\bar{Q}_{22}\boldsymbol{\phi}}{(\Delta s)^2}+\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2}\cdot\frac{\Delta \boldsymbol{\phi}}{\Delta s}\right\}\frac{\Delta \boldsymbol{f}}{\Delta s}-\frac{8\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{3}(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i)\boldsymbol{\tau}_i\right\|_{\mathbb{R}^n}$$
$$\leq C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}.$$

Hence Corollary 2.1 implies that there exists an L^2 function $\boldsymbol{N}_{2222}(\boldsymbol{f})[\boldsymbol{\phi}]$ such that

$$\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}(-\boldsymbol{h}_{2222})\cdot\Delta\boldsymbol{\psi}\,ds_1ds_2=\langle \boldsymbol{N}_{2222}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2},$$

and there exists a constant $C(\|\boldsymbol{f}\|_{H^{3-\alpha}})$ such that the estimate

$$\|\boldsymbol{N}_{2222}(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^2}\leq C(\|\boldsymbol{f}\|_{H^{3-\alpha}})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}$$

holds. □

4.3 The L^2-representation of K_{23}

Lemma 4.3 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{23}:H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^{3-\alpha}$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\boldsymbol{H}_{23}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2=\langle \boldsymbol{N}_{23}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover

$$||\boldsymbol{N}_{23}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2}\leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}$$

holds.

Proof Observe that

$$
\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon} H_{23}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 \\
&=-\iint_{|\Delta s|\geq\varepsilon} \frac{2\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}ds_1ds_2 \\
&=-\iint_{|\Delta s|\geq\varepsilon} \frac{2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\left(\frac{\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot\tilde{Q}_1\boldsymbol{\phi}}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^2}-\frac{2\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^2}\right)\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&=\iint_{|\Delta s|\geq\varepsilon}\left\{-\frac{(\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}-\frac{(\tilde{Q}_2\boldsymbol{f}\cdot\tilde{Q}_1\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}+\frac{4\mathscr{M}_2(\boldsymbol{f})(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\right\}\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&=-\iint_{|\Delta s|\geq\varepsilon}\frac{(\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\cdot\Delta\boldsymbol{\psi}ds_1ds_2-\iint_{|\Delta s|\geq\varepsilon}\frac{(\tilde{Q}_2\boldsymbol{f}\cdot\tilde{Q}_1\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&\quad+\iint_{|\Delta s|\geq\varepsilon}\frac{4\mathscr{M}_2(\boldsymbol{f})(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&=-\iint_{|\Delta s|\geq\varepsilon}\frac{(\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\cdot\Delta\boldsymbol{\psi}ds_1ds_2+\iint_{|\Delta s|\geq\varepsilon}\frac{(\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\cdot(-\Delta\boldsymbol{\psi})ds_2ds_1 \\
&\quad+\iint_{|\Delta s|\geq\varepsilon}\frac{4\mathscr{M}_2(\boldsymbol{f})(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&=-\iint_{|\Delta s|\geq\varepsilon}\frac{2(\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\cdot\Delta\boldsymbol{\psi}ds_1ds_2+\iint_{|\Delta s|\geq\varepsilon}\frac{4\mathscr{M}_2(\boldsymbol{f})(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&=\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{231}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2+\iint_{|\Delta s|\geq\varepsilon}\boldsymbol{h}_{232}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2,
\end{aligned}
$$

where

$$
\boldsymbol{h}_{231}(s_1,s_2)=-\frac{2(\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4},
$$

$$
\boldsymbol{h}_{232}(s_1,s_2)=-\frac{4\mathscr{M}_2(\boldsymbol{f})(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}.
$$

Now, we have

$$
\begin{aligned}
\boldsymbol{h}_{231}(s_1,s_2)&=-\frac{2(\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4} \\
&=-\frac{2}{\Delta s}\frac{(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\left(\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{f}}{\Delta s} \\
&=-\frac{2}{\Delta s}\{1+(\Delta s)^2\mathscr{M}^4(\boldsymbol{f})\}\left(\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{f}}{\Delta s} \\
&=-\frac{2}{\Delta s}\left(\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{f}}{\Delta s}-2\mathscr{M}^4(\boldsymbol{f})\left(\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}\right)\Delta\boldsymbol{f}
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{h}_{232}(s_1,s_2) &= -\frac{4\mathscr{M}_2(\boldsymbol{f})(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4} \\
&= -\frac{4}{\Delta s}\frac{(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}}^4}\mathscr{M}_2(\boldsymbol{f})\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{f}}{\Delta s} \\
&= -\frac{4}{\Delta s}\{1+(\Delta s\}^2\mathscr{M}^4(\boldsymbol{f}))\mathscr{M}_2(\boldsymbol{f})\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{f}}{\Delta s} \\
&= -\frac{4}{\Delta s}\mathscr{M}_2(\boldsymbol{f})\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{f}}{\Delta s} - 4\mathscr{M}^4(\boldsymbol{f})\mathscr{M}_2(\boldsymbol{f})\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\Delta\boldsymbol{f}.
\end{aligned}
$$

Using Lemmas 2.1, 2.6 and 2.5, we know that there exists a constant $C(\boldsymbol{f})$ such that

$$
\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left|\left(\frac{\tilde{Q}_1\boldsymbol{f}}{\Delta s}\cdot\frac{\tilde{Q}_2\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{f}}{\Delta s}-(\boldsymbol{\kappa}_i\cdot\boldsymbol{\phi}_i'')\boldsymbol{\tau}_i\right| \leq C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{H^{3-\alpha}},
$$

$$
\begin{aligned}
\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left|\mathscr{M}(\boldsymbol{f})\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta\boldsymbol{f}}{\Delta s}-(-\frac{1}{2})\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i)\boldsymbol{\tau}_i\right| &\leq C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{C^1} \\
&\leq C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}
\end{aligned}
$$

for $i\in\{1,2\}$. Since $\boldsymbol{h}_{231}$ and $\boldsymbol{h}_{232}$ satisfy the assumption of Corollary 2.1, we obtain the claimed assertions.

□

4.4 The L^2-representation of K_{24}

We decompose $H_{24}[\boldsymbol{\phi},\boldsymbol{\psi}]$ as follows:

$$
\begin{aligned}
H_{24} =& -\frac{2\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} \\
=& -\frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\left\{\frac{\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\psi}+\tilde{Q}_2\boldsymbol{f}\cdot\tilde{Q}_1\boldsymbol{\psi}}{2\|\Delta\boldsymbol{f}\|^2}-\frac{2\mathscr{M}_2(\boldsymbol{f})(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right\} \\
=&\frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left[\tilde{Q}_1\boldsymbol{f}\cdot\left\{\boldsymbol{\psi}_2'-\frac{(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_2)\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right\}\right. \\
&\left.-\tilde{Q}_2\boldsymbol{f}\cdot\left\{\boldsymbol{\psi}_1'-\frac{(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right\}\right]+\frac{4(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\cdot\Delta\boldsymbol{\psi} \\
=&\left[\frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left\{-(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_2)\tilde{Q}_1\boldsymbol{f}+(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)\tilde{Q}_2\boldsymbol{f}\right\}+\frac{4(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\right]\cdot\Delta\boldsymbol{\psi} \\
&-\frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\tilde{Q}_2\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\cdot\boldsymbol{\psi}_1'+\frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\tilde{Q}_1\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\cdot\boldsymbol{\psi}_2'.
\end{aligned}
$$

