Interpolation between the isoperimetric ratio and curvature for plane curves and an application to curvature flows with non-local terms

（平面閉曲線に対する等周比を用いた補間不等式とその非局所曲率流への応用）

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Abstract

Several inequalities for the isoperimetric ratio for plane curves are derived. In particular, we obtain interpolation inequalities between the deviation of curvature and the isoperimetric ratio. As applications, we study the large-time behavior of some geometric flows of closed plane curves without a convexity assumption.
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1 Introduction

It is an interesting problem to study the behavior of plane curves evolving in time. A famous and basic problem is the curve-shortening flow. Since the first variation of length is the curvature, the flow is governed by

$$\partial_t \boldsymbol{f} = -\delta L(\boldsymbol{f}) = \boldsymbol{\kappa}. \tag{1.1}$$

Here $\boldsymbol{f} : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^2$ is a function which image represents the curve, L is its total length, and $\boldsymbol{\kappa}$ is the curvature vector. This equation is also called the curvature flow. It was proposed to describe the motion of grain boundary in annealing of metal by Mullins [14]. Annealing is slowly warming and cooling down. Metal is generally not a single structure as a whole because the orientation of the crystal may be different, even though the composition is the same. Part of a single structure is called the grain, and its boundary is called the grain boundary. By annealing, small grains are vaporized and it can be deformed into a metal close to a single structure only with large grains. The mean curvature flow is used to describe the grain boundary motion at this time.

Thereafter the equation (1.1) was first studied by Gage-Hamilton [7] and Grayson [9]. They proved that a simple closed convex initial curve remains so along the flow, and the evolving curve becomes more and more circular and shrinks to a point in a finite time.

Then a number of papers have been devoted to the study of curvature flow under some geometric constrains, for example, area-preserving or length-preserving, which prevent the curve shrink into a point. The curvature flow with area constrain is

$$\partial_t \boldsymbol{f} = \boldsymbol{\kappa} - \frac{1}{L}\left(\int_0^L \boldsymbol{\kappa}\cdot\boldsymbol{\nu}\, ds\right)\boldsymbol{\nu}, \tag{1.2}$$

and the flow with length constrain is

$$\partial_t \boldsymbol{f} = \boldsymbol{\kappa} - \left(\frac{1}{2\pi} \int_0^L \|\boldsymbol{\kappa}\|^2 ds \right) \boldsymbol{\nu}, \tag{1.3}$$

where s is arc-length parameter, and $\boldsymbol{\nu}$ is the inner unit normal vector. Besides these, there is a flow like the following,

$$\partial_t \boldsymbol{f} = \boldsymbol{\kappa} - \frac{L}{2A} \boldsymbol{\nu}. \tag{1.4}$$

This is the gradient flow of the isoperimetric ratio. Along these flows, the curve is driven by the curvature together with non-local term. Hence they are called the non-local curvature flows. Jiang-Pan [11] and Gage [6] proved that a simple closed convex initial curve remains so along (1.4) and (1.2) respectively, and the evolving curve converges to a circle in each non-local curvature flows, see Section 4. To put it simply, their method is the following. Since changing the tangential component of velocity does not affect the shapes of the evolving curves, they choose a suitable component so that tangential angle does not depend on time t. Moreover the curve can be parameterized by tangential angle θ, because of convexity. That is, let define the tangential angle θ by

$$\cos\theta(s) = \boldsymbol{f}'(s) \cdot \boldsymbol{e}_1.$$

When the curve is strictly convex, the function $\theta(s)$ is monotone decreasing. Therefore we use it as a spacial variable instead of s. For example the curvature flow equations (1.2), (1.3) and (1.4) are transported into

$$\partial_t \kappa = \kappa^2 \partial_\theta^2 \kappa + \kappa^3 - \frac{2\pi\kappa^2}{L}, \tag{1.5}$$

$$\partial_t \kappa = \kappa^2 \partial_\theta^2 \kappa + \kappa^2 \left(\kappa - \frac{1}{2\pi} \int_0^L \kappa^2 ds \right), \tag{1.6}$$

$$\partial_t \kappa = \kappa^2 \partial_\theta^2 \kappa + \kappa^3 - \frac{L}{2A} \kappa^2, \tag{1.7}$$

where $\kappa = \boldsymbol{\kappa} \cdot \boldsymbol{\nu}$ is the curvature. These are parabolic equations of second order for κ. They obtained conclusions, utilizing general properties of parabolic equations, for example the maximal principle. The key of this method is to parametrize the curve by tangential angle. Therefore it can not been utilized when initial curve is not convex. Now there is little results in non-local curvature flows, when initial curve is not convex. However, since the grain boundaries described above are not always convex, we would like to know the behavior of evolving curves not assuming convexity. To do this, we consider as follows.

Let $\boldsymbol{f} = (f_1, f_2) : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^2$ be a function such that $\mathrm{Im}\boldsymbol{f}$ is a closed plane curve with rotation number 1 and the variable of $\boldsymbol{f}$ is the arc-length parameter. The unit tangent vector is $\boldsymbol{\tau} = (f_1', f_2')$. Let $\boldsymbol{\nu} = (-f_2', f_1')$ be the inward unit normal vector, and let $\boldsymbol{\kappa} = \boldsymbol{f}''$ be the curvature vector. The (signed) area A is given by

$$A = -\frac{1}{2}\int_0^L \boldsymbol{f} \cdot \boldsymbol{\nu}\, ds.$$

The curvature $\kappa = \boldsymbol{\kappa} \cdot \boldsymbol{\nu}$ is positive when $\mathrm{Im}\boldsymbol{f}$ is convex. Since the curve has rotation number 1, the deviation of curvature is

$$\tilde{\kappa} = \kappa - \frac{1}{L}\int_0^L \kappa\, ds = \kappa - \frac{2\pi}{L}.$$

For a non-negative integer ℓ, we set

$$I_\ell = L^{2\ell+1}\int_0^L |\tilde{\kappa}^{(\ell)}|^2 ds,$$

which is a scale invariant quantity (cf. [2]). It is important to estimate I_ℓ for the global analysis of evolving curves. We get the Gagliardo-Nirenberg inequalities

$$I_\ell \leq C I_m^{\frac{\ell}{m}} I_0^{1-\frac{\ell}{m}},$$

where $0 \leqq \ell \leqq m$ and C is constant and independent of L. These are very useful but only they are not sufficient to estimate I_0 because these inequalities use I_0. Hence we need a different type of inequality to estimate I_ℓ for $\ell \geqq 0$.

The purpose of this paper is to prove the new interpolation inequalities by using isoperimetric ratio. Hereafter we call $\dfrac{4\pi A}{L^2}$ the isoperimetric ratio, not $\dfrac{L^2}{4\pi A}$. The curve along the flow (1.2) or (1.3) is expected to converge to a circle when the initial curve is close to a circle (in some sense) even if it is not convex. If it is true, the isoperimetric ratio converges to 1 as $t \to \infty$. Taking it into consideration, we introduce the quantity

$$I_{-1} = 1 - \frac{4\pi A}{L^2},$$

which is also scale invariant, and is non-negative by the isoperimetric inequality.

We assume it holds that

$$\frac{d}{dt}I_0 + C_1 I_1 \leqq I_0$$

for some evolving curve, where C_1 is a constant. Because of Wirtinger's inequality

$$I_0 \leqq \frac{I_1}{4\pi^2},$$

there exists $\lambda > 0$ such that

$$I_0 \leqq Ce^{-\lambda t}$$

if $C_1 > \dfrac{1}{4\pi^2}$. However we can not prove exponential decay of I_0 as above when $C_1 \leqq \dfrac{1}{4\pi^2}$. If we can estimate I_0 using I_{-1}, then it is expected that we can prove exponential decay of I_0 regardless of C_1 when $I_{-1} \to 0$.

Of course, $\tilde{\kappa} \equiv 0$ implies $\mathrm{Im}\boldsymbol{f}$ is a round circle, which attains the minimum $I_{-1} = 0$. This suggests that I_{-1} can be dominated by certain quantities involving $\tilde{\kappa}$. Indeed, we have we have

$$\begin{aligned} I_{-1} = \frac{L^2 - 4\pi A}{L^2} &= \frac{1}{L^2}\int_0^L (-L\boldsymbol{f}\cdot\boldsymbol{\kappa} + 2\pi\boldsymbol{f}\cdot\boldsymbol{\nu})\,ds \\ &= -\frac{1}{L}\int_0^L \tilde{\kappa}(\boldsymbol{f}\cdot\boldsymbol{\nu})ds \\ &= -\frac{1}{L}\int_0^L \tilde{\kappa}\left(\boldsymbol{f} - \frac{1}{L}\int_0^L \boldsymbol{f}\,ds\right)\cdot\boldsymbol{\nu}\,ds \end{aligned}$$

and

$$\left|\boldsymbol{f} - \frac{1}{L}\int_0^L \boldsymbol{f}\,ds\right| \leqq L$$

from which we obtain

$$0 \leqq I_{-1} \leqq I_0^{\frac{1}{2}}.$$

However, we can write

$$\begin{aligned} I_{-1} = &-\frac{1}{L}\int_0^L \tilde{\kappa} \\ &\times\left\{\left(\boldsymbol{f} - \frac{1}{L}\int_0^L \boldsymbol{f}\,ds\right)\cdot\boldsymbol{\nu} - \frac{1}{L}\int_0^L \left(\boldsymbol{f} - \frac{1}{L}\int_0^L \boldsymbol{f}\,ds\right)\cdot\boldsymbol{\nu}\,ds\right\}ds. \end{aligned}$$

Since

$$\left(\boldsymbol{f} - \frac{1}{L}\int_0^L \boldsymbol{f}\,ds\right)\cdot\boldsymbol{\nu} - \frac{1}{L}\int_0^L \left(\boldsymbol{f} - \frac{1}{L}\int_0^L \boldsymbol{f}\,ds\right)\cdot\boldsymbol{\nu}\,ds = 0$$

when $\tilde{\kappa} = 0$ vanished identically, it seems that the above inequality can be improved. In Section 2, we will show an improved version

$$0 \leqq I_{-1} \leqq \frac{I_0}{8\pi^2}$$

in Theorem 2.1.

The converse inequality seems not to hold; the reason will be clarified in Section 2. However, I_0 can be estimated by use of I_{-1} with the help of κ and its derivative

$$I_0 \leqq I_{-1}^{\frac{1}{2}} \left[L^3 \int_0^L \left\{ \kappa^3 \tilde{\kappa} + (\tilde{\kappa}')^2 \right\} ds \right]^{\frac{1}{2}}$$

(see Theorem 2.2). Combining this inequality and the Gagliardo-Nirenberg inequality, in Theorem 3.1 we will show interpolation inequalities satisfied by I_{-1}, I_ℓ and I_m for $0 \leqq \ell \leqq m$:

$$I_\ell \leqq C \left(I_{-1}^{\frac{m-\ell}{2}} I_m + I_{-1}^{\frac{m-\ell}{m+1}} I_m^{\frac{\ell+1}{m+1}} \right).$$

Here the constant C depends on ℓ and m but not on $\tilde{\kappa}$ nor L.

In section 4, we give applications of our inequalities to the analysis of the large-time behavior of some non-local curvature flows for closed plane curves.

In the final section, we consider the higher order curvature flow

$$\partial_t \boldsymbol{f} = (-1)^m (\partial_s^{2m} \tilde{\kappa}) \boldsymbol{\nu}. \tag{1.8}$$

This flow is the area-preserving flow (1.2) when $m = 0$.

When $m = 1$, the flow (1.8) is

$$\partial_t \boldsymbol{f} = -(\partial_s^2 \tilde{\kappa}) \boldsymbol{\nu}. \tag{1.9}$$

This flow was proposed by Mullins [14] and we call it *curve diffusion flow*. The flow is a fourth-order parabolic partial differential equation. Hence we do not expect convexity to be preserved along the flow. Indeed, Giga and Ito [8] showed the existence of a simple closed strictly convex plane curve that becomes non-convex in finite time under the flow. Also, Escher–Ito [4] and Chou [1] proved that evolving curves may develop singularities in finite time even when the initial curve is smooth.

On the other hand, there are some results for large-time behavior. Chou [1] showed that the evolving curve converges exponentially to a circle assuming the global existence of the flow. Moreover Elliott–Garcke [3] and Wheeler [19] showed the global existence and investigated the large-time behavior for initial data close to a circle.

Hence we investigate the large-time behavior of (1.8) assuming the global existence of the flow.

2 Preliminaries

For the vector-valued function $\boldsymbol{f} = (f_1, f_2) : \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^2$, we define a complex-valued function by

$$f = f_1 + if_2.$$

We expand f by the Fourier series

$$f = \sum_{k\in\mathbb{Z}} \hat{f}(k)\varphi_k,$$

where

$$\varphi_k(s) = \frac{1}{\sqrt{L}} \exp\left(\frac{2\pi iks}{L}\right), \quad \hat{f}(k) = \int_0^L f\overline{\varphi_k}ds.$$

The series $\displaystyle\sum_{k\in\mathbb{Z}} k^\ell|\hat{f}(k)|^2$ is related to $\displaystyle\left(\frac{L}{2\pi}\right)^\ell \int_0^L \kappa^\ell ds$. To see this we need some expression of $f^{(\ell-1)}\overline{f'}$ in terms of κ and its derivatives. Set

$$F_\ell = f^{(\ell-1)}\overline{f'}.$$

Lemma 2.1 *It holds that*

$$F_1 = \boldsymbol{f}\cdot\boldsymbol{\tau} + i\boldsymbol{f}\cdot\boldsymbol{\nu}, \quad F_\ell = i\kappa F_{\ell-1} + F'_{\ell-1} \quad \text{for} \quad \ell \geqq 2. \tag{2.1}$$

Proof. Firstly, since $\boldsymbol{\tau} = (f_1', f_2')$ and $\boldsymbol{\nu} = (-f_2', f_1')$, we have

$$F_1 = (f_1 + if_2)(f_1' - if_2') = \boldsymbol{f}\cdot\boldsymbol{\tau} + i\boldsymbol{f}\cdot\boldsymbol{\nu}.$$

The recurrence relation is derived from

$$F_\ell = f^{(\ell-1)}\overline{f'} = \left(f^{(\ell-2)}\overline{f'}\right)' - f^{(\ell-2)}\overline{f''} = F'_{\ell-1} - f^{(\ell-2)}\overline{f'}f'\overline{f''} = F'_{\ell-1} - F_{\ell-1}\overline{F_3}$$

and

$$F_3 = f''\overline{f'} = (f_1''+if_2'')(f_1'-if_2') = \frac{1}{2}\left(|f'|^2\right)' + i(-f_1''f_2' + f_2''f_1') = i\boldsymbol{f}''\cdot\boldsymbol{\nu} = i\kappa.$$

$\square$

Proposition 2.1 *For $\ell \geqq 2$, it holds that*

$$\sum_{k\in\mathbb{Z}} k^\ell|\hat{f}(k)|^2 = -i^{1-\ell}\left(\frac{L}{2\pi}\right)^\ell \int_0^L \kappa F_{\ell-1}ds. \tag{2.2}$$

Proof. It follows from the recurrence relation in Lemma2.1 that

$$\int_0^L F_\ell ds = i \int_0^L \kappa F_{\ell-1} ds.$$

On the other hand,

$$\int_0^L F_\ell ds = \langle f^{(\ell-1)}, f' \rangle_{L^2} = -i^\ell \left(\frac{2\pi}{L} \right)^\ell \sum_{k \in \mathbb{Z}} k^\ell |\hat{f}(k)|^2.$$

$\square$

Corollary 2.1 *We have*

$$\sum_{k \in \mathbb{Z}} k |\hat{f}(k)|^2 = \frac{LA}{\pi}, \tag{2.3}$$

$$\sum_{k \in \mathbb{Z}} k^2 |\hat{f}(k)|^2 = \left(\frac{L}{2\pi} \right)^2 \int_0^L \kappa^0 ds = \frac{L^3}{4\pi^2}, \tag{2.4}$$

$$\sum_{k \in \mathbb{Z}} k^3 |\hat{f}(k)|^2 = \left(\frac{L}{2\pi} \right)^3 \int_0^L \kappa \, ds = \frac{L^3}{4\pi^2}, \tag{2.5}$$

$$\sum_{k \in \mathbb{Z}} k^4 |\hat{f}(k)|^2 = \left(\frac{L}{2\pi} \right)^4 \int_0^L \kappa^2 ds, \tag{2.6}$$

$$\sum_{k \in \mathbb{Z}} k^5 |\hat{f}(k)|^2 = \left(\frac{L}{2\pi} \right)^5 \int_0^L \kappa^3 ds, \tag{2.7}$$

$$\sum_{k \in \mathbb{Z}} k^6 |\hat{f}(k)|^2 = \left(\frac{L}{2\pi} \right)^6 \int_0^L \left\{ \kappa^4 + (\kappa')^2 \right\} ds. \tag{2.8}$$

Remark 2.1 If $\kappa > 0$ everywhere, it holds that

$$\sum_{k \in \mathbb{Z}} k |\hat{f}(k)|^2 = \frac{L}{2\pi} \int_0^L \frac{1}{\kappa} \left\{ 1 - \left(f' \bar{f} \right)' \right\} ds.$$

In particular if κ is a constant, then

$$\sum_{k \in \mathbb{Z}} k |\hat{f}(k)|^2 = \frac{L}{2\pi} \int_0^L \frac{ds}{\kappa}.$$

Proof of Corollary 2.1. Since

$$\int_0^L F_1 ds = i\int_0^L \boldsymbol{f}\cdot\boldsymbol{\nu}\, ds = -2iA,$$

we obtain

$$\sum_{k\in\mathbb{Z}} k|\hat{f}(k)|^2 = \frac{L}{2\pi i}\int_0^L f'\bar{f}\, ds = \frac{L}{2\pi i}\int_0^L \overline{F_1} ds = \frac{LA}{\pi}.$$

Thus (2.3) follows. The relations (2.4)–(2.8) are consequence of (2.2) and (2.1). Indeed,

$$\sum_{k\in\mathbb{Z}} k^2|\hat{f}(k)|^2 = -i^{-1}\left(\frac{L}{2\pi}\right)^2\int_0^L \kappa F_1 ds = i\left(\frac{L}{2\pi}\right)^2\int_0^L \kappa\left(\boldsymbol{f}\cdot\boldsymbol{\tau} + i\boldsymbol{f}\cdot\boldsymbol{\nu}\right) ds,$$

and

$$\int_0^L \kappa\boldsymbol{f}\cdot\boldsymbol{\tau}\, ds = -\int_0^L \boldsymbol{f}\cdot\boldsymbol{\nu}' ds = \int_0^L \boldsymbol{\tau}\cdot\boldsymbol{\nu}\, ds = 0,$$
$$\int_0^L \kappa\boldsymbol{f}\cdot\boldsymbol{\nu}\, ds = \int_0^L \boldsymbol{f}\cdot\boldsymbol{\tau}'\, ds = -\int_0^L \boldsymbol{\tau}\cdot\boldsymbol{\tau}\, ds = -L.$$

Thus (2.4) holds. Since $\boldsymbol{f}$ is parametrized by the arc-length, we have $F_2 = |f'|^2 = \|\boldsymbol{f}'\|^2 = 1$. It follows from (2.1) that

$$F_3 = i\kappa, \quad F_4 = -\kappa^2 + i\kappa', \quad F_5 = -3\kappa\kappa' + i\left(-\kappa^3 + \kappa''\right).$$

Hence we obtain

$$\int_0^L \kappa F_2 ds = \int_0^L \kappa\, ds = 2\pi, \quad \int_0^L \kappa F_3 ds = i\int_0^L \kappa^2 ds,$$
$$\int_0^L \kappa F_4 ds = -\int_0^L \kappa^3 ds, \quad \int_0^L \kappa F_5 ds = -i\int_0^L \left\{\kappa^4 + (\kappa')^2\right\} ds.$$

Consequently (2.5)–(2.8) are obtained from (2.2).

Corollary 2.2

$$I_{-1} = \frac{4\pi^2}{L^3}\sum_{k\in\mathbb{Z}} k(k-1)|\hat{f}(k)|^2.$$

Proof. We obtain

$$I_{-1} = 1 - \frac{4\pi A}{L^2} = \frac{4\pi^2}{L^3}\left(\frac{L^3}{4\pi^2} - \frac{LA}{\pi}\right) = \frac{4\pi^2}{L^3}\sum_{k\in\mathbb{Z}} k(k-1)|\hat{f}(k)|^2$$

from (2.4) and (2.3). □

Since $k(k-1) \geqq 0$ for $k \in \mathbb{Z}$, we obtain the isoperimetric inequality $I_{-1} \geqq 0$ from this corollary, which is essentially the proof by Hurwitz [10].

Corollary 2.3

$$I_0 = \frac{16\pi^4}{L^3} \sum_{k\in\mathbb{Z}} k^3(k-1)|\hat{f}(k)|^2.$$

Proof. The assertion is derived as

$$\begin{aligned} I_0 &= L\int_0^L \tilde{\kappa}^2 ds = L\int_0^L \kappa\tilde{\kappa}\, ds = L\left(\int_0^L \kappa^2 ds - \frac{2\pi}{L}\int_0^L \kappa\, ds\right) \\ &= \frac{16\pi^4}{L^3} \sum_{k\in\mathbb{Z}} k^3(k-1)|\hat{f}(k)|^2, \end{aligned}$$

using (2.6) and (2.5). □

Since $k(k-1) \leqq k^3(k-1)$ for $k \in \mathbb{Z}$, we have

$$I_{-1} \leqq \frac{4\pi^2}{L^3} \sum_{k\in\mathbb{Z}} k^3(k-1)|\hat{f}(k)|^2 = \frac{I_0}{4\pi^2}. \tag{2.9}$$

We set

$$g = \sum_{k\in\mathbb{Z}} \sqrt{k(k-1)}\hat{f}(k)\varphi_k,$$

and then (2.9) is

$$\|g\|_{L^2}^2 \leqq \frac{L^2}{4\pi^2}\|g'\|_{L^2}^2.$$

This is Wirtinger's inequality with the best constant. Therefore it is reasonable to think that (2.9) cannot be sharpened. However, the function f is not an arbitrary one, but satisfies $|f'| \equiv 1$, and this suggests that we may be able to improve the constant in (2.9). Indeed, we can show an improved version by use of (2.4) and (2.5).

Theorem 2.1 *We have*

$$I_{-1} \leqq \frac{I_0}{8\pi^2}.$$

Equality never holds except the trivial case $\tilde{\kappa} \equiv 0$.

Proof. First observe that (2.4) and (2.5) imply

$$\sum_{k\in\mathbb{Z}} k^2|\hat{f}(k)|^2 = \sum_{k\in\mathbb{Z}} k^3|\hat{f}(k)|^2.$$

Hence

$$I_{-1} = \frac{4\pi^2}{L^3} \sum_{k\in\mathbb{Z}} k(k^2-1)|\hat{f}(k)|^2 = \frac{4\pi^2}{L^3} \sum_{k\in\mathbb{Z}} k(k-1)(k+1)|\hat{f}(k)|^2,$$
$$I_0 = \frac{16\pi^4}{L^3} \sum_{k\in\mathbb{Z}} k^2(k^2-1)|\hat{f}(k)|^2 = \frac{16\pi^4}{L^3} \sum_{k\in\mathbb{Z}} k^2(k-1)(k+1)|\hat{f}(k)|^2.$$

Consequently we obtain

$$\frac{I_0}{8\pi^2} - I_{-1} = \frac{2\pi^2}{L^3} \sum_{k\in\mathbb{Z}} k(k-2)(k-1)(k+1)|\hat{f}(k)|^2 \geqq 0,$$

because $k(k-2)(k-1)(k+1) \geqq 0$ for $k \in \mathbb{Z}$.

For the equality case, assume that f satisfies $I_{-1} = \dfrac{I_0}{8\pi^2}$. It follows from the previous paragraph that

$$f = \sum_{k=-1}^{2} \hat{f}(k)\varphi_k.$$

Since $|f'|^2 \equiv 1$ and since $\varphi_k\overline{\varphi_\ell} = L^{\frac{1}{2}}\varphi_{k-\ell}$, we have

$$1 = \left|\sum_{k=-1}^{2} \frac{2\pi i k}{L}\hat{f}(k)\varphi_k\right|^2 = \frac{4\pi^2}{L^2} \sum_{k,\ell=-1}^{2} k\ell\hat{f}(k)\overline{\hat{f}(\ell)}\varphi_k\overline{\varphi_\ell}$$
$$= \frac{4\pi^2}{L^{\frac{3}{2}}} \sum_{m=-3}^{3} \sum_{\substack{k-\ell=m \\ -1\leqq k\leqq 2,\, -1\leqq \ell\leqq 2}} k\ell\hat{f}(k)\overline{\hat{f}(\ell)}\varphi_m.$$

From this we find that, in particular,

$$0 = \sum_{\substack{k-\ell=3 \\ -1\leqq k\leqq 2,\, -1\leqq \ell\leqq 2}} k\ell\hat{f}(k)\overline{\hat{f}(\ell)} = -2\hat{f}_2\overline{\hat{f}_{-1}}.$$

On the other hand, it follows from (2.4) and (2.6) that

$$0 = \sum_{k=-1}^{2} k^2(k-1)|\hat{f}(k)|^2 = -2|\hat{f}_{-1}|^2 + 4|\hat{f}_2|^2.$$

Therefore $\hat{f}_{-1} = \hat{f}_2 = 0$. Consequently $f = \hat{f}_0\varphi_0 + \hat{f}_1\varphi_1$, which implies $\mathrm{Im}\boldsymbol{f}$ is a round circle. Hence $\tilde{\kappa} \equiv 0$. $\square$

We remark that it is impossible to show

$$k^3(k-1) \leqq Ck(k-1) \quad \text{for} \quad k \in \mathbb{Z},$$

and this implies that there is no hope to see $I_0 \leqq CI_{-1}$. Thereupon we give an estimate of I_0 in terms of I_{-1} with the help of κ and its derivative.