Setting

$$
\boldsymbol{h}_{241}(s_1,s_2)=\frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)\tilde{Q}_2\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6},
$$

$$
\boldsymbol{h}_{242}(s_1,s_2)=\frac{4(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4},
$$

$$\boldsymbol{h}_{243}(s_1,s_2) = -\frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\tilde{Q}_2\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4},$$

we have

$$\begin{aligned}
\boldsymbol{h}_{241}(s_2,s_1) =& \frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})(-\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_2)(-\tilde{Q}_1\boldsymbol{f})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6} \\
=& \frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_2)(\tilde{Q}_1\boldsymbol{f})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}, \\
\boldsymbol{h}_{243}(s_2,s_1) =& -\frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})(-\tilde{Q}_1\boldsymbol{f})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4} \\
=& \frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\tilde{Q}_1\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}.
\end{aligned}$$

Then, we can calculate

$$\begin{aligned}
&\iint_{|\Delta s|\geq\varepsilon} H_{24}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2 \\
=&\iint_{|\Delta s|\geq\varepsilon} (-\boldsymbol{h}_{241}(s_2,s_1)+\boldsymbol{h}_{241}(s_1,s_2)+\boldsymbol{h}_{242}(s_1,s_2))\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&+\iint_{|\Delta s|\geq\varepsilon} (\boldsymbol{h}_{243}(s_1,s_2)\cdot\boldsymbol{\psi}_1'+\boldsymbol{h}_{243}(s_2,s_1)\cdot\boldsymbol{\psi}_2')ds_1ds_2 \\
=&\iint_{|\Delta s|\geq\varepsilon} (-\boldsymbol{h}_{241}(s_2,s_1))\cdot\Delta\boldsymbol{\psi}ds_1ds_2 + \iint_{|\Delta s|\geq\varepsilon} \boldsymbol{h}_{241}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&+\iint_{|\Delta s|\geq\varepsilon} \boldsymbol{h}_{242}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&+\iint_{|\Delta s|\geq\varepsilon} \boldsymbol{h}_{243}(s_1,s_2)\cdot\boldsymbol{\psi}_1'ds_1ds_2 + \iint_{|\Delta s|\geq\varepsilon} \boldsymbol{h}_{243}(s_2,s_1)\cdot\boldsymbol{\psi}_2'ds_1ds_2 \\
=&\iint_{|\Delta s|\geq\varepsilon} (-\boldsymbol{h}_{241}(s_1,s_2))\cdot(-\Delta\boldsymbol{\psi})ds_1ds_2 + \iint_{|\Delta s|\geq\varepsilon} \boldsymbol{h}_{241}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&+\iint_{|\Delta s|\geq\varepsilon} \boldsymbol{h}_{242}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&+\iint_{|\Delta s|\geq\varepsilon} \boldsymbol{h}_{243}(s_1,s_2)\cdot\boldsymbol{\psi}_1'ds_1ds_2 + \iint_{|\Delta s|\geq\varepsilon} \boldsymbol{h}_{243}(s_1,s_2)\cdot\boldsymbol{\psi}_1'ds_1ds_2 \\
=&\iint_{|\Delta s|\geq\varepsilon} 2\boldsymbol{h}_{241}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2 + \iint_{|\Delta s|\geq\varepsilon} \boldsymbol{h}_{242}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2 \\
&+\iint_{|\Delta s|\geq\varepsilon} 2\boldsymbol{h}_{243}(s_1,s_2)\cdot\boldsymbol{\psi}_1'ds_1ds_2.
\end{aligned}$$

Lemma 4.4 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{241}: H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^{3-\alpha}$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} 2\boldsymbol{h}_{241}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2 = -\langle(\boldsymbol{\tau}\cdot\boldsymbol{\phi}')L_2\boldsymbol{f},\boldsymbol{\psi}\rangle_{L^2}+\langle\boldsymbol{N}_{241}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover

$$||\boldsymbol{N}_{241}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2}\leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}$$

holds.

Proof We decompose the integrand $\boldsymbol{h}_{241}$ as

$$\begin{aligned}\boldsymbol{h}_{241}(s_1,s_2)=&\frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)\tilde{Q}_2\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\\ =&\frac{2}{(\Delta s)^2}\frac{(\Delta s)^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\frac{\tilde{Q}_2\boldsymbol{f}}{\Delta s}\\ =&-\frac{2}{(\Delta s)^2}\frac{(\Delta s)^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\frac{\tilde{Q}_{22}\boldsymbol{f}}{\Delta s}\\ &-\frac{2}{\Delta s}\frac{(\Delta s)^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2}\\ =&\boldsymbol{h}_{2411}+\boldsymbol{h}_{2412},\end{aligned}$$

where $\boldsymbol{h}_{2411}$ is the term of $\mathcal{O}(\Delta s)^{-2}$, and $\boldsymbol{h}_{2412}$ contains the remaining term. That is,

$$\boldsymbol{h}_{2411}(s_1,s_2)=-\frac{2(\Delta s)^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\frac{\tilde{Q}_{22}\boldsymbol{f}}{(\Delta s)^3},$$

$$\boldsymbol{h}_{2412}(s_1,s_2)=-\frac{2}{\Delta s}\frac{(\Delta s)^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2}.$$

Lemmas 2.1 and 2.5 imply that

$$\begin{aligned}&\frac{2(\Delta s)^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\\ =&2(1+\mathcal{O}(\Delta s)^2)\left[\left\{\boldsymbol{\tau}_i+(\Delta s)\frac{(-1)^i}{2}\boldsymbol{\kappa}_i+\mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right\}\cdot\left\{\boldsymbol{\phi}_i'+(\Delta s)\frac{(-1)^i}{2}\boldsymbol{\phi}_i''+\mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right\}\right]\\ &\times(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_1+\mathcal{O}(\Delta s)^2)\\ =&2(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i')+(\Delta s)(-1)^i(\boldsymbol{\kappa}_i\cdot\boldsymbol{\phi}_i'+\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i'')+\mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha},\end{aligned}$$

and

$$\begin{aligned}\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left|\frac{(\Delta s)^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2}-\frac{\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{3}(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i')\boldsymbol{\tau}_i\right|&\leq C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{C^1}\\ &\leq C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}\end{aligned}$$

for $i\in\{1,2\}$. Applying Lemma 2.12 and Corollary 2.1 to $\boldsymbol{h}_{2411}$ and $\boldsymbol{h}_{2412}$ respectively, we can conclude that there exists an L^2 function $\boldsymbol{N}_{241}(\boldsymbol{f})[\boldsymbol{\phi}]$ such that

$$\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}2\boldsymbol{h}_{241}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2=-\langle(\boldsymbol{\tau}\cdot\boldsymbol{\phi}')L_2\boldsymbol{f},\boldsymbol{\psi}\rangle_{L^2}+\langle\boldsymbol{N}_{241}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover

$$||\boldsymbol{N}_{241}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2}\leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}$$

holds.

□

Lemma 4.5 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{242}:H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^{3-\alpha}$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}\boldsymbol{h}_{242}(s_1,s_2)\cdot\Delta\boldsymbol{\psi}ds_1ds_2=\langle\boldsymbol{N}_{242}(\boldsymbol{f})[\boldsymbol{\phi}]\cdot\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover

$$||\boldsymbol{N}_{242}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \le C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}$$

holds.