Theorem 2.2 *The integral* $\displaystyle\int_0^L \left\{\kappa^3\tilde{\kappa} + (\tilde{\kappa}')^2\right\} ds$ *is non-negative, and it holds that*

$$I_0 \leqq I_{-1}^{\frac{1}{2}} \left[L^3 \int_0^L \left\{\kappa^3\tilde{\kappa} + (\tilde{\kappa}')^2\right\} ds \right]^{\frac{1}{2}}.$$

Equality never holds except the trivial case $\tilde{\kappa} \equiv 0$.

Proof. By Cauchy's inequality we have

$$\begin{aligned} I_0 &\leqq \frac{16\pi^4}{L^3} \left\{ \sum_{k\in\mathbb{Z}} k(k-1)|\hat{f}(k)|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{k\in\mathbb{Z}} k^5(k-1)|\hat{f}(k)|^2 \right\}^{\frac{1}{2}} \\ &= \frac{8\pi^3}{L^{\frac{3}{2}}} I_{-1}^{\frac{1}{2}} \left\{ \sum_{k\in\mathbb{Z}} k^5(k-1)|\hat{f}(k)|^2 \right\}^{\frac{1}{2}}, \end{aligned}$$

and (2.7) and (2.8) show that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} k^5(k-1)|\hat{f}(k)|^2 &= \left(\frac{L}{2\pi}\right)^6 \int_0^L \left\{ \kappa^4 + (\kappa')^2 - \frac{2\pi}{L}\kappa^3 \right\} ds \\ &= \left(\frac{L}{2\pi}\right)^6 \int_0^L \left\{\kappa^3\tilde{\kappa} + (\tilde{\kappa}')^2\right\} ds. \end{aligned}$$

Assume that f satisfies the equality case in the assertion. It follows from the equality condition of Cauchy's inequality that $\hat{f}_k = 0$ except $k = 0$ and 1. Consequently $\mathrm{Im}\boldsymbol{f}$ is a round circle, and $\tilde{\kappa} \equiv 0$. $\square$

3 Interpolation inequalities

In this section we derive several interpolation inequalities from Theorem 2.2.

Theorem 3.1 *Let* $0 \leqq \ell \leqq m$. *There exists a positive constant* $C = C(\ell, m)$ *independent of* L *such that*

$$I_\ell \leqq C \left(I_{-1}^{\frac{m-\ell}{2}} I_m + I_{-1}^{\frac{m-\ell}{m+1}} I_m^{\frac{\ell+1}{m+1}} \right)$$

holds.

Proof. When $\ell = m$, the assertion is clear.

Let $\ell < m$. Then $m \geqq 1$. The Gagliardo-Nirenberg inequality shows

$$\left(L^{(j+1)p-1}\int_0^L |\tilde{\kappa}^{(j)}|^p ds\right)^{\frac{1}{p}} \leqq C(j,m,p) I_m^{\frac{1}{2m}\left(j-\frac{1}{p}+\frac{1}{2}\right)} I_0^{\frac{1}{2}\left\{1-\frac{1}{m}\left(j-\frac{1}{p}+\frac{1}{2}\right)\right\}} \tag{3.1}$$

for $p \geqq 2$ and $j \leqq m$. Here $C(j,m,p)$ is independent of L. Combining this and Wirtinger's inequality $I_0 \leqq \dfrac{I_1}{4\pi^2}$, we have

$$L^3\int_0^L \tilde{\kappa}^4 ds \leqq C I_1^{\frac{1}{2}} I_0^{\frac{3}{2}} \leqq C I_1^2, \quad L_2\int_0^L |\tilde{\kappa}|^3 ds \leqq C I_1^{\frac{1}{4}} I_0^{\frac{5}{4}} \leqq C I_1^{\frac{3}{2}}.$$

Therefore

$$\begin{aligned} L^3\int_0^L \kappa^3\tilde{\kappa}\, ds &= L^3\int_0^L \left(\tilde{\kappa}+\frac{2\pi}{L}\right)^3 \tilde{\kappa}\, ds \\ &\leqq C\left(L^3\int_0^L \tilde{\kappa}^4 ds + L^2\int_0^L |\tilde{\kappa}|^3 ds + L\int_0^L \tilde{\kappa}^2 ds\right) \\ &\leqq C\left(I_1^2 + I_1^{\frac{3}{2}} + I_1\right). \end{aligned}$$

Consequently the inequality in Theorem 2.2 implies

$$I_0 \leqq C I_{-1}^{\frac{1}{2}}\left(I_1 + I_1^{\frac{1}{2}}\right),$$

which is the assertion with $\ell = 0$, $m = 1$.

Putting $p = 2$ in (3.1), we have

$$I_j \leqq C(j,m) I_m^{\frac{j}{m}} I_0^{1-\frac{j}{m}}. \tag{3.2}$$

Combining these with $j = 1$ and Young's inequality, we have

$$I_0 \leqq C I_{-1}^{\frac{1}{2}}\left(I_m^{\frac{1}{m}} I_0^{1-\frac{1}{m}} + I_m^{\frac{1}{2m}} I_0^{\frac{1}{2}\left(1-\frac{1}{m}\right)}\right) \leqq \epsilon I_0 + C_\epsilon\left(I_{-1}^{\frac{m}{2}} I_m + I_{-1}^{\frac{m}{m+1}} I_m^{\frac{1}{m+1}}\right),$$

where ϵ is an arbitrary positive number. Consequently we obtain

$$I_0 \leqq C\left(I_{-1}^{\frac{m}{2}} I_m + I_{-1}^{\frac{m}{m+1}} I_m^{\frac{1}{m+1}}\right),$$

which is the assertion with $\ell = 0$, $m \geqq 2$.

Let $\ell \geqq 1$. Using the above inequality and (3.2) with $j = \ell$, we obtain

$$I_\ell \leqq C I_m^{\frac{\ell}{m}}\left(I_{-1}^{\frac{m}{2}} I_m + I_{-1}^{\frac{m}{m+1}} I_m^{\frac{1}{m+1}}\right)^{1-\frac{\ell}{m}} \leqq C\left(I_{-1}^{\frac{m-\ell}{2}} I_m + I_{-1}^{\frac{m-\ell}{m+1}} I_m^{\frac{\ell+1}{m+1}}\right).$$

□

4 Applications to non-local flows

We give applications of our inequalities to the asymptotic analysis of geometric flows of closed plane curves. Two of the flows are the flow (1.4) studied by Jiang-Pan [11], and the area-preserving curvature flow (1.2) considered by Gage [6]. We also consider the length-preserving curvature flow (1.3). If the initial curve is convex, then the flows exist for all time keeping the convexity, and the curve approaches a round circle; this was shown in [11, 6, 12]. The local existence of flows without a convexity assumption was shown by Ševčovič-Yazaki [18]. However, the large-time behavior for this case is still open. It seems finite-time blow-up may occur for some non-convex initial curves [13], but, on the other hand, the global existence for a certain initial non-convex curve was shown in [18]. Escher-Simonett [5] showed the global existence and investigated the large-time behavior of the area-preserving curvature flow for initial data close to a circle and without a convexity assumption. In this section, we investigate the large-time behavior of the flow without a convexity assumption *assuming* the global existence.

Firstly we consider the general flows

$$\partial_t \boldsymbol{f} = h\boldsymbol{\nu}, \tag{4.1}$$

where $h = \kappa + \varphi(L, A, W)$. Here $W := \dfrac{1}{2}\displaystyle\int_0^L \kappa^2 ds$ is the elastic energy and φ is a smooth function of L, A and W. We assume the global existence of solutions. Observe that the equation which $\boldsymbol{f}$ satisfies is

$$\partial_t \boldsymbol{f} = \partial_s^2 \boldsymbol{f} - \varphi(L, A, W) R \partial_s \boldsymbol{f},$$

where

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since this is a parabolic equation with a non-local term, we may assume that $\boldsymbol{f}$ is smooth for $t > 0$ as long as the solution exists. Hence by shifting the initial time, we may assume that the initial data is smooth.

Let $P_n^k(\tilde{\kappa})$ be any linear combination of the type

$$P_n^k(\tilde{\kappa}) = \sum_{i_1+\cdots+i_n=k} c_{i_1,\ldots,i_n} \partial_s^{i_1}\tilde{\kappa} \cdots \partial_s^{i_n}\tilde{\kappa}$$

with universal, constant coefficients $c_{i_1,\ldots,i_n}$ for $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Similarly we define P_0^k as a universal constant.

For the flows (1.2), (1.3), we can show the following.

(1) I_{-1} decays exponentially as $t \to \infty$: $I_{-1} \leqq C_{-1}e^{-\lambda t}$.

(2) I_0 is integrable on $[0, \infty)$: $\displaystyle\int_0^\infty I_0 dt < \infty$.

(3) $L(t)$ is strictly positive on $[0, \infty)$ and there exists positive constant L_∞ such that $L(t)$ converges to L_∞.

(4) $A(t)$ is strictly positive on $[0, \infty)$ and there exists positive constant A_∞ such that $A(t)$ converges to A_∞.

(5) There exist positive constants $C_1, \cdots, C_4$ independent of initial data and t such that

$$\frac{d}{dt}I_0 + \frac{C_1}{L^2}I_1 + \frac{C_2}{L^2}I_0^2 \leqq \frac{C_3}{L^2}I_0^3 + \frac{C_4}{L^2}I_{-1}.$$

(6) There exist $1 \leqq \alpha < 5, 1 \leqq \beta < 5, \gamma \geqq 0, \sigma \geqq 0$ such that

$$\begin{aligned} \frac{d}{dt}I_\ell + \frac{2}{L^2}I_{\ell+1} \leqq {} & L^{2\ell+1}\int_0^L \partial_s^\ell \tilde{\kappa} \sum_{n=1}^{\alpha} L^{-(\alpha-n)} P_n^\ell(\tilde{\kappa})ds \\ & + L^{2\ell+1}\left(\frac{L}{A}\right)^\gamma \left(\int_0^L \partial_s^\ell \tilde{\kappa} \sum_{n=1}^{\beta} L^{-(\beta-n)} P_n^\ell(\tilde{\kappa})ds\right)\left(\int_0^L \tilde{\kappa}^2 ds\right)^\sigma. \end{aligned}$$

These are ingredients for proving the convergence of flow to a circle. We shall show this fact in subsection 4.1. And we shall see that (1)–(6) hold true for the flow (1.2), (1.3) respectively in the subsection 4.1–4.3.

4.1 General flows

Here we consider the large-time behavior of solutions to (4.1) under the assumption (1)–(6). Firstly we prove the exponential decay of I_ℓ for $\ell \in \mathbb{N} \cup \{0\}$.

Theorem 4.1 *Assume that* $\boldsymbol{f}$ *is a global solution of* (4.1) *such that the initial rotation number is* 1 *and the initial* (*signed*) *area is positive. Moreover we assume* (1)*–*(6). *Then, for each* $\ell \in \mathbb{N} \cup \{0\}$, *there exist* $C_\ell > 0$ *and* $\lambda_\ell > 0$ *such that*

$$I_\ell(t) \leqq C_\ell e^{-\lambda_\ell t}.$$

Proof. We firstly consider when $\ell = 0$. It is obvious that there exists T_1 satisfying

$$\int_{T_1}^{\infty} \frac{C_4}{L^2} e^{-\lambda t} dt < \frac{C_2}{2C_3}.$$

Furthermore, there exists $T_2 \geqq T_1$ such that $I_0(T_2) < \dfrac{C_2}{2C_3}$, because of (2). We would like to show $I_0(t) < \dfrac{C_2}{C_3}$ for $t \geqq T_2$. To do this, we argue by contradiction. Then there exists $T_3 > T_2$ such that

$$I_0(t) < \frac{C_2}{C_3} \text{ for } t \in [T_2, T_3), \text{ and } I_0(T_3) = \frac{C_2}{C_3}.$$

It follows from (1) and (5) that

$$\frac{d}{dt} I_0 \leqq \frac{C_4}{L^2} e^{-\lambda t}$$

for $t \in [T_2, T_3]$. Hence

$$I_0(T_3) = I_0(T_2) + \int_{T_2}^{T_3} \frac{d}{dt} I_0 dt < \frac{C_2}{2C_3} + \int_{T_1}^{\infty} \frac{C_4}{L^2} e^{-\lambda t} dt < \frac{C_2}{C_3}.$$

This contradicts $I_0(T_3) = \dfrac{C_2}{C_3}$. Consequently I_0 is uniformly bounded, and (5) implies

$$\frac{d}{dt} I_0 + \frac{C_1}{L^2} I_1 \leqq \frac{C_4}{L^2} e^{-\lambda t}.$$

Thus the assertion for $\ell = 0$ with some positive λ_0 has been proved.

Next we show the exponential decay of I_ℓ for $\ell \in \mathbb{N}$. Set

$$J_{k,p} = \left\{ L^{(1+k)p-1} \int_0^L |\partial_s^k \tilde{\kappa}|^p ds \right\}^{\frac{1}{p}}.$$

By the Gagliardo-Nirenberg inequalities we have

$$J_{k,p} \leqq C J_{m,2}^{\theta} J_{0,2}^{1-\theta} = C I_m^{\frac{\theta}{2}} I_0^{\frac{1-\theta}{2}} \tag{4.2}$$

for $k \in \{0, 1, \ldots, m\}$, $p \geqq 2$. Here C is independent of L, and $\theta = \frac{1}{m}\left(k - \frac{1}{p} + \frac{1}{2}\right) \in [0, 1]$.

Since $P_1^\ell(\tilde\kappa) = c\partial_s^\ell \tilde\kappa$ and L, A are uniformly bounded by (3)–(4), we have

$$L^{2\ell+1}\left|\int_0^L (\partial_s^\ell\tilde\kappa) L^{-(\alpha-1)} P_1^\ell(\tilde\kappa)\, ds\right| = \frac{|c|}{L^2} I_\ell,$$

$$L^{2\ell+1}\left|\int_0^L (\partial_s^\ell\tilde\kappa)\left(\frac{L}{A}\right)^\gamma L^{-(\alpha-1)} P_1^\ell(\tilde\kappa)\, ds\right|\left(\int_0^L \tilde\kappa^2 ds\right)^\sigma \leqq \frac{C}{L^2} I_\ell^{\sigma+1}.$$

When $\alpha \geqq 2$, by Hölder's inequality, we have

$$\begin{aligned}\left|L^{2\ell+1}\int_0^L \partial_s^\ell\tilde\kappa \sum_{n=2}^{\alpha} L^{-(\alpha-n)} P_n^\ell(\tilde\kappa) ds\right| &= \left|\sum_{n=2}^{\alpha} L^{2\ell+1}\int_0^L (\partial_s^\ell\tilde\kappa) L^{-(\alpha-n)} P_n^\ell(\tilde\kappa) ds\right| \\ &\leqq \sum_{\substack{j_1+\cdots+j_n=n\\ j_1\geqq 0,\cdots j_n\geqq 0}} \frac{C}{L^2} J_{\ell,n+1} J_{j_1,n+1}\cdots J_{j_n,n+1},\end{aligned}$$

and (4.2) yields

$$J_{j,n+1} \leqq C I_{\ell+1}^{\frac{\theta(j,n+1)}{2}} I_0^{\frac{1-\theta(j,n+1)}{2}}, \quad \theta(j,n+1) = \frac{1}{m}\left(j - \frac{1}{n+1} + \frac{1}{2}\right).$$

Hence applying Young's inequality, we obtain

$$\begin{aligned}\left|L^{2\ell+1}\int_0^L \partial_s^\ell\tilde\kappa \sum_{n=2}^{\alpha} L^{-(\alpha-n)} P_n^\ell(\tilde\kappa) ds\right| &\leqq \frac{C}{L^2} I_{\ell+1}^{\frac{2\ell+\frac{n-1}{2}}{2(\ell+1)}} I_0^{\frac{\ell(n-1)+\frac{n+3}{2}}{2(\ell+1)}} \\ &\leqq \frac{\epsilon}{L^2} I_{\ell+1} + \frac{C_\epsilon}{L^2} I_0^{\frac{2\ell(n-1)+n+3}{5-n}}\end{aligned}$$

for any $\epsilon > 0$. The second term is a function decaying exponentially. When $\beta \geqq 2$, we have

$$\begin{aligned}&\left|L^{2\ell+1}\left(\frac{L}{A}\right)^\gamma \int_0^L \partial_s^\ell\tilde\kappa \sum_{n=2}^{\beta} L^{-(\alpha-n)} P_n^\ell(\tilde\kappa) ds\right|\left(\int_0^L \tilde\kappa^2 ds\right)^\sigma \\ &\qquad \leqq \left(\frac{\epsilon}{L^2} I_{\ell+1} + \frac{C_\epsilon}{L^2} I_0^{\frac{2\ell(n-1)+n+3}{5-n}}\right) I_0^\sigma\end{aligned}$$

for any $\epsilon > 0$ same ways as above. Hence we obtain

$$\begin{aligned}\frac{d}{dt} I_\ell &+ \frac{2}{L^2} I_{\ell+1} \\ &\leqq \frac{C}{L^2} I_\ell + \frac{C}{L^2} I_\ell^{\sigma+1} + \frac{\epsilon}{L^2} I_{\ell+1} + \frac{C_\epsilon}{L^2} I_0^{\frac{2\ell(n-1)+n+3}{5-n}} \\ &\quad + \frac{\epsilon}{L^2} I_{\ell+1} I_0^\sigma + \frac{C_\epsilon}{L^2} I_0^{\frac{2\ell(n-1)+n+3}{5-n}} I_0^\sigma.\end{aligned} \tag{4.3}$$

Taking ϵ small and t large, the terms $\dfrac{\epsilon}{L^2} I_{\ell+1}, \dfrac{\epsilon}{L^2} I_{\ell+1} I_0^\sigma$ are included into the left-hand side of (4.3) from exponential decay of I_0. The terms $\dfrac{C_\epsilon}{L^2} I_0^{\frac{2\ell(n-1)+n+3}{5-n}}$, $\dfrac{C_\epsilon}{L^2} I_0^{\frac{2\ell(n-1)+n+3}{5-n}} I_0^\sigma$ are functions decaying exponentially. Furthermore we have from Theorem 3.1

$$\frac{C}{L^2} I_\ell \leqq \frac{C}{L^2} \left(I_{-1}^{\frac{1}{2}} I_{\ell+1} + I_{-1}^{\frac{1}{\ell+2}} I_{\ell+1}^{\frac{\ell+1}{\ell+2}} \right) \leqq \frac{C}{L^2} \left\{ \left(I_{-1}^{\frac{1}{2}} + \epsilon \right) I_{\ell+1} + C_\epsilon I_{-1} \right\}$$

$$\begin{aligned} \frac{C}{L^2} I_\ell^{\sigma+1} &\leqq \frac{C}{L^2} \left(I_{-1}^{\frac{1}{2}} I_{\ell+1} + I_{-1}^{\frac{1}{\ell+2}} I_{\ell+1}^{\frac{\ell+1}{\ell+2}} \right)^{\sigma+1} \\ &\leqq \frac{C}{L^2} \left\{ \left(I_{-1}^{\frac{1}{2}} + \epsilon \right)^{\sigma+1} I_{\ell+1}^{\sigma+1} + C_\epsilon^{\sigma+1} I_{-1}^{\sigma+1} \right\} \end{aligned}$$

for any $\epsilon > 0$. Taking ϵ small and t large, the terms $\dfrac{C}{L^2} \left\{ \left(I_{-1}^{\frac{1}{2}} + \epsilon \right) I_{\ell+1} \right\}$, $\dfrac{C}{L^2} \left\{ \left(I_{-1}^{\frac{1}{2}} + \epsilon \right)^{\sigma+1} I_{\ell+1}^{\sigma+1} \right\}$ can be absorbed into the left-hand side of (4.3). The terms $C_\epsilon I_{-1}, C_\epsilon^{\sigma+1} I_{-1}^{\sigma+1}$ are functions decaying exponentially. Therefore (4.3) and Wirtinger's inequality imply

$$\frac{d}{dt} I_\ell + \frac{C}{L^2} I_\ell \leqq C e^{-\mu t},$$

which shows the exponential decay of I_ℓ. $\square$

It follows from Theorem 4.1 that $\tilde{\kappa}$ uniformly converges to 0 as $t \to \infty$. This implies that Im$\boldsymbol{f}$ converges to a circle. The precise statement is as follows.

Theorem 4.2 *Let $\boldsymbol{f}$ be as in Theorem 4.1, and let $f(s,t) = \sum_{k \in \mathbb{Z}} \hat{f}(k)(t) \varphi_k(s)$ be the Fourier expansion for any fixed $t > 0$. Set*

$$\boldsymbol{c}(t) = \frac{1}{\sqrt{L(t)}} (\Re \hat{f}(0)(t), \Im \hat{f}(0)(t)),$$

and define $r(t) \geqq 0$ and $\sigma(t) \in \mathbb{R}/2\pi\mathbb{Z}$ by

$$\hat{f}(1)(t) = \sqrt{L(t)} r(t) \exp\left(i \frac{2\pi\sigma(t)}{L(t)} \right).$$

Furthermore we set

$$\tilde{\boldsymbol{f}}(\theta, t) = \boldsymbol{f}(L(t)\theta - \sigma(t), t), \quad \textit{for} \quad (\theta, t) \in \mathbb{R}/\mathbb{Z} \times [0, \infty).$$

We assume that there exists $\boldsymbol{c}_\infty \in \mathbb{R}^2$ *such that*

$$\|\boldsymbol{c}(t) - \boldsymbol{c}_\infty\| \leqq Ce^{-\gamma t}.$$

Then the following claims hold.

(A) *The function* $r(t)$ *converges exponentially to the constant* $\dfrac{L_\infty}{2\pi}$ *as* $t \to \infty$*:*

$$\left| r(t) - \frac{L_\infty}{2\pi} \right| \leqq Ce^{-\gamma t}.$$

(B) *There exists* $\sigma_\infty \in \mathbb{R}/2\pi\mathbb{Z}$ *such that*

$$|\sigma(t) - \sigma_\infty| \leqq Ce^{-\gamma t}.$$

(C) *For any* $k \in \mathbb{N} \cup \{0\}$ *there exist* $C_k > 0$ *and* $\gamma_k > 0$ *such that*

$$\|\tilde{\boldsymbol{f}}(\cdot, t) - \tilde{\boldsymbol{f}}_\infty\|_{C^k(\mathbb{R}/\mathbb{Z})} \leqq C_k e^{-\gamma_k t},$$

where

$$\tilde{\boldsymbol{f}}_\infty(\theta) = \boldsymbol{c}_\infty + \frac{L_\infty}{2\pi}(\cos 2\pi\theta, \sin 2\pi\theta).$$

(D) *For sufficiently large* t, $\mathrm{Im}\,\tilde{\boldsymbol{f}}(\cdot, t)$ *is the boundary of a bounded domain* $\Omega(t)$*. Furthermore, there exists* $T_* \geqq 0$ *such that* $\Omega(t)$ *is strictly convex for* $t \geqq T_*$*.*

(E) *Let* $D_{r_\infty}(\boldsymbol{c}_\infty)$ *be the closed disk with center* $\boldsymbol{c}_\infty$ *and radius* r_∞*. Then we have*

$$d_H(\overline{\Omega(t)}, D_{r_\infty}(\boldsymbol{c}_\infty)) \leqq Ce^{-\gamma t},$$

where d_H *is the Hausdorff distance.*

(F) *Let* $\boldsymbol{b}(t) = \dfrac{1}{A(t)} \displaystyle\iint_{\Omega(t)} \boldsymbol{x}\, d\boldsymbol{x}$ *be the barycenter of* $\Omega(t)$*. Then we have*

$$\|A(t)(\boldsymbol{b}(t) - \boldsymbol{c}(t))\| \leqq Ce^{-\gamma t}.$$

Proof. (A) It follows from Proposition 2.1 and Lemma 2.1 that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} k^\ell (k-1)|\hat{f}(k)|^2 =& -i^{-\ell}\left(\frac{L}{2\pi}\right)^{\ell+1}\int_0^L \kappa F_\ell ds \\ &+ i^{1-\ell}\left(\frac{L}{2\pi}\right)^{\ell}\int_0^L \kappa F_{\ell-1} ds \\ =& -i^{-\ell}\left(\frac{L}{2\pi}\right)^{\ell+1}\int_0^L \kappa F_\ell ds \\ &+ i^{-\ell}\left(\frac{L}{2\pi}\right)^{\ell}\int_0^L F_\ell ds \\ =& -i^{-\ell}\left(\frac{L}{2\pi}\right)^{\ell+1}\int_0^L \tilde{\kappa} F_\ell ds\end{aligned}$$

for $\ell \geqq 2$. Since F_2 is a constant, and since F_ℓ with $\ell \geqq 3$ is a polynomial function of κ and its derivatives up to the $(\ell-3)$rd order, they are bounded functions of (s,t). Also L is bounded and $\tilde{\kappa}$ decays exponentially as $t \to \infty$. Therefore when ℓ is odd,

$$\left|\sum_{k\neq 0,1} k^{\ell+1}|\hat{f}(k)|^2\right| \leqq C\left|\sum_{k\in\mathbb{Z}} k^\ell(k-1)|\hat{f}(k)|^2\right| \leqq C_\ell e^{-\gamma_\ell t}.$$