Proof We have

$$\begin{aligned}\boldsymbol{h}_{242}(s_1,s_2) =& \frac{4(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4} \\ =& \frac{4}{\Delta s}\frac{(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\mathscr{M}_2(\boldsymbol{f})\frac{\Delta\boldsymbol{f}}{\Delta s},\end{aligned}$$

and Lemmas 2.1 and 2.6 imply that

$$\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left|\frac{4(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\mathscr{M}_2(\boldsymbol{f})\frac{\Delta\boldsymbol{f}}{\Delta s} - 4(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i)(-\frac{\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{2})\boldsymbol{\tau}_i\right| \le C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}$$

for $i\in\{1,2\}$. Hence the assertion follows from Corollary 2.1.

□

Lemma 4.6 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{243}: H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^{3-\alpha}$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} 2\boldsymbol{h}_{243}(s_1,s_2)\cdot\boldsymbol{\psi}_1' ds_1 ds_2 = \langle 3(\boldsymbol{\tau}\cdot\boldsymbol{\phi}')L_2\boldsymbol{f} + 3(\boldsymbol{\tau}\cdot\boldsymbol{\phi}')(\boldsymbol{\tau}\cdot L_2\boldsymbol{f})\boldsymbol{\tau},\boldsymbol{\psi}\rangle_{L^2} + \langle\boldsymbol{N}_{243}(\boldsymbol{f})[\boldsymbol{\phi}]\cdot\boldsymbol{\psi}_1\rangle_{L^2}.$$

Moreover

$$||\boldsymbol{N}_{243}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \le C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^2}$$

holds.

Proof By a simple calculation, we have

$$\begin{aligned}\boldsymbol{h}_{243}(s_1,s_2)\cdot\Delta\boldsymbol{\psi} =& \frac{2(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})(\tilde{Q}_2\boldsymbol{f}\cdot\Delta\boldsymbol{\psi})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4} \\ =& \frac{2(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left(\frac{\tilde{Q}_2\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\psi}}{\Delta s}\right) \\ =& 2(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1')(\boldsymbol{\kappa}_1\cdot\boldsymbol{\psi}_1') + \mathcal{O}(\Delta s)^{\frac{1}{2}-\alpha}.\end{aligned}$$

Hence we have

$$\begin{aligned}&\iint_{|\Delta s|\ge\varepsilon}\boldsymbol{h}_{243}(s_1,s_2)\cdot\boldsymbol{\psi}_1' ds_1 ds_2 \\ =& -\iint_{|\Delta s|\ge\varepsilon}\left(\frac{\partial}{\partial s_1}\boldsymbol{h}_{243}\right)(s_1,s_2)\cdot\Delta\boldsymbol{\psi} ds_1 ds_2 + \mathcal{O}(\varepsilon^{\frac{1}{2}-\alpha})\end{aligned}$$

from Lemma 2.13. Using Lemma 2.5, we can calculate the derivative of $\boldsymbol{h}_{243}$ as

$$
\begin{aligned}
\frac{\partial}{\partial s_1}\boldsymbol{h}_{243}(s_1,s_2) =& -\frac{2}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left\{(\boldsymbol{\tau}_1\cdot\Delta\boldsymbol{\phi}+\Delta\boldsymbol{f}\cdot\boldsymbol{\phi}_1')\tilde{Q}_2\boldsymbol{f}+(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\frac{\partial\tilde{Q}_2\boldsymbol{f}}{\partial s_1}\right\} \\
&+\frac{8(\Delta\boldsymbol{f}\cdot\boldsymbol{\tau}_1)(\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi})\tilde{Q}_2\boldsymbol{f}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6} \\
=& -\frac{2}{(\Delta s)^2}\frac{|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\boldsymbol{\tau}_1\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\phi}_1'\right)\frac{\tilde{Q}_2\boldsymbol{f}}{\Delta s} \\
&-\frac{2}{(\Delta s)^2}\frac{|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left\{\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{f}}{\Delta s}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\right. \\
&\left.+\frac{2(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{f}}{\Delta s}+\hat{Q}_2\boldsymbol{f}\right\} \\
&+\frac{8}{(\Delta s)^2}\frac{|\Delta s|^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\tilde{Q}_2\boldsymbol{f}}{\Delta s} \\
=&\frac{2}{(\Delta s)^2}\frac{|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\boldsymbol{\tau}_1\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\phi}_1'\right)\frac{\tilde{Q}_{22}\boldsymbol{f}}{\Delta s} \\
&+\frac{2}{(\Delta s)^2}\frac{|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\boldsymbol{\tau}_1\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\phi}_1'\right)\frac{\bar{Q}_{22}\boldsymbol{f}}{\Delta s} \\
&-\frac{2}{(\Delta s)^2}\frac{|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left\{\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{f}}{\Delta s}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\right. \\
&\left.+\frac{2(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{f}}{\Delta s}+\hat{Q}_2\boldsymbol{f}\right\} \\
&-\frac{8}{(\Delta s)^2}\frac{|\Delta s|^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\tilde{Q}_{22}\boldsymbol{f}}{\Delta s} \\
&-\frac{8}{(\Delta s)^2}\frac{|\Delta s|^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\bar{Q}_{22}\boldsymbol{f}}{\Delta s} \\
=&\boldsymbol{h}_{2431}(s_1,s_2)+\boldsymbol{h}_{2432}(s_1,s_2),
\end{aligned}
$$

where $\boldsymbol{h}_{2431}$ consists of terms of order $\mathcal{O}(\Delta s)^{-2}$, and $\boldsymbol{h}_{2432}$ contains the remaining terms. That is,

$$
\begin{aligned}
\boldsymbol{h}_{2431} =&2\frac{|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\boldsymbol{\tau}_1\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\phi}_1'\right)\frac{\tilde{Q}_{22}\boldsymbol{f}}{(\Delta s)^3} \\
&-2\frac{|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\left\{\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{(\Delta s)^3}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{f}}{\Delta s}+\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\tilde{Q}_{21}\boldsymbol{f}}{(\Delta s)^3}\right. \\
&\left.+\frac{2(\Delta s)^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\tilde{Q}_{21}\boldsymbol{f}}{(\Delta s)^3}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_2\right)\frac{\Delta\boldsymbol{f}}{\Delta s}\right\} \\
&-8\frac{|\Delta s|^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\tilde{Q}_{22}\boldsymbol{f}}{(\Delta s)^3}, \\
\boldsymbol{h}_{2432} =&\frac{2}{\Delta s}\frac{|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\boldsymbol{\tau}_1\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\phi}_1'\right)\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2} \\
&-\frac{8}{\Delta s}\frac{|\Delta s|^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2}-\frac{2}{\Delta s}\frac{|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\frac{\hat{Q}_2\boldsymbol{f}}{\Delta s}.
\end{aligned}
$$