By the Parseval identity and the Sobolev embedding theorem we have

$$\left\| f(\cdot,t) - \hat{f}(0)(t)\varphi_0(\cdot) - \hat{f}(1)(t)\varphi_1(\cdot)\right\|_{C^k(\mathbb{R}/L(t)\mathbb{Z})} \leqq C_k e^{-\gamma'_k t}$$

for any k. Using the expression of $\mathbb{R}^2$-valued functions, we have

$$\boldsymbol{f}(s,t) = \boldsymbol{c}(t) + r(t)\left(\cos\frac{2\pi(s+\sigma(t))}{L(t)}, \sin\frac{2\pi(s+\sigma(t))}{L(t)}\right) + \boldsymbol{\rho}(s,t),$$

$$\|\boldsymbol{\rho}(\cdot,t)\|_{C^k(\mathbb{R}/L(t)\mathbb{Z})} \leqq C_k e^{-\gamma'_k t}.$$

Since

$$\partial_s \boldsymbol{f}(s,t) = \frac{2\pi r(t)}{L(t)}\left(-\sin\frac{2\pi(s+\sigma(t))}{L(t)}, \cos\frac{2\pi(s+\sigma(t))}{L(t)}\right) + \partial_s\boldsymbol{\rho}(s,t),$$

we have

$$\begin{aligned}\left|r(t) - \frac{L(t)}{2\pi}\right| = \frac{L(t)}{2\pi}\left|\frac{2\pi r(t)}{L(t)} - 1\right| &= \frac{L(t)}{2\pi}\left|\|\partial_s\boldsymbol{f}(s,t) - \partial_s\boldsymbol{\rho}(s,t)\| - 1\right| \\ &\leqq Ce^{-\gamma_1 t}.\end{aligned}$$

Therefore $r(t)$ converges to $r_\infty = \dfrac{L_\infty}{2\pi}$ exponentially as $t \to \infty$.

(B) First we clarify the meaning of $\partial_t \boldsymbol{f}$, it is not $\lim_{h\to 0} \dfrac{\boldsymbol{f}(s,t+h)-\boldsymbol{f}(s,t)}{h}$ as one might expect. The variable s in $f(s,t)$ is an element of $\mathbb{R}/L(t)\mathbb{Z}$, on the other hand, the s in $f(s,t+h)$ is in $\mathbb{R}/L(t+h)\mathbb{Z}$. Hence the above quotient is not well-defined. To address this, let us introduce a function $\bar{\boldsymbol{f}}$ on $\mathbb{R}/2\pi\mathbb{Z}\times[0,\infty)$ given by

$$\bar{\boldsymbol{f}}(u,t) = \boldsymbol{f}\left(\frac{L(t)u}{2\pi}, t\right).$$

Then the variable u is independent of t, and $\partial_t \boldsymbol{f}$ is given by

$$\partial_t \boldsymbol{f} = \lim_{h\to 0} \frac{\bar{\boldsymbol{f}}(u,t+h)-\bar{\boldsymbol{f}}(u,t)}{h}.$$

We define a complex-valued function $\bar{f}$ by

$$\bar{\boldsymbol{f}}(u,t) = (\Re \bar{f}(u,t), \Im \bar{f}(u,t)),$$

and note that the Fourier expansion of $\bar{f}$ is

$$\bar{f}(u,t) = \sum_{k\in\mathbb{Z}} \hat{f}(k)(t)\varphi_k\left(\frac{L(t)u}{2\pi}\right) = \sqrt{2\pi}\sum_{k\in\mathbb{Z}} \frac{\hat{f}(k)(t)}{\sqrt{L(t)}}\phi_k(u),$$

where

$$\phi_k(u) = \frac{1}{\sqrt{2\pi}} e^{iku}.$$

Therefore we have

$$\partial_t \bar{f} = \sqrt{2\pi}\sum_{k\in\mathbb{Z}} \frac{d}{dt}\frac{\hat{f}(k)(t)}{\sqrt{L(t)}}\phi_k(u),$$

and

$$\int_0^{2\pi} |\partial_t \bar{f}|^2 du = 2\pi \sum_{k\in\mathbb{Z}} \left| \frac{d}{dt}\frac{\hat{f}(k)(t)}{\sqrt{L(t)}} \right|^2$$

by the Parseval identity. Since

$$|\partial_t \bar{f}|^2 = \|\partial_t \bar{\boldsymbol{f}}\|^2 = \|\partial_t \boldsymbol{f}\|^2 = \left\| \boldsymbol{\kappa} - \frac{L}{2A}\boldsymbol{\nu} \right\|^2 = \left| \tilde{\kappa} - \frac{L}{2A} I_{-1} \right|^2$$

decays exponentially and uniformly in spatial variable as $t\to\infty$, we have

$$\sum_{k\in\mathbb{Z}} \left| \frac{d}{dt}\frac{\hat{f}(k)(t)}{\sqrt{L(t)}} \right|^2 \leqq Ce^{-2\gamma t}$$

for some $C > 0$ and $\gamma > 0$. In particular,

$$\left| \frac{d}{dt} \frac{\hat{f}(1)(t)}{\sqrt{L(t)}} \right|^2 \leqq Ce^{-2\gamma t}.$$

Also, it is not difficult to see that

$$\left| \frac{d}{dt} \frac{\hat{f}(1)(t)}{\sqrt{L(t)}} \right|^2 = \left| \frac{d}{dt} r(t) \right|^2 + 4\pi^2 r(t)^2 \left| \frac{d}{dt} \left(\frac{\sigma(t)}{L(t)} \right) \right|^2.$$

Since $r(t)$ and $L(t)$ converge exponentially to positive constants, so does $\sigma(t)$ to some $\sigma_\infty \in \mathbb{R}/2\pi\mathbb{Z}$.

(C) We have

$$\begin{aligned} \tilde{\boldsymbol{f}}(\theta, t) &= \boldsymbol{c}(t) + r(t) (\cos 2\pi\theta, \sin 2\pi\theta) + \tilde{\boldsymbol{\rho}}(\theta, t) \\ &= \tilde{\boldsymbol{f}}_\infty(\theta) + \boldsymbol{c}(t) - \boldsymbol{c}_\infty + (r(t) - r_\infty) (\cos 2\pi\theta, \sin 2\pi\theta) + \tilde{\boldsymbol{\rho}}(\theta, t), \end{aligned}$$

where

$$\tilde{\boldsymbol{\rho}}(\theta, t) = \boldsymbol{\rho}(L(t)\theta - \sigma(t), t).$$

Therefore the estimates for $\boldsymbol{c}(t)$, $r(t)$, and $\boldsymbol{\rho}(\cdot, t)$ yield

$$\|\tilde{\boldsymbol{f}}(\cdot, t) - \tilde{\boldsymbol{f}}_\infty\|_{C^k(\mathbb{R}/\mathbb{Z})} \leqq C_k e^{-\tilde{\gamma}_k t}.$$

(D) The above estimate implies that $\mathrm{Im}\tilde{\boldsymbol{f}}(\cdot, t)$ is the boundary of a bounded domain $\Omega(t)$ when t is sufficiently large. Since $\tilde{\kappa}$ converges to 0 uniformly, and since L goes to a positive constant L_∞ as $t \to \infty$ uniformly in s,

$$\kappa = \frac{2\pi}{L} + \tilde{\kappa}$$

is strictly positive for large t. Consequently $\partial\Omega(t)$ is a strictly convex curve.

(E) Let $D_{r(t)}(\boldsymbol{c}(t))$ be the closed disk with center $\boldsymbol{c}(t)$ and the radius $r(t)$. We have

$$\begin{aligned} & d_H(\overline{\Omega(t)}, D_{r_\infty}(\boldsymbol{c}_\infty)) \\ & \quad \leqq d_H(\overline{\Omega(t)}, D_{r(t)}(\boldsymbol{c}(t))) + d_H(D_{r(t)}(\boldsymbol{c}(t)), D_{r(t)}(\boldsymbol{c}_\infty)) \\ & \qquad + d_H(D_{r(t)}(\boldsymbol{c}_\infty), B_{r_\infty}(\boldsymbol{c}_\infty)) \end{aligned}$$

For sufficiently large t we show

$$\|\tilde{\boldsymbol{\rho}}(\cdot,t)\|_{C^0(\mathbb{R}/\mathbb{Z})} \leqq |r(t)|.$$

Hence $D_{r(t)-\|\tilde{\boldsymbol{\rho}}(\cdot,t)\|_{C^0(\mathbb{R}/\mathbb{Z})}}(\boldsymbol{c}(t)) \subseteq \overline{\Omega(t)} \subseteq D_{r(t)+\|\tilde{\boldsymbol{\rho}}(\cdot,t)\|_{C^0(\mathbb{R}/\mathbb{Z})}}(\boldsymbol{c}(t))$. Let $S_{r(t)}(\boldsymbol{c}(t))$ be the closed circle with center $\boldsymbol{c}(t)$ and the radius $r(t)$. For any $\boldsymbol{x} \in \mathrm{Im}\tilde{\boldsymbol{f}}(\cdot,t) \setminus \Big(\mathrm{Im}\tilde{\boldsymbol{f}}(\cdot,t) \cap D_{r(t)}(\boldsymbol{c}(t))\Big)$ we have

$$\begin{aligned} d(\boldsymbol{x}, D_{r(t)}(\boldsymbol{c}(t))) &\leqq d(S_{r(t)}(\boldsymbol{c}(t)), S_{r(t)+\|\tilde{\boldsymbol{\rho}}(\cdot,t)\|_{C^0(\mathbb{R}/\mathbb{Z})}}(\boldsymbol{c}(t))) \\ &= \|\tilde{\boldsymbol{\rho}}(\cdot,t)\|_{C^0(\mathbb{R}/\mathbb{Z})}. \end{aligned}$$

Furthermore, for any $\boldsymbol{x} \in \mathrm{Im}\tilde{\boldsymbol{f}}(\cdot,t) \cap D_{r(t)}(\boldsymbol{c}(t))$, we have

$$\begin{aligned} d(\boldsymbol{x}, S_{r(t)}(\boldsymbol{c}(t))) &\leqq d(S_{r(t)}(\boldsymbol{c}(t)), S_{r(t)-\|\tilde{\boldsymbol{\rho}}(\cdot,t)\|_{C^0(\mathbb{R}/\mathbb{Z})}}(\boldsymbol{c}(t))) \\ &= \|\tilde{\boldsymbol{\rho}}(\cdot,t)\|_{C^0(\mathbb{R}/\mathbb{Z})}. \end{aligned}$$

Therefore we obtain

$$d_H(\overline{\Omega(t)}, D_{r(t)}(\boldsymbol{c}(t))) \leqq \|\tilde{\boldsymbol{\rho}}(\cdot,t)\|_{C^0(\mathbb{R}/\mathbb{Z})}.$$

Moreover we clearly find that

$$\begin{aligned} d_H(D_{r(t)}(\boldsymbol{c}(t)), D_{r(t)}(\boldsymbol{c}_\infty)) &\leqq \|\boldsymbol{c}(t) - \boldsymbol{c}_\infty\|, \\ d_H(D_{r(t)}(\boldsymbol{c}_\infty), D_{r_\infty}(\boldsymbol{c}_\infty)) &\leqq |r(t) - r_\infty|. \end{aligned}$$

Hence we show

$$d_H(\overline{\Omega(t)}, D_{r_\infty}(\boldsymbol{c}_\infty)) \leqq Ce^{-\gamma t}$$

for some $C > 0$ and $\gamma > 0$.

(F) Clearly we have

$$A(\boldsymbol{b} - \boldsymbol{c}) = \iint_{\Omega(t)} (\boldsymbol{x} - \boldsymbol{c})\, d\boldsymbol{x}.$$

We define

$$\boldsymbol{b} = (b_1, b_2), \quad \boldsymbol{c} = (c_1, c_2), \quad b = b_1 + ib_2, \quad c = c_1 + ic_2.$$

From the divergence theorem, we have

$$\begin{aligned} A(b-c) &= \iint_{\Omega(t)} \{(x_1 - c_1) + i(x_2 - c_2)\} d\boldsymbol{x} \\ &= \frac{1}{2} \iint_{\Omega(t)} \mathrm{div}\big((x_1 - c_1)^2, i(x_2 - c_2)^2\big)\, d\boldsymbol{x} \\ &= -\frac{1}{2} \int_0^L \big((f_1 - c_1)^2, i(f_2 - c_2)^2\big) \cdot \boldsymbol{\nu}\, ds \\ &= -\frac{1}{2} \int_0^L \big\{(f_1 - c_1)^2(-\partial_s f_2) + i(f_2 - c_2)^2 \partial_s f_1\big\}\, ds. \end{aligned}$$

Since, for $j = 1,\ 2$,

$$\int_0^L (f_j - c_j)^2 \partial_s f_j \, ds = 0,$$

we obtain

$$\begin{aligned} A(b-c) = & -\frac{i}{2}\int_0^L \big\{(f_1-c_1)^2(\partial_s f_1 + i\partial_s f_2) \\ & \qquad + (f_2-c_2)^2(\partial_s f_1 + i\partial_s f_2)\big\} ds \\ = & -\frac{i}{2}\int_0^L |f-c|^2 \partial_s f \, ds. \end{aligned}$$

It holds that

$$|f-c|^2 = \left| re^{\frac{2\pi i(s+\sigma)}{L}} + \rho \right|^2 = r^2 + 2r\Re \rho e^{\frac{2\pi i(s+\sigma)}{L}} + |\rho|^2$$

and r is independent of s. Hence we show

$$A(b-c) = -\frac{i}{2}\int_0^L \left\{ 2r\Re\rho e^{\frac{2\pi i(s+\sigma)}{L}} + |\rho|^2 \right\} \partial_s f \, ds.$$

Therefore we have

$$\|A(\boldsymbol{b}-\boldsymbol{c})\| \leqq |A(b-c)| \leqq \frac{1}{2}\int_0^L \left(2r|\rho| + |\rho|^2\right) ds \leqq Ce^{-\gamma t}.$$

Thus we have shown each of the claims in the theorem. □

In the following subsection, the assumptions (1)–(6) in Theorem 4.1 hold for the gradient flow of the isoperimetric ratio (4.4) below, the area-preserving flow (1.2), and the length-preserving flow (1.3). As a consequence, these global flows converge to a circle in the sense of Theorem 4.2 even if the initial curve is not convex.

4.2 The gradient flow of the isoperimetric ratio

We consider the large-time behavior of the flow (1.4) of closed plane curves, when $h = \kappa - \dfrac{L}{2A}$. Assume that $\boldsymbol{f} : \bigcup_{t \geqq 0} (\mathbb{R}/L(t)\mathbb{Z} \times \{t\}) \to \mathbb{R}^2$ is a global solution with initial rotation number 1. This is the gradient flow of $\dfrac{L^2}{4\pi A}$

studied by Jiang-Pan [11]. Therefore along the flow, $\dfrac{4\pi A}{L^2}$ is non-decreasing. Furthermore the (signed) area A is also non-decreasing, if the initial (signed) area is positive. Indeed,

$$\begin{aligned}(4.4)\qquad \frac{d}{dt}\frac{4\pi A}{L^2} &= \frac{4\pi}{L^2}\frac{dA}{dt} - \frac{8\pi A}{L^3}\frac{dL}{dt} = \frac{8\pi A}{L^3}\int_0^L \partial_t \boldsymbol{f} \cdot \left(-\frac{L}{2A}\boldsymbol{\nu} + \boldsymbol{\kappa}\right) ds \\ &= \frac{8\pi A}{L^3}\int_0^L \left(\kappa - \frac{L}{2A}\right)^2 ds \geqq 0,\end{aligned}$$

$$(4.5)\qquad \frac{dA}{dt} = -\int_0^L \partial_t \boldsymbol{f} \cdot \boldsymbol{\nu}\, ds = \int_0^L \left(-\kappa + \frac{L}{2A}\right) ds = \frac{-4\pi A + L^2}{2A} \geqq 0.$$

Since

$$(4.6)\qquad \frac{dL}{dt} = -\int_0^L \partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa}\, ds = -\int_0^L \kappa^2 ds + \frac{\pi L}{A},$$

we find that L is non-increasing by Gage's inequality if $\mathrm{Im}\boldsymbol{f}$ is convex. Here we do not assume convexity. We can prove the following theorem.

Theorem 4.3 *Assume that* $\boldsymbol{f}$ *is a global solution of* (1.4) *such that the initial rotation number is* 1 *and the initial* (*signed*) *area is positive. Then* $\boldsymbol{f}$ *satisfies* (1)–(6).

Proof. It follows from (4.5) that

$$\frac{d}{dt}A^2 = L^2 - 4\pi A \geqq 0.$$

The second derivative is

$$\begin{aligned}(4.7)\qquad \frac{d^2}{dt^2}A^2 &= \frac{d}{dt}(L^2 - 4\pi A) = 2L\frac{dL}{dt} - 4\pi\frac{dA}{dt} \\ &= \int_0^L \partial_t \boldsymbol{f} \cdot (-2L\boldsymbol{\kappa} + 4\pi\boldsymbol{\nu})\, ds = \int_0^L \left(\kappa - \frac{L}{2A}\right)(-2L\tilde{\kappa})\, ds \\ &= -2L\int_0^L \tilde{\kappa}^2 ds \leqq 0.\end{aligned}$$

Therefore

$$0 \leqq \frac{d}{dt}A^2 \leqq \left.\frac{d}{dt}A^2\right|_{t=0}.$$

We put $C_0 = \left.\dfrac{d}{dt}A^2\right|_{t=0}$. Since the initial (signed) area is positive, so is A, and

$$A^2 \leqq C_0 t + A(0)^2.$$

Hence

$$\int_0^t \frac{dt}{A} \geqq \int_0^t \frac{dt}{\sqrt{C_0 t + A(0)^2}} \to \infty \quad \text{as} \quad t \to \infty. \tag{4.8}$$

It follows from (4.4) that

$$\frac{d}{dt}I_{-1} = -\frac{8\pi A}{L^3}\int_0^L \left(\tilde{\kappa} + \frac{2\pi}{L} - \frac{L}{2A}\right)^2 ds = -\frac{2\pi}{A}I_{-1}^2 - \frac{8\pi A}{L^3}\int_0^L \tilde{\kappa}^2 ds.$$

Solving the differential inequality $\dfrac{d}{dt}I_{-1} \leqq -\dfrac{2\pi}{A}I_{-1}^2$, we have

$$0 \leqq I_{-1} \leqq \frac{I_{-1}(0)}{1 + 2\pi I_{-1}(0)\displaystyle\int_0^t \frac{dt}{A}} \to 0 \quad (t \to \infty).$$

Also, (4.7) and Theorem 2.1 give us

$$\frac{d}{dt}(L^2 - 4\pi A) + \frac{16\pi^2}{L^2}(L^2 - 4\pi A) \leqq \frac{d}{dt}(L^2 - 4\pi A) + 2L\int_0^L \tilde{\kappa}^2 ds = 0.$$

Using $I_{-1} \to 0$ as $t \to \infty$ and (4.8), we obtain

$$0 \leqq L^2 - 4\pi A \leqq (L(0)^2 - 4\pi A(0))\exp\left(-16\pi^2\int_0^t \frac{dt}{L^2}\right) \to 0 \quad \text{as} \quad t \to \infty.$$

Dividing both sides by L^2, we have

$$0 \leqq I_{-1} \leqq C\frac{16\pi^2}{L^2}\exp\left(-16\pi^2\int_0^t \frac{dt}{L^2}\right) = -C\frac{d}{dt}\exp\left(-16\pi^2\int_0^t \frac{dt}{L^2}\right).$$

Integrating this with respect to t, we obtain

$$0 \leqq \int_0^t I_{-1}dt \leqq C\left\{1 - \exp\left(-16\pi^2\int_0^t \frac{dt}{L^2}\right)\right\} \leqq C.$$

Integrating (4.5), we have

$$A = A(0) + \int_0^t \frac{L^2}{2A}I_{-1}dt \leqq C.$$

Since A is non-decreasing and bounded from above, there exists a finite limit $A_\infty = \lim_{t\to\infty} A \in (0,\infty)$. Hence $\lim_{t\to\infty} L = 2\sqrt{\pi A_\infty} \in (0,\infty)$ exists, say L_∞. Consequently, we obtain

$$0 \leqq L^2 - 4\pi A \leqq (L(0)^2 - 4\pi A(0)) \exp\left(-16\pi^2 \int_0^t \frac{dt}{L^2}\right) \leqq Ce^{-\lambda t},$$
$$0 \leqq I_{-1} = \frac{L^2 - 4\pi A}{L^2} \leqq Ce^{-\lambda t},$$
$$0 \leqq A_\infty - A \leqq \frac{1}{A_\infty + A} \int_t^\infty \frac{d}{dt} A^2 dt = \frac{1}{A_\infty + A} \int_t^\infty (L^2 - 4\pi A)\, dt \leqq Ce^{-\lambda t},$$
$$|L - L_\infty| = \frac{|L^2 - 4\pi A + 4\pi(A - A_\infty)|}{L + L_\infty} \leqq Ce^{-\lambda t}$$

for some $C > 0$ and $\lambda > 0$. Hence we have (1), (3) and (4). Furthermore, Integrating (4.7), we have

$$\int_0^\infty I_0 dt \leqq C.$$

Hence (2) holds.

Next we consider the behavior of I_0. Since

$$I_0 = L \int_0^L \left(\kappa - \frac{2\pi}{L}\right)^2 ds = L \int_0^L \kappa^2 ds - 4\pi^2,$$

we have

$$
\begin{aligned}
\frac{d}{dt}I_0 =& \frac{dL}{dt}\int_0^L \kappa^2 ds + L\frac{d}{dt}\int_0^L \kappa^2 ds \\
=& \int_0^L \left\{ -\boldsymbol{\kappa}\int_0^L \kappa^2 ds + L\left(2\nabla_s^2\boldsymbol{\kappa} + \|\boldsymbol{\kappa}\|_{\mathbb{R}^2}^2\boldsymbol{\kappa}\right)\right\}\cdot \partial_t \boldsymbol{f}\, ds \\
=& \int_0^L \left\{ -\kappa\int_0^L \kappa^2 ds + L\left(2\partial_s^2\kappa + \kappa^3\right)\right\}\left(\kappa - \frac{L}{2A}\right) ds \\
=& -\left(\int_0^L \kappa^2 ds\right)^2 + \frac{\pi L}{A}\int_0^L \kappa^2 ds + L\int_0^L \left\{-2(\partial_s\kappa)^2 + \kappa^4 - \frac{L}{2A}\kappa^3\right\} ds \\
=& -2L\int_0^L (\partial_s\tilde{\kappa})^2 ds + L\left\{\int_0^L \kappa^4 ds - \frac{1}{L}\left(\int_0^L \kappa^2 ds\right)^2\right\} \\
& - \frac{L^2}{2A}\int_0^L \kappa^2\tilde{\kappa}\, ds \\
=& -2L\int_0^L (\partial_s\tilde{\kappa})^2 ds - \left(\int_0^L \tilde{\kappa}^2 ds\right)^2 \\
& + \int_0^L \left\{ L\tilde{\kappa}^4 + \left(8\pi - \frac{L^2}{2A}\right)\tilde{\kappa}^3 + \left(\frac{16\pi^2}{L} - \frac{2\pi L}{A}\right)\tilde{\kappa}^2\right\} ds.
\end{aligned}
$$

By virtue of the Gagliardo-Nirenberg inequalities, we have

$$
\begin{aligned}
\frac{d}{dt}I_0 + \frac{1}{L^2}I_0^2 + \frac{2}{L^2}I_1 \leqq& \frac{C}{L^2}\int_0^L \left(L^3\tilde{\kappa}^4 + L^2|\tilde{\kappa}|^3 + L\tilde{\kappa}^2\right) ds \\
\leqq& \frac{C}{L^2}\left(I_1^{\frac{1}{2}}I_0^{\frac{3}{2}} + I_1^{\frac{1}{4}}I_0^{\frac{5}{4}} + I_0\right).
\end{aligned}
\tag{4.9}
$$

Applying Young's and Wirtinger's inequalities, Theorem 2.1 and Theorem 3.1, we have

$$
\begin{aligned}
I_1^{\frac{1}{2}}I_0^{\frac{3}{2}} &\leqq \epsilon I_1 + C_\epsilon I_0^3, \\
I_1^{\frac{1}{4}}I_0^{\frac{5}{4}} &\leqq \epsilon I_1 + C_\epsilon I_0^{\frac{5}{3}} \leqq \epsilon(I_1 + I_0) + C_\epsilon I_0^3 \leqq C\epsilon I_1 + C_\epsilon I_0^3, \\
I_0 &\leqq I_{-1}^{\frac{1}{2}}\left(I_1 + I_1^{\frac{1}{2}}\right) \leqq \left(I_{-1}^{\frac{1}{2}} + \epsilon\right) I_1 + C_\epsilon I_{-1}
\end{aligned}
$$

for any $\epsilon > 0$. Hence we show (5).