Using Lemmas 2.1, 2.2 and 2.4, we have

$$\begin{aligned}
&\frac{2|\Delta s|^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\boldsymbol{\tau}_1 \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s} + \frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\phi}_1'\right) \\
=&2(1+\mathcal{O}(\Delta s)^2)\left[\left\{\boldsymbol{\tau}_i + (\Delta s)\boldsymbol{\kappa}_i + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right\} \cdot \left\{\boldsymbol{\phi}_i' + (\Delta s)\frac{(-1)^i}{2}\boldsymbol{\phi}_i'' + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right\}\right. \\
&\left.+\left\{\boldsymbol{\tau}_i + (\Delta s)\frac{(-1)^i}{2}\boldsymbol{\kappa}_i + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right\} \cdot \left\{\boldsymbol{\phi}_i' + (\Delta s)\boldsymbol{\phi}_i'' + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right\}\right] \\
=&2\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i' + (\Delta s)\left\{1 + \frac{(-1)^i}{2}\right\}(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i'' + \boldsymbol{\kappa}_i \cdot \boldsymbol{\phi}_i'),
\end{aligned}$$

$$\begin{aligned}
&\frac{-2|\Delta s|^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s}\right)\frac{\Delta \boldsymbol{f}}{\Delta s} \\
=&-2(1+\mathcal{O}(\Delta s)^2)\left\{(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i') + (\Delta s)\frac{(-1)^i}{2}(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i'') + (\Delta s)\frac{(-1)^i}{2}(\boldsymbol{\kappa}_i \cdot \boldsymbol{\phi}_i') + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right\} \\
&\times\left(\boldsymbol{\tau}_i + (\Delta s)\frac{(-1)^i}{2}\boldsymbol{\kappa}_i + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right) \\
&-2(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i')\boldsymbol{\tau}_i - (\Delta s)(-1)^i\{(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i'' + \boldsymbol{\kappa}_i \cdot \boldsymbol{\phi}_i')\boldsymbol{\tau}_i - \boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i'\boldsymbol{\kappa}_i\} + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha},
\end{aligned}$$

$$\begin{aligned}
\frac{-2|\Delta s|^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2\right) =& -2(1+\mathcal{O}(\Delta s)^2)(1+\mathcal{O}(\Delta s)^2) \\
=& -2 + \mathcal{O}(\Delta s)^2,
\end{aligned}$$

$$\begin{aligned}
&\frac{-2|\Delta s|^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s}\right)\frac{2|\Delta s|^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_2\right)\frac{\Delta \boldsymbol{f}}{\Delta s} \\
=&-2(1+\mathcal{O}(\Delta s)^2)\left\{(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i') + (\Delta s)\frac{(-1)^i}{2}(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i'') + (\Delta s)\frac{(-1)^i}{2}(\boldsymbol{\kappa}_i \cdot \boldsymbol{\phi}_i') + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right\} \\
&\times 2(1+\mathcal{O}(\Delta s)^2)(1+\mathcal{O}(\Delta s)^2)\left\{\boldsymbol{\tau}_i + (\Delta s)\frac{(-1)^i}{2}\boldsymbol{\kappa}_i + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right\} \\
=&-4(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i')\boldsymbol{\tau}_i - 2(\Delta s)(-1)^i\{(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i'' + \boldsymbol{\kappa}_i \cdot \boldsymbol{\phi}_i')\boldsymbol{\tau}_i - \boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i'\boldsymbol{\kappa}_i\} + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha},
\end{aligned}$$

$$\begin{aligned}
&\frac{-8|\Delta s|^4}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \boldsymbol{\tau}_1\right)\left(\frac{\Delta \boldsymbol{f}}{\Delta s} \cdot \frac{\Delta \boldsymbol{\phi}}{\Delta s}\right) \\
=&-8(1+\mathcal{O}(\Delta s)^2)(1+\mathcal{O}(\Delta s)^2) \\
&\times\left\{(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i') + (\Delta s)\frac{(-1)^i}{2}(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i'') + (\Delta s)\frac{(-1)^i}{2}(\boldsymbol{\kappa}_i \cdot \boldsymbol{\phi}_i') + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha}\right\} \\
=&-8(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i') - 4(\Delta s)(-1)^i(\boldsymbol{\tau}_i \cdot \boldsymbol{\phi}_i'' + \boldsymbol{\kappa}_i \cdot \boldsymbol{\phi}_i') + \mathcal{O}(\Delta s)^{\frac{3}{2}-\alpha},
\end{aligned}$$

thus we know that each term of $\boldsymbol{h}_{2431}$ satisfies the assertions of Lemma 2.12. Hence, we obtain that there exists an L^2 function $\boldsymbol{N}_{2431}(\boldsymbol{f})[\boldsymbol{\phi}]$ such that

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}(-2\boldsymbol{h}_{2431}) \cdot \Delta\boldsymbol{\psi}\, ds_1 ds_2 \\
=&-2\langle(\boldsymbol{\tau} \cdot \boldsymbol{\phi}')L_2\boldsymbol{f}, \boldsymbol{\psi}\rangle_{L^2} \\
&+\langle(\boldsymbol{\tau} \cdot \boldsymbol{\phi}')(\boldsymbol{\tau} \cdot L_2\boldsymbol{f})\boldsymbol{\tau} + (\boldsymbol{\tau} \cdot \boldsymbol{\phi}')L_2\boldsymbol{f} + 2(\boldsymbol{\tau} \cdot \boldsymbol{\phi}')(\boldsymbol{\tau} \cdot L_2\boldsymbol{f})\boldsymbol{\tau}, \boldsymbol{\psi}\rangle_{L^2} \\
&+4\langle(\boldsymbol{\tau} \cdot \boldsymbol{\phi}')L_2\boldsymbol{f}, \boldsymbol{\psi}\rangle_{L^2} + \langle\boldsymbol{N}_{2431}(\boldsymbol{f})[\boldsymbol{\phi}], \boldsymbol{\psi}\rangle_{L^2} \\
=&\langle 3(\boldsymbol{\tau} \cdot \boldsymbol{\phi}')L_2\boldsymbol{f} + 3(\boldsymbol{\tau} \cdot \boldsymbol{\phi}')(\boldsymbol{\tau} \cdot L_2\boldsymbol{f})\boldsymbol{\tau}, \boldsymbol{\psi}\rangle_{L^2} + \langle\boldsymbol{N}_{2431}(\boldsymbol{f})[\boldsymbol{\phi}], \boldsymbol{\psi}\rangle_{L^2},
\end{aligned}$$

and there exists a constant C such that the estimate

$$||\boldsymbol{N}_{2431}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^2}$$

holds. On the other hand, Lemmas 2.1, 2.5 and estimate of $\hat{Q}_2\boldsymbol{f}$ imply that there exists a constant $C(\boldsymbol{f})$ such that

$$\begin{aligned}&\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left\|\frac{2|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\left(\boldsymbol{\tau}_1\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}+\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\phi}_1'\right)\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2}-\frac{4\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{3}(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i')\right\|_{\mathbb{R}^n}\\&\leq C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{H^{3-\alpha}},\end{aligned}$$

$$\begin{aligned}&\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left\|-\frac{8|\Delta s|^6}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^6}\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\boldsymbol{\tau}_1\right)\left(\frac{\Delta\boldsymbol{f}}{\Delta s}\cdot\frac{\Delta\boldsymbol{\phi}}{\Delta s}\right)\frac{\bar{Q}_{22}\boldsymbol{f}}{(\Delta s)^2}-\frac{2|\Delta s|^4}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}\frac{\hat{Q}_2\boldsymbol{f}}{\Delta s}\right.\\&\left.+\frac{8\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{3}(\boldsymbol{\tau}_i\cdot\boldsymbol{\phi}_i')\boldsymbol{\tau}_i-\frac{2\|\boldsymbol{\kappa}_i\|_{\mathbb{R}^n}^2}{3}\boldsymbol{\tau}_i\right\|_{\mathbb{R}^n}\\&\leq C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}\end{aligned}$$

for $i\in\{1,2\}$ Corollary 2.1 implies that there exists an L^2 function $\boldsymbol{N}_{2432}(\boldsymbol{f})[\boldsymbol{\phi}]$ such that

$$\lim_{\varepsilon\downarrow 0}\iint_{|\Delta s|\geq\varepsilon}(-\boldsymbol{h}_{2432})\cdot\Delta\boldsymbol{\psi}ds_1ds_2=\langle\boldsymbol{N}_{2432}(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle_{L^2},$$

and there exists a constant C such that the estimate

$$||\boldsymbol{N}_{2432}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^2}$$

holds. From the above estimates, the claimed assertion follows.