Next we show the exponential decay of I_ℓ for $\ell \in \mathbb{N}$. Now observe that

$$
\begin{aligned}
\frac{d}{dt}I_\ell =& (2\ell+1)L^{2\ell}\frac{dL}{dt}\int_0^L (\partial_s^\ell\tilde{\kappa})^2 ds + L^{2\ell+1}\int_0^L (\partial_s^\ell\tilde{\kappa})^2\partial_t(ds) \\
& + 2L^{2\ell+1}\int_0^L (\partial_s^\ell\tilde{\kappa})(\partial_t\partial_s^\ell\tilde{\kappa})\, ds.
\end{aligned}
$$

It follows from (4.6) and Theorem 4.3 that

$$
\begin{aligned}
(2\ell+1)L^{2\ell}\frac{dL}{dt}\int_0^L (\partial_s^\ell \tilde{\kappa})^2 ds &= (2\ell+1)L^{2\ell}\left(-\int_0^L \kappa^2 ds + \frac{\pi L}{A}\right)\int_0^L (\partial_s^\ell \tilde{\kappa})^2 ds \\
&\leqq \frac{C}{L^2} I_\ell .
\end{aligned}
$$

Since $\partial_t(ds) = -\partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa}\, ds$, we have

$$
\begin{aligned}
L^{2\ell+1}\int_0^L (\partial_s^\ell \tilde{\kappa})^2 \partial_t(ds) &= L^{2\ell+1}\int_0^L (\partial_s^\ell \tilde{\kappa})^2 \left\{-\kappa\left(\kappa - \frac{L}{2A}\right)\right\} ds \\
&\leqq \frac{L^{2\ell+2}}{2A}\int_0^L (\partial_s^\ell \tilde{\kappa})^2 \left(\tilde{\kappa} + \frac{2\pi}{L}\right) ds \\
&\leqq \frac{L^{2\ell+2}}{2A}\int_0^L (\partial_s^\ell \tilde{\kappa})^2 \tilde{\kappa}\, ds + \frac{C}{L^2} I_\ell .
\end{aligned} \tag{4.10}
$$

We can show

$$
\partial_t \partial_s^k \tilde{\kappa} = \partial_s^{k+2}\tilde{\kappa} + \sum_{n=0}^{3} L^{-(3-n)} P_n^k(\tilde{\kappa}) + \frac{L}{A}\sum_{n=0}^{2} L^{-(2-n)} P_n^k(\tilde{\kappa}) \tag{4.11}
$$

by induction on k. Indeed, we have

$$
\begin{aligned}
\partial_t \kappa &= \partial_s^2 \kappa + \kappa^3 - \frac{L}{2A}\kappa^2 \\
&= \partial_s^2 \tilde{\kappa} + \sum_{n=0}^{3} L^{-(3-n)} P_n^0(\tilde{\kappa}) + \frac{L}{A}\sum_{n=0}^{2} L^{-(2-n)} P_n^0(\tilde{\kappa}).
\end{aligned}
$$

Hence (4.11) holds when $k = 0$. Next we assume that

$$
\partial_t \partial_s^k \tilde{\kappa} = \partial_s^{k+2}\tilde{\kappa} + \sum_{n=0}^{3} L^{-(3-n)} P_n^k(\tilde{\kappa}) + \frac{L}{A}\sum_{n=0}^{2} L^{-(2-n)} P_n^k(\tilde{\kappa})
$$

holds. Since

$$
\partial_t \partial_s = \partial_s \partial_t + \left(\kappa^2 - \frac{L}{2A}\kappa\right)\partial_s ,
$$

we show

$$\begin{aligned}
\partial_t \partial_s^{k+1}\kappa &= \partial_s\partial_t\partial_s^k\kappa + \left(\kappa^2 - \frac{L}{2A}\kappa\right)\partial_s^{k+1}\kappa \\
&= \partial_s^{k+2}\tilde{\kappa} + \sum_{n=0}^{3} L^{-(3-n)}P_n^k(\tilde{\kappa}) + \frac{L}{A}\sum_{n=0}^{2} L^{-(2-n)}P_n^k(\tilde{\kappa}) \\
&\quad + \sum_{n=0}^{3} L^{-(3-n)}P_n^k(\tilde{\kappa}) + \frac{L}{A}\sum_{n=0}^{2} L^{-(2-n)}P_n^k(\tilde{\kappa}) \\
&= \partial_s^{k+2}\tilde{\kappa} + \sum_{n=0}^{3} L^{-(3-n)}P_n^k(\tilde{\kappa}) + \frac{L}{A}\sum_{n=0}^{2} L^{-(2-n)}P_n^k(\tilde{\kappa}).
\end{aligned}$$

Hence (4.11) holds for $k \geqq 0$. Since $P_0^\ell(\tilde{\kappa})$ is a constant,

$$\int_0^L (\partial_s^\ell\tilde{\kappa})P_0^\ell(\tilde{\kappa})\,ds = 0.$$

Therefore we have

$$\begin{aligned}
&2L^{2\ell+1}\int_0^L (\partial_s^\ell\tilde{\kappa})(\partial_t\partial_s^\ell\tilde{\kappa})\,ds \\
&\quad = -\frac{2}{L^2}I_{\ell+1} + L^{2\ell+1}\int_0^L (\partial_s^\ell\tilde{\kappa})\left(\sum_{n=1}^{3} L^{-(3-n)}P_n^\ell(\tilde{\kappa}) + \frac{L}{A}\sum_{n=0}^{2} L^{-(2-n)}P_n^\ell(\tilde{\kappa})\right)ds.
\end{aligned}$$

Here the term $\dfrac{L^{2\ell+2}}{2A}\displaystyle\int_0^L (\partial_s^\ell\tilde{\kappa})^2\tilde{\kappa}\,ds$ on the last line of (4.10) is included into $L^{2\ell+1}\displaystyle\int_0^L (\partial_s^\ell\tilde{\kappa})\frac{L}{A}P_2^\ell(\tilde{\kappa})\,ds$. Hence we obtain

$$\begin{aligned}
\frac{d}{dt}I_\ell + \frac{2}{L^2}I_{\ell+1} \leqq{}& L^{2\ell+1}\int_0^L \partial_s^\ell\tilde{\kappa}\sum_{n=1}^{3} L^{-(3-n)}P_n^\ell(\tilde{\kappa})ds \\
&+ L^{2\ell+1}\frac{L}{A}\left(\int_0^L \partial_s^\ell\tilde{\kappa}\sum_{n=1}^{2} L^{-(2-n)}P_n^\ell(\tilde{\kappa})ds\right).
\end{aligned}$$

Therefore we show (6) when $\alpha = 3, \beta = 2, \gamma = 1, \sigma = 0$.

Theorem 4.4 *Let $\boldsymbol{f}$ be as in Theorem 4.3, and we set $f, \boldsymbol{c}, r, \sigma, \tilde{\boldsymbol{f}}$ same as Theorem 4.2. Then we show that there exists $\boldsymbol{c}_\infty \in \mathbb{R}^2$ such that*

$$\|\boldsymbol{c}(t) - \boldsymbol{c}_\infty\| \leqq Ce^{-\gamma t}.$$

Proof. First observe that

$$\boldsymbol{c} = \frac{1}{L}\int_0^L \boldsymbol{f}\, ds.$$

Since

$$\partial_t \boldsymbol{f} = \partial_s \left(\partial_s \boldsymbol{f} - \frac{L}{2A} R \boldsymbol{f} \right),$$

we have

$$\int_0^L \partial_t \boldsymbol{f}\, ds = \boldsymbol{o}.$$

Therefore the time-derivative of $\boldsymbol{c}$ is

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{c} &= \frac{1}{L}\int_0^L \boldsymbol{f}\partial_t(ds) - \frac{1}{L^2}\frac{dL}{dt}\int_0^L \boldsymbol{f}\, ds \\
&= -\frac{1}{L}\int_0^L \left\{ (\partial_t \boldsymbol{f}\cdot\boldsymbol{\kappa}) - \frac{1}{L}\int_0^L (\partial_t \boldsymbol{f}\cdot\boldsymbol{\kappa})\, ds \right\} \boldsymbol{f}\, ds \\
&= -\frac{1}{L}\int_0^L \left\{ (\partial_t \boldsymbol{f}\cdot\boldsymbol{\kappa}) - \frac{1}{L}\int_0^L (\partial_t \boldsymbol{f}\cdot\boldsymbol{\kappa})\, ds \right\} \left(\boldsymbol{f} - \frac{1}{L}\int_0^L \boldsymbol{f}\, ds \right) ds.
\end{aligned}$$

Since

$$\partial_t \boldsymbol{f}\cdot\boldsymbol{\kappa} = \kappa^2 - \frac{L}{2A}\kappa = \tilde{\kappa}^2 - \frac{L}{2A}(2I_{-1}-1)\tilde{\kappa} - \frac{\pi}{A}I_{-1}$$

decays exponentially as $t \to \infty$, and because

$$\left\| \boldsymbol{f} - \frac{1}{L}\int_0^L \boldsymbol{f}\, ds \right\|_{\mathbb{R}^2} \leqq L \leqq C,$$

we find that $\boldsymbol{c}$ converges exponentially to a vector, say $\boldsymbol{c}_\infty$, as $t \to \infty$. Consequently $\mathrm{Im}\boldsymbol{f}$ converges to a circle with center at $\boldsymbol{c}_\infty$. □

From Theorem 4.2 and Theorem 4.4, the claims (A)–(F) in Theorem 4.2 hold also for global solution of (1.4). Hence it converges exponentially to a circle with center $\boldsymbol{c}_\infty$ and radius $\dfrac{L_\infty}{2\pi}$.

4.3 The area-preserving curvature flow

In this subsection we consider the area-preserving flow (1.2) when $h = \kappa - \dfrac{1}{L}\displaystyle\int_0^L \kappa\, ds$. Assume that $\boldsymbol{f} : \displaystyle\bigcup_{t \geqq 0} (\mathbb{R}/L(t)\mathbb{Z} \times \{t\}) \to \mathbb{R}^2$ is a global solution with initial rotation number 1. Since the rotation number 1, the integral of

κ is 2π. It is well-known that if the initial curve is convex, then any solution of (1.2) converges to a round circle as $t \to \infty$ as proved by Gage [6]. In this subsection, we give a proof of this fact without the convexity assumption *assuming* the global existence.

Theorem 4.5 *Assume that* $\boldsymbol{f}$ *is a global solution of* (1.2) *such that the initial rotation number is* 1 *and the initial* (*signed*) *area is positive. Then* $\boldsymbol{f}$ *satisfies* (1)–(6).

Proof. We find (4) holds clearly bacause (1.2) is the area-preserving flow. Since

$$\frac{dL}{dt} = -\int_0^L \partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa} ds = -\int_0^L \tilde{\kappa}^2 ds,$$

we have

$$L \leqq L(0).$$

From this, the area-preserving property, and Theorem 2.2, we have

$$\begin{aligned} \frac{d}{dt}(L^2 - 4\pi A) = 2L\frac{dL}{dt} &= -2L\int_0^L \tilde{\kappa}^2 ds = -2I_0 \\ &\leqq -\frac{16\pi^2}{L^2}(L^2 - 4\pi A) \leqq -\frac{16\pi^2}{L(0)^2}(L^2 - 4\pi A). \end{aligned} \tag{4.12}$$

Therefore

$$L^2 - 4\pi A \leqq (L(0)^2 - 4\pi A(0)) \exp\left(-\frac{16\pi^2}{L(0)^2}t\right).$$

Hence we show (1). Therefore we have $\lim_{t\to\infty} L = 2\sqrt{\pi A(0)}$, and

$$\left|L - 2\sqrt{\pi A(0)}\right| \leqq \frac{L^2 - 4\pi A}{L + 2\sqrt{\pi A(0)}} \leqq \frac{L(0)^2 - 4\pi A(0)}{4\sqrt{\pi A(0)}} \exp\left(-\frac{16\pi^2}{L(0)^2}t\right).$$

Hence we obtain (3). Furthermore, from (4.12), we have (2).