□

4.5 The L^2-representation of K_{25}

Lemma 4.7 *Let $\alpha\in(0,\frac{1}{2})$. Then there exists a mapping $\boldsymbol{N}_{25}:H^{3-\alpha}\to\mathscr{B}(H^{3-\alpha},L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^{3-\alpha}$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^{3-\alpha}$, we have*

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2}H_{25}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2=\langle\boldsymbol{N}_{25}(\boldsymbol{f})[\boldsymbol{\phi}]\cdot\boldsymbol{\psi}\rangle_{L^2}.$$

Moreover

$$||\boldsymbol{N}_{25}(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2} \leq C\left(||\boldsymbol{f}||_{H^{3-\alpha}}\right)||\boldsymbol{\phi}||_{H^{3-\alpha}}$$

holds.

Proof Observe that

$$\iint_{|\Delta s|\geq\varepsilon}H_{25}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]ds_1ds_2=\iint_{|\Delta s|\geq\varepsilon}(\boldsymbol{h}_{25}(s_1,s_2)\cdot\Delta\boldsymbol{\psi})ds_1ds_2,$$

where

$$\boldsymbol{h}_{25}(s_1,s_2)=-\frac{2\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|^2}.$$

Now, we have

$$\begin{aligned}\boldsymbol{h}_{25}(s_1,s_2)=&-\frac{2\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|^2}\\=&-\frac{1}{\Delta s}2\mathscr{M}_2(\boldsymbol{f})\frac{(\Delta s)^2}{\|\Delta\boldsymbol{f}\|^2}\frac{\Delta\boldsymbol{\phi}}{\Delta s}\\=&-\frac{2}{\Delta s}\mathscr{M}_2(\boldsymbol{f})\frac{\Delta\boldsymbol{\phi}}{\Delta s}-\frac{2}{\Delta s}\mathscr{M}_2(\boldsymbol{f})(\Delta s)^2\mathscr{M}(\boldsymbol{f})\frac{\Delta\boldsymbol{\phi}}{\Delta s}\\=&-\frac{2}{\Delta s}\mathscr{M}_2(\boldsymbol{f})\frac{\Delta\boldsymbol{\phi}}{\Delta s}-2\mathscr{M}_2(\boldsymbol{f})\mathscr{M}(\boldsymbol{f})\Delta\boldsymbol{\phi}.\end{aligned}$$

Lemmas 2.1 and 2.6 imply that there exists a constant $C(\boldsymbol{f})$ such that

$$\begin{aligned}\sup_{s_1\neq s_2}\frac{1}{|\Delta s|^{\frac{1}{2}-\alpha}}\left|\mathscr{M}_2(\boldsymbol{f})\frac{\Delta\boldsymbol{\phi}}{\Delta s}-\left(-\frac{1}{2}\|\boldsymbol{\kappa}_i\|^2_{\mathbb{R}^n}\boldsymbol{\phi}_i'\right)\right|\leq& C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{C^1}\\\leq& C(\boldsymbol{f})\|\boldsymbol{\phi}\|_{H^{3-\alpha}}\end{aligned}$$

for $i\in\{1,2\}$. Hence we get the claimed assertions from Corollary 2.1.

□

4.6 The L^2-representation of the second variation of $\mathcal{E}_2$

From Proposition 4.1 to Lemma 4.7, we can conclude that $\delta^2\mathcal{E}_2$ has the L^2-representaion.

Theorem 4.1 *There exists a mapping $\boldsymbol{N}_2 : H^3\to\mathscr{B}(H^3,L^2)$ such that, for any bi-Lipschitz function $\boldsymbol{f}\in H^3$ and functions $\boldsymbol{\phi},\boldsymbol{\psi}\in H^3$, we have*

$$\delta^2\mathcal{E}_2(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]=\langle P^\perp_{\boldsymbol{\tau}}L_2\boldsymbol{\phi}-(L_2\boldsymbol{f}\cdot\boldsymbol{\phi}')\boldsymbol{\tau}+2(\boldsymbol{\tau}\cdot\boldsymbol{\phi}')L_2\boldsymbol{f}+\boldsymbol{N}_2(\boldsymbol{f})[\boldsymbol{\phi}],\boldsymbol{\psi}\rangle.$$

Moreover, $\boldsymbol{N}_2$ satisfies

$$||\boldsymbol{N}_2(\boldsymbol{f})[\boldsymbol{\phi}]||_{L^2}\leq C\left(||\boldsymbol{f}||_{H^3}\right)||\boldsymbol{\phi}||_{H^3}.$$

Furthermore, we can extend the domain of the linear form

$$\boldsymbol{\psi}\mapsto\delta^2\mathcal{E}_2(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]$$

to $\boldsymbol{\psi}\in L^2$.

5 Proof of the Łojasiewicz inequalities

In this section, we prove the Łojasiewicz inequality for each functional $\mathcal{E}_i$. We define $\boldsymbol{H}_i(\boldsymbol{f})\in\mathscr{B}(H^3,L^2)$ as

$$\boldsymbol{H}_i(\boldsymbol{f})[\boldsymbol{\phi}]=P^\perp_{\boldsymbol{\tau}}L_i\boldsymbol{\phi}-(L_i\boldsymbol{f}\cdot\boldsymbol{\phi}')\boldsymbol{\tau}+2(\boldsymbol{\tau}\cdot\boldsymbol{\phi}')L_i\boldsymbol{f}+\boldsymbol{N}_i(\boldsymbol{f})[\boldsymbol{\phi}],$$

for any $\boldsymbol{\phi}\in H^3(\mathbb{R}/\mathbb{Z})$. For $\boldsymbol{f}_0\in C^1(\mathbb{R}/\mathbb{Z})$, we define function spaces

$$\begin{aligned}H^3(\mathbb{R}/\mathbb{Z})^\perp_{\boldsymbol{f}_0'}=&\left\{P^\perp_{\boldsymbol{f}_0'}\boldsymbol{f}\,\middle|\,\boldsymbol{f}\in H^3(\mathbb{R}/\mathbb{Z})\right\},\\L^2(\mathbb{R}/\mathbb{Z})^\perp_{\boldsymbol{f}_0'}=&\left\{P^\perp_{\boldsymbol{f}_0'}\boldsymbol{f}\,\middle|\,\boldsymbol{f}\in L^2(\mathbb{R}/\mathbb{Z})\right\}.\end{aligned}$$

Lemma 5.1 *Assume that the bi-Lipschitz continuous function $\boldsymbol{f}_0 \in C^\infty(\mathbb{R}/\mathbb{Z})$ is a stationary point of $\mathcal{E}_i$. Then $\boldsymbol{H}_i(\boldsymbol{f}_0)$ is a Fredholm operator from $H^3(\mathbb{R}/\mathbb{Z})^\perp_{\boldsymbol{f}_0'}$ to $L^2_{\boldsymbol{f}_0'}(\mathbb{R}/\mathbb{Z})^\perp$.*

Proof Using the results of §3 and §4, the linear map $\boldsymbol{H}_1(\boldsymbol{f}_0), \boldsymbol{H}_2(\boldsymbol{f}_0)$ are represented by

$$\boldsymbol{H}_1(\boldsymbol{f}_0)[\boldsymbol{\phi}] = 2\pi P^\perp_{\boldsymbol{f}_0'}(-\Delta_s)^{\frac{3}{2}}\boldsymbol{\phi} + R_1(\boldsymbol{f}_0)\boldsymbol{\phi},$$

$$\boldsymbol{H}_2(\boldsymbol{f}_0)[\boldsymbol{\phi}] = -\frac{4}{3}\pi P^\perp_{\boldsymbol{f}_0'}(-\Delta_s)^{\frac{3}{2}}\boldsymbol{\phi} + R_2(\boldsymbol{f}_0)\boldsymbol{\phi},$$

where each $R_i(\boldsymbol{f}_0)$ is a pseudo-differential operator whose order is less than 3.

First, we show that $(-\Delta_s)^{\frac{3}{2}}$ is a Fredholm operator from $H^3(\mathbb{R}/\mathbb{Z})^\perp_{\boldsymbol{f}_0'}$ to $L^2(\mathbb{R}/\mathbb{Z})^\perp_{\boldsymbol{f}_0'}$. For a function $u \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $\boldsymbol{v} \in H^3(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, if we define

$$(-\Delta_s)^{\frac{3}{2}}(u\boldsymbol{v}) - \left((-\Delta_s)^{\frac{3}{2}}u\right)\boldsymbol{v} = F(u, \boldsymbol{v}),$$

then we have

$$\|F(u, \boldsymbol{v})\|_{L^2} \leq C\|u\|_{H^2}\|\boldsymbol{v}\|_{H^3}.$$

Now, let $\{\boldsymbol{\nu}_i\}_{i=1}^{n-1}$ be a family of C^∞ functions which constitute a basis of the normal plane to the vector $\boldsymbol{f}_0'(s)$ for each parameter s. Then, for any function $\boldsymbol{\phi} \in H^3(\mathbb{R}/\mathbb{Z})^\perp_{\boldsymbol{f}_0'}$, we obtain

$$\boldsymbol{\phi} = \sum_{i=1}^{n-1}\phi_i\boldsymbol{\nu}_i, \quad \phi_i = \boldsymbol{\phi}\cdot\boldsymbol{\nu}_i.$$

Hence, we have

$$\begin{aligned}
P^\perp_{\boldsymbol{f}_0'}(-\Delta_s)^{\frac{3}{2}}\boldsymbol{\phi} &= P^\perp_{\boldsymbol{f}_0'}\left\{(-\Delta_s)^{\frac{3}{2}}\left(\sum_{i=1}^{n-1}\phi_i\boldsymbol{\nu}_i\right)\right\} \\
&= P^\perp_{\boldsymbol{f}_0'}\left\{\sum_{i=1}^{n-1}\left((-\Delta_s)^{\frac{3}{2}}\phi_i\right)\boldsymbol{\nu}_i\right\} \\
&\quad + P^\perp_{\boldsymbol{f}_0'}\left[\sum_{i=1}^{n-1}\left\{(-\Delta_s)^{\frac{3}{2}}(\phi_i\boldsymbol{\nu}_i) - \left((-\Delta_s)^{\frac{3}{2}}\phi_i\right)\boldsymbol{\nu}_i\right\}\right] \\
&= \sum_{i=1}^{n-1}\left((-\Delta_s)^{\frac{3}{2}}\phi_i\right)\boldsymbol{\nu}_i + P^\perp_{\boldsymbol{f}_0'}\sum_{i=1}^{n-1}F(\phi_i, \boldsymbol{\nu}_i).
\end{aligned}$$

Now we define a mapping from $H^3(\mathbb{R}/\mathbb{Z})^\perp_{\boldsymbol{f}_0'}$ to $L^2(\mathbb{R}/\mathbb{Z})^\perp_{\boldsymbol{f}_0'}$ by

$$\boldsymbol{\phi} \mapsto \sum_{i=1}^{n-1}\left((-\Delta_s)^{\frac{3}{2}}\phi_i\right)\boldsymbol{\nu}_i.$$

Then, $\boldsymbol{\phi} \mapsto \sum_{i=1}^{n-1}\left((-\Delta_s)^{\frac{3}{2}}\phi_i\right)\boldsymbol{\nu}_i$ is a Fredholm operator. In fact, considering the ordinary differential equation

$$(-\Delta_s)^{\frac{3}{2}}\boldsymbol{\phi} = 0,$$

we obtain that $\boldsymbol{\phi} = \text{const}$. Therefore, we can conclude

$$\mathrm{Ker}((-\Delta_s)^{\frac{3}{2}}) = \{c \in H^3(\mathbb{R}/\mathbb{Z}) | \, c \text{ is a constant function}\}.$$

Next, using

$$f = (f + C(f)) - C(f), \quad C(f) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx,$$

we can get a direct sum decomposition

$$L^2(\mathbb{R}/\mathbb{Z}) = M_1 \oplus M_2,$$

$$M_1 = \{f \in L^2(\mathbb{R}/\mathbb{Z}) \,|\, C(f) = 0\}, \quad M_2 = \{c \in L^2(\mathbb{R}/\mathbb{Z}) \,|\, c \text{ is a constant function}\}.$$

Here, we have

$$M_1 = \mathrm{Rg}((-\Delta_s)^{\frac{3}{2}}),$$

and thus

$$\mathrm{Coker}((-\Delta_s)^{\frac{3}{2}}) = M_2.$$

Since

$$\mathrm{Ker}((-\Delta_s)^{\frac{3}{2}}), \quad \mathrm{Coker}((-\Delta_s)^{\frac{3}{2}})$$

are finite dimensional subspaces of $L^2(\mathbb{R}/\mathbb{Z})$, we can conclude that $(-\Delta_s)^{\frac{3}{2}}$ is a Fredholm operator from $H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ to $L^2(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$.

Finally, note that $R_i\boldsymbol{\phi}$, $F(\phi_i, \boldsymbol{\nu}_i)$ are pseudo-differential operators of order less than 3 with respect to $\boldsymbol{\phi}$. From the Rellich-Kondrachov compactness theorem, we have $H^1(\mathbb{R}/\mathbb{Z}) \Subset L^2(\mathbb{R}/\mathbb{Z})$. Hence, a pseudo-differential operator of order less than 3 is a compact operator from $H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ to $L^2(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$. Moreover, a sum of a Fredholm operator and a compact operator is a Fredholm operator.

Now, $\boldsymbol{H}_i(\boldsymbol{f}_0)[\boldsymbol{\phi}]$ is a sum of a Fredholm operator and a compact operator, hence is a Fredholm operator from $H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ to $L^2(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$. □

If we let $\left\| \dfrac{d\boldsymbol{f}}{dx} \right\|_{\mathbb{R}^n} = \gamma(x)$, then $ds = \gamma(x)dx$ holds.

Lemma 5.2 *Assume that the bi-Lipschitz continuous function $\boldsymbol{f}_0 \in C^\infty(\mathbb{R}/\mathbb{Z})$ is a stationary point of $\mathcal{E}_i$. Then $\boldsymbol{G}_i$ is an analytic function from $H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ to $L^2(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ on some neighbourhood of $\boldsymbol{f}_0$ in $\|\cdot\|_{H^3}$.*

Proof Let $\boldsymbol{f} \in H^3(\mathbb{R}/\mathbb{Z})$. If we choose $\sigma > 0$ sufficiently small, under the condition $\|\boldsymbol{f} - \boldsymbol{f}_0\|_{H^3} < \delta$, $\boldsymbol{f}$ is bi-Lipschitz continuous. That is, there exists $b > 0$ such that

$$\frac{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}{|\Delta s|} \geq b.$$

Here, $s = s(x)$ is the arc-length parameter. Note that the mapping $H^3(\mathbb{R}/\mathbb{Z}) \ni \boldsymbol{f} \mapsto s \in C^1(\mathbb{R}/\mathbb{Z})$ is analytic. According to [11], we have

$$\begin{aligned}\boldsymbol{G}_1(\boldsymbol{f}) = L_1\boldsymbol{f} &- 2\int_{\mathbb{R}/\mathbb{Z}} \left\{ \frac{2}{(\Delta s)^2}(T_2^4\boldsymbol{f} \cdot \boldsymbol{\tau}(s_1))\Delta\boldsymbol{\tau} - \mathscr{M}(\boldsymbol{f})\boldsymbol{\kappa}(s_1) \right\} \gamma(x_2)dx_2 \\ &- 4\int_{\mathbb{R}/\mathbb{Z}} \left\{ \frac{\mathscr{M}_1(\boldsymbol{f})}{\Delta s} + \frac{1}{\Delta s}\left(\mathscr{M}_1(\boldsymbol{f}) - \frac{1}{2}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n} \right) \boldsymbol{\tau}(s_1) \right\} \gamma(x_2)dx_2.\end{aligned}$$

Since the linear term L_1 has an L^2-estimate

$$\|L_1 \boldsymbol{f}\|_{L^2} \leq C\|\boldsymbol{f}\|_{H^3},$$

the mapping $L_1 : H^3(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z})$ is analytic.

Next, we note that the mapping

$$\boldsymbol{f} \longmapsto \frac{\Delta \boldsymbol{f}}{\Delta s},$$

from $H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ to $C((\mathbb{R}/\mathbb{Z})^2)$ is analytic, and since $x \mapsto x^{-1}$ is analytic on $x \in [b, \infty)$, we obtain that the mapping from $H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ to $C((\mathbb{R}/\mathbb{Z})^2)$ given by

$$\boldsymbol{f} \longmapsto \frac{(\Delta s)^2}{\|\Delta \boldsymbol{f}\|^2_{\mathbb{R}^n}}$$

is analytic. Next, if we take a function $\boldsymbol{u} \in C((\mathbb{R}/\mathbb{Z})^2)$, then we have

$$\begin{aligned}\left\| \int_{\mathbb{R}/\mathbb{Z}} \boldsymbol{u}(\cdot, x_2)\gamma(x_2)dx_2 \right\|_{L^2} &\leq C \left\| \int_{\mathbb{R}/\mathbb{Z}} \boldsymbol{u}(\cdot, x_2)\gamma(x_2)dx_2 \right\|_{L^\infty} \\ &\leq C\|\boldsymbol{u}\|_{L^\infty((\mathbb{R}/\mathbb{Z})^2)}.\end{aligned}$$

Therefore, the mapping

$$\boldsymbol{u} \longmapsto \int_{\mathbb{R}/\mathbb{Z}} \boldsymbol{u}(\cdot, x_2)\gamma(x_2)dx_2$$

is an analytic mapping from $C((\mathbb{R}/\mathbb{Z})^2)$ to $L^2(\mathbb{R}/\mathbb{Z})$.

Finally, we define a mapping from $H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ to $L^2(\mathbb{R}/\mathbb{Z})$ by

$$\boldsymbol{f} \longmapsto \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{\Delta s}\left(\frac{\Delta \boldsymbol{\tau}}{\Delta s} + \boldsymbol{\kappa}(s_1)\right) \cdot \left(\frac{\Delta \boldsymbol{\tau}}{\Delta s} - \boldsymbol{\kappa}(s_1)\right)\gamma(x_2)dx_2.$$

Then the abave mapping is quadratic and we have

$$\begin{aligned}&\left\| \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{\Delta s}\left(\frac{\Delta \boldsymbol{\tau}}{\Delta s} + \boldsymbol{\kappa}(s_1)\right) \cdot \left(\frac{\Delta \boldsymbol{\tau}}{\Delta s} - \boldsymbol{\kappa}(s_1)\right)\gamma(x_2)dx_2 \right\|_{L^2} \\ &\leq C \left\| \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{|\Delta s|^{1/2+\alpha}} \left| \frac{\Delta \boldsymbol{\tau}}{\Delta s} + \boldsymbol{\kappa}(s_1) \right| \left(\int_{\mathbb{R}/\mathbb{Z}} \frac{|\boldsymbol{\kappa}(s_3) - \boldsymbol{\kappa}(s_1)|}{|\Delta s|^{1/2-\alpha}} |\gamma(x_3)dx_3| \right) |\gamma(x_2)|dx_2 \right\|_{L^\infty} \\ &\leq C \left\| \frac{\Delta \boldsymbol{\tau}}{\Delta s} + \boldsymbol{\kappa}(s_1) \right\|_{L^\infty((\mathbb{R}/\mathbb{Z})^2)} \left\| \frac{\Delta \boldsymbol{\kappa}}{|\Delta s|^{1/2-\alpha}} \right\|_{L^\infty((\mathbb{R}/\mathbb{Z})^2)} \\ &\leq C \left\| \boldsymbol{f} \right\|_{C^2} \left\| \boldsymbol{f} \right\|_{C^{2,1/2-\alpha}} \\ &\leq C \left\| \boldsymbol{f} \right\|^2_{H^3},\end{aligned}$$

which implies that it is bounded. Hence, the mapping is analytic.

Now, $T^i_j \boldsymbol{f}$, $T^i_j \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1)$, $\mathscr{M}(\boldsymbol{f})$, $\mathscr{M}_1(\boldsymbol{f})$ can be expressed by use of four arithmetic operations with $\boldsymbol{f}, \boldsymbol{\tau}, \frac{\Delta \boldsymbol{f}}{\Delta s}, \frac{\Delta \boldsymbol{\tau}}{\Delta s}, \frac{(\Delta s)}{||\boldsymbol{f}||^2_{\mathbb{R}^n}}$ and there holds

$$\begin{aligned}&\frac{1}{\Delta s}\left(\mathscr{M}_1(\boldsymbol{f}) - \frac{1}{2}\|\boldsymbol{\kappa}(s_1)\|^2_{\mathbb{R}^n}\right) \\ &= \frac{\|\Delta \boldsymbol{\tau}\|^2_{\mathbb{R}^n}}{2\Delta s}\mathscr{M}(\boldsymbol{f}) + \frac{\|\Delta \boldsymbol{\tau}\|^2_{\mathbb{R}^n}}{2\Delta s}\left(\frac{\Delta \boldsymbol{\tau}}{\Delta s} + \boldsymbol{\kappa}(s_1)\right) \cdot \left(\frac{\Delta \boldsymbol{\tau}}{\Delta s} - \boldsymbol{\kappa}(s_1)\right).\end{aligned}$$

Noting these facts, we obatain that $\boldsymbol{G}_1$ is an analytic mapping from $H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ to $L^2(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$.

On the other hand, it is shown that

$$\boldsymbol{G}_2(\boldsymbol{f}) = L_2\boldsymbol{f} + \boldsymbol{N}_2^{(2)},$$

$$\begin{aligned}
\boldsymbol{N}_2^{(2)}(\boldsymbol{f}) = & -4\int_{\mathbb{R}/\mathbb{Z}} \frac{(\Delta s)^2}{||\Delta \boldsymbol{f}||^2_{\mathbb{R}^n}} \left\{ \frac{T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1)}{(\Delta s)^2}\frac{T_2^0\boldsymbol{f}}{\Delta s} + \frac{T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2)}{(\Delta s)^2}\frac{T_1^0\boldsymbol{f}}{\Delta s}\right\} ds_2 \\
& -4\int_{\mathbb{R}/\mathbb{Z}} \left[\frac{T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1)}{(\Delta s)^2}\frac{T_2^0\boldsymbol{f}}{\Delta s} + \frac{T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2)}{(\Delta s)^2}\frac{T_1^4\boldsymbol{f}}{\Delta s} \right. \\
& \left. +2\{(T_2^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))+1\}\frac{T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1)}{(\Delta s)^2}\frac{T_1^4\boldsymbol{f}}{\Delta s}\right] ds_2 \\
& -4\int_{\mathbb{R}/\mathbb{Z}} \frac{1}{(\Delta s)^3}\left[T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1) - T_2^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2) \right. \\
& \left. +2\{(T_2^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2))+1\}(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1)) + T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}(s_2) - \frac{(\Delta s)^2}{6}||\boldsymbol{\kappa}(s_1)||^2_{\mathbb{R}}\right]\boldsymbol{\tau}(s_1) ds_2 \\
& -4\int_{\mathbb{R}/\mathbb{Z}} \left[\mathscr{M}_2(\boldsymbol{f})\frac{T_1^2\boldsymbol{f}}{\Delta s} + \frac{1}{\Delta s}\left\{\mathscr{M}_2(\boldsymbol{f}) + \frac{1}{2}||\boldsymbol{\kappa}(s_1)||^2_{\mathbb{R}}\right\}\right] ds_2,
\end{aligned}$$

in [11]. Note that $\mathscr{M}_2(\boldsymbol{f})$ can be expressed by four arithmetic operations or these combination of $\boldsymbol{f}, \boldsymbol{\tau}, \frac{\Delta \boldsymbol{f}}{\Delta s}, \frac{\Delta \boldsymbol{\tau}}{\Delta s}$. A similar argument to the case of $\boldsymbol{G}_1$ lead to the analiticity of $\boldsymbol{G}_2$. □

We state a lemma by Chill [3] to prove the Łojasiewicz inequalities.

Lemma 5.3 [3, Chill, Corollary 3.11.] *Let V and W be a pair of Banach spaces which satisfy $V \hookrightarrow W \hookrightarrow V'$, and let E be a functional near 0 in V. E'' denotes the second order Frechét derivative of E. If*

$$V_0 = \mathrm{Ker}(E''(0))$$

and P denotes a projection of V onto V_0, then we assume

$$V = V_0 \oplus V_1, \quad V_1 = \mathrm{Ker}(P).$$

In addition, we give conditions as follows:

1. $P'(W) \subset W$,
2. $E' \in C^1(V, W)$,
3. $\mathrm{Rg}(E''(0)) = \mathrm{Ker}(P') \cap W$,
4. *E' is an analytic mapping from V to W,*
5. $\mathrm{dim} V_0 < \infty$.

Then the functional E satisfies the Łojasiewicz inequality near 0 for an exponent $\theta \in (0, \frac{1}{2})$.

Theorem 1.15 *Assume that the bi-Lipschitz function $\boldsymbol{f}_0 \in C^\infty(\mathbb{R}/\mathbb{Z})$ is a stationary point of $\mathcal{E}_i$. Then there exist $\theta \in (0, \frac{1}{2}), \sigma > 0, c > 0$ such that if a bi-Lipschitz function $\boldsymbol{f} \in H^3(\mathbb{R}/\mathbb{Z})$ satisfies $\|\boldsymbol{f} - \boldsymbol{f}_0\|_{H^3} \leq \sigma$, the inequality*

$$|\mathcal{E}_i(\boldsymbol{f}) - \mathcal{E}_i(\boldsymbol{f}_0)|^{1-\theta} \leq c\|\boldsymbol{G}_i(\boldsymbol{f})\|_{L^2}$$

holds.

Proof Applying Lemma 5.3 to the case $V = H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$, $W = L^2(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$, we obtain that $\mathcal{E}_i : H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'} \to \mathbb{R}$ satisfies the Łojasiewicz inequality. That is, there exist $\theta \in (0, \frac{1}{2})$, $\tilde{\sigma} > 0$ such that if a bi-Lipschitz function $\boldsymbol{f} \in H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ satisfies $\|\boldsymbol{\phi}\|_{H^3} \leq \tilde{\sigma}$, it holds that

$$|\mathcal{E}_i(\boldsymbol{f}_0 + \boldsymbol{\phi}) - \mathcal{E}_i(\boldsymbol{f}_0)|^{1-\theta} \leq c\|\boldsymbol{G}_i(\boldsymbol{f}_0 + \boldsymbol{\phi})\|_{L^2}.$$

Using [2, Lemma A.9], if there exists $\sigma > 0$ such that $\|\boldsymbol{f}(x) - \boldsymbol{f}_0(x)\|_{\mathbb{R}^n} \leq \sigma$ for any $x \in \mathbb{R}/\mathbb{Z}$, then

$$\|\boldsymbol{f}(x)\|_{\mathbb{R}^n} \geq \frac{1}{2}$$

for any $x \in \mathbb{R}/\mathbb{Z}$, and there exists a diffeomorphism Ψ on $\mathbb{R}/\mathbb{Z}$ and a function $\boldsymbol{\phi} \in H^3(\mathbb{R}/\mathbb{Z})^{\perp}_{\boldsymbol{f}_0'}$ such that

$$\boldsymbol{f} \circ \Psi = \boldsymbol{f}_0 + \boldsymbol{\phi},$$

$$\|\boldsymbol{\phi}\|_{H^3} \leq C\|\boldsymbol{f} - \boldsymbol{f}_0\|_{H^3}.$$

If we choose $\sigma > 0$ sufficiently small, we have

$$\left\| \frac{d}{dx}(\boldsymbol{f} \circ \Psi) \right\|_{\mathbb{R}^n} > \frac{1}{2},$$

and hence we get

$$\begin{aligned}
|\mathcal{E}_i(\boldsymbol{f}) - \mathcal{E}_i(\boldsymbol{f}_0)|^{1-\theta} &= |\mathcal{E}_i(\boldsymbol{f}_0 + \boldsymbol{\phi}) - \mathcal{E}_i(\boldsymbol{f}_0)|^{1-\theta} \\
&\leq c \left(\int_{\mathbb{R}/\mathbb{Z}} \|\boldsymbol{G}_i(\boldsymbol{f}_0 + \boldsymbol{\phi})(x)\|^2_{\mathbb{R}^n} dx \right)^{1/2} \\
&\leq 2c \left(\int_{\mathbb{R}/\mathbb{Z}} \|\boldsymbol{G}_i(\boldsymbol{f}_0 + \boldsymbol{\phi})(x)\|^2_{\mathbb{R}^n} \left\| \frac{d}{dx}(\boldsymbol{f} \circ \Psi)(x) \right\|_{\mathbb{R}^n} dx \right)^{1/2} \\
&= 2c\|\boldsymbol{G}_i(\boldsymbol{f})\|_{L^2}.
\end{aligned}$$

□

Acknowledgement. The author would like to express his gratitude to Professor Neal Bez for English language editing.

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