Next we consider when $\ell = 0$. We have

$$
\begin{aligned}
\frac{d}{dt} I_0 &= \frac{d}{dt}\left(L \int_0^L \kappa^2 ds - 4\pi^2 \right) \\
&= \frac{dL}{dt} \int_0^L \kappa^2 ds + L \frac{d}{dt} \int_0^L \kappa^2 ds \\
&= \int_0^L \left\{ -\boldsymbol{\kappa} \int_0^L \kappa^2 ds + L \left(2\nabla_s^2 \boldsymbol{\kappa} + \|\boldsymbol{\kappa}\|_{\mathbb{R}^2}^2 \boldsymbol{\kappa} \right) \right\} \cdot \partial_t \boldsymbol{f}\, ds \\
&= \int_0^L \left\{ -\kappa \int_0^L \kappa^2 ds + L \left(2\partial_s^2 \kappa + \kappa^3 \right) \right\} \tilde{\kappa}\, ds \\
&= -\int_0^L \kappa^2 ds \int_0^L \tilde{\kappa}^2 ds + L \int_0^L \kappa^3 \tilde{\kappa}\, ds + -2L \int_0^L (\partial_s \kappa)^2 ds \\
&= -2L \int_0^L (\partial_s \tilde{\kappa})^2 ds + \int_0^L \tilde{\kappa}^2 ds \left(L \int_0^L \kappa^2 ds - 4\pi^2 \right) + L \int_0^L \kappa^3 \tilde{\kappa}\, ds \\
&= -2L \int_0^L (\partial_s \tilde{\kappa})^2 ds - \left(\int_0^L \tilde{\kappa}^2 ds \right)^2 - \frac{4\pi^2}{L} \int_0^L \tilde{\kappa}^2 ds \\
&\quad + L \int_0^L \tilde{\kappa} \left(\tilde{\kappa}^3 + \frac{6\pi}{L} \tilde{\kappa}^2 + \frac{12\pi^2}{L^2} \tilde{\kappa} + \frac{8\pi^3}{L^3} \right) ds.
\end{aligned}
$$

By calculation, we have

$$
\frac{d}{dt} I_0 + \frac{4\pi^2}{L^2} I_0 + \frac{1}{L^2} I_0 + \frac{2}{L^2} I_1 \leqq \frac{C}{L^2} \int_0^L \left(L^3 \tilde{\kappa}^4 + L^2 |\tilde{\kappa}|^3 + L \tilde{\kappa}^2 \right) ds.
$$

Since we have the same form as (4.9), we have (5).

Next we consider the behavior of I_ℓ for $\ell \in \mathbb{N}$. We can show

$$
\partial_t \partial_s^k \tilde{\kappa} = \partial_s^{k+2} \tilde{\kappa} + \sum_{n=1}^{3} L^{-(3-n)} P_n^k(\tilde{\kappa}) \tag{4.13}
$$

by induction on k. Indeed, we have

$$
\begin{aligned}
\partial_t \kappa &= \partial_s^2 \tilde{\kappa} + \tilde{\kappa} \kappa^2 \\
&= \partial_s^2 \tilde{\kappa} + \sum_{n=1}^{3} L^{-(3-n)} P_n^0(\tilde{\kappa}).
\end{aligned}
$$

Hence (4.13) holds when $k = 0$. Next we assume that

$$
\partial_t \partial_s^k \tilde{\kappa} = \partial_s^{k+2} \tilde{\kappa} + \sum_{n=1}^{3} L^{-(3-n)} P_n^k(\tilde{\kappa})
$$

hold. Since

$$\partial_t \partial_s = \partial_s \partial_t + \tilde{\kappa}\kappa \partial_s,$$

we show

$$\begin{aligned}
\partial_t \partial_s^{k+1} \kappa =& \partial_s \partial_t \partial_s^k \kappa + \tilde{\kappa}\kappa \partial_s^{k+1} \kappa \\
=& \partial_s^{k+2} \tilde{\kappa} + \sum_{n=1}^{3} L^{-(3-n)} P_n^{k+1}(\tilde{\kappa}) + \sum_{n=2}^{3} L^{-(3-n)} P_n^{k+1}(\tilde{\kappa}) \\
=& \partial_s^{k+2} \tilde{\kappa} + \sum_{n=1}^{3} L^{-(3-n)} P_n^{k+1}(\tilde{\kappa}).
\end{aligned}$$

Hence (4.13) holds for $k \geqq 0$. Hence we have

$$\begin{aligned}
\frac{d}{dt} I_\ell =& (2\ell+1) L^{2\ell} \frac{dL}{dt} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 ds + 2L^{2\ell+1} \int_0^L \partial_s^\ell \tilde{\kappa} \partial_t \partial_s^\ell \tilde{\kappa}\, ds \\
&+ L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 \partial_t (ds) \\
=& -(2\ell+1) \int_0^L \tilde{\kappa}^2 ds \int_0^L (\partial_s^\ell \tilde{\kappa})^2 ds - 2L^{2\ell+1} \int_0^L (\partial_s^{\ell+1} \tilde{\kappa})^2 ds \\
&+ L^{2\ell+1} \int_0^L \partial_s^\ell \tilde{\kappa} \sum_{n=1}^{3} L^{-(3-n)} P_n^\ell(\tilde{\kappa}) ds - L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 \tilde{\kappa}\kappa\, ds.
\end{aligned}$$

By calculation, we have

$$\frac{d}{dt} I_\ell + \frac{2}{L^2} I_{\ell+1} \leqq L^{2\ell+1} \int_0^L \partial_s^\ell \tilde{\kappa} \sum_{n=1}^{3} L^{-(3-n)} P_n^\ell(\tilde{\kappa}) ds.$$

Therefore we have (6) when $\alpha = 3, \beta = 1, \gamma = 0, \sigma = 0$. □

Theorem 4.6 *Let $\boldsymbol{f}$ be as in Theorem 4.5, and we set $f, \boldsymbol{c}, r, \sigma, \tilde{\boldsymbol{f}}$ same as Theorem 4.2. Then we show that there exists $\boldsymbol{c}_\infty \in \mathbb{R}^2$ such that*

$$\|\boldsymbol{c}(t) - \boldsymbol{c}_\infty\| \leqq C e^{-\gamma t}.$$

Proof. The time-derivative of $\boldsymbol{c}$ is

$$\begin{aligned}\frac{d}{dt}\boldsymbol{c} &= \frac{1}{L}\int_0^L \boldsymbol{f}\partial_t(ds) - \frac{1}{L^2}\frac{dL}{dt}\int_0^L \boldsymbol{f}\,ds \\ &= -\frac{1}{L}\int_0^L \left\{(\partial_t\boldsymbol{f}\cdot\boldsymbol{\kappa}) - \frac{1}{L}\int_0^L (\partial_t\boldsymbol{f}\cdot\boldsymbol{\kappa})\,ds\right\}\boldsymbol{f}\,ds \\ &= -\frac{1}{L}\int_0^L \left\{(\partial_t\boldsymbol{f}\cdot\boldsymbol{\kappa}) - \frac{1}{L}\int_0^L (\partial_t\boldsymbol{f}\cdot\boldsymbol{\kappa})\,ds\right\}\left(\boldsymbol{f} - \frac{1}{L}\int_0^L \boldsymbol{f}\,ds\right)ds.\end{aligned}$$

Since

$$\partial_t\boldsymbol{f}\cdot\boldsymbol{\kappa} = \kappa\tilde{\kappa} = \tilde{\kappa}^2 + \frac{2\pi}{L}\tilde{\kappa}$$

decays exponentially as $t\to\infty$, we find that $\boldsymbol{c}$ converges exponentially to a vector, say $\boldsymbol{c}_\infty$, as $t\to\infty$ in the same way as Theorem 4.4. Consequently $\mathrm{Im}\boldsymbol{f}$ converges to a circle with center at $\boldsymbol{c}_\infty$. □

From Theorem 4.5 and Theorem 4.6, the claims (A)–(F) in Theorem 4.2 hold also for global solution of (1.2). Hence it converges exponentially to a circle with center $\boldsymbol{c}_\infty$ and radius $\dfrac{L_\infty}{2\pi}$.

4.4 The length-preserving curvature flow

In this subsection we consider the length-preserving flow (1.3) when $h = \kappa - \dfrac{1}{2\pi}\displaystyle\int_0^L \kappa^2 ds$. Assume that $\boldsymbol{f} : \displaystyle\bigcup_{t\geqq 0}(\mathbb{R}/L(t)\mathbb{Z}\times\{t\}) \to \mathbb{R}^2$ is a global solution with initial rotation number 1. Ma-Zhu [12] proved that if the initial curve is convex, then any solutions of this flow converge to a round circle as $t\to\infty$. We have the same results for area-preserving curvature flow without convexity assumption *assuming* the global existence. In this subsection, we give a proof of this fact.

Theorem 4.7 *Assume that $\boldsymbol{f}$ is a global solution of* (1.3) *such that the initial rotation number is* 1 *and the initial* (*signed*) *area is positive. Then $\boldsymbol{f}$ satisfies* (1)–(6).

Proof. We find (3) holds clearly because (1.3) is the length-preserving flow. Since

$$\frac{dA}{dt} = -\int_0^L \partial_t\boldsymbol{f}\cdot\boldsymbol{\nu}ds = \frac{L}{2\pi}\int_0^L \tilde{\kappa}^2 ds = \frac{I_0}{2\pi}, \tag{4.14}$$

from Theorem 2.1, we have

$$\frac{d}{dt}I_{-1} = -\frac{4\pi}{L^2}\frac{dA}{dt} = -\frac{2}{L^2}I_0 \leqq -\frac{16\pi^2}{L(0)^2}I_{-1}.$$

Therefore we have (1) and find $\lim_{t\to\infty} A = \dfrac{L(0)^2}{4\pi}$. Hence we have (4). Integrating (4.14), we obtain (2).

Next we consider the behavior of I_0. By calculation, we have

$$\begin{aligned}
\frac{d}{dt}I_0 &= \frac{d}{dt}\left(L\int_0^L \kappa^2 ds\right)\\
&= L\int_0^L \left(2\nabla_s^2\boldsymbol{\kappa} + \|\boldsymbol{\kappa}\|_{\mathbb{R}^2}^2\boldsymbol{\kappa}\right)\cdot\partial_t \boldsymbol{f}\, ds\\
&= L\int_0^L 2(\partial_s^2\tilde{\kappa} + \kappa^3)\left(\tilde{\kappa} - \frac{1}{2\pi}\int_0^L \tilde{\kappa}^2 ds\right) ds\\
&= -2L\int_0^L (\partial_s\tilde{\kappa})^2 ds + L\int_0^L \kappa^3\tilde{\kappa}\, ds - \frac{1}{2\pi}\int_0^L \kappa^3 ds\int_0^L \tilde{\kappa}^2 ds\\
&= -\frac{2}{L^2}I_1 + L\int_0^L\left(\tilde{\kappa}^4 + \frac{6\pi}{L}\tilde{\kappa}^3 + \frac{12\pi^2}{L^2}\tilde{\kappa}^2\right) ds\\
&\quad - \frac{L}{2\pi}\int_0^L\left(\tilde{\kappa}^3 + \frac{6\pi}{L}\tilde{\kappa}^2 + \frac{8\pi^3}{L^3}\right) ds\int_0^L \tilde{\kappa}^2 ds.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\frac{d}{dt}I_0 + \frac{3}{L^2}I_0^2 + \frac{2}{L^2}I_1\\
&\quad = \frac{1}{L^2}\int_0^L (L^3\tilde{\kappa}^4 + 6\pi L^2\tilde{\kappa}^3 + 8\pi^2 L\tilde{\kappa}^2) ds\\
&\qquad - \frac{1}{2\pi L^2}\left(L^2\int_0^L \tilde{\kappa}^3 ds\right)\left(L\int_0^L \tilde{\kappa}^2 ds\right).
\end{aligned}$$

The first term on the right-hand side is estimated above by

$$\left(I_{-1}^{\frac{1}{2}} + \epsilon\right) I_1 + \frac{C_\epsilon}{L^2}(I_0^3 + I_{-1})$$

in the same way as Theorem 4.3. Moreover we have, by Young's inequality,

$$\begin{aligned}
-\frac{1}{2\pi L^2}\left(L^2\int_0^L \tilde{\kappa}^3 ds\right)\left(L\int_0^L \tilde{\kappa}^2 ds\right) &\leqq \frac{C}{L^2}I_1^{\frac{1}{4}}I_0^{\frac{5}{4}}I_0 = \frac{C}{L^2}I_1^{\frac{1}{4}}I_0^{\frac{9}{4}}\\
&\leqq \frac{\epsilon}{L^2}I_1 + \frac{C_\epsilon}{L^2}I_0^3.
\end{aligned}$$

Taking ϵ sufficiently small, we have

$$\frac{d}{dt}I_0 + \frac{3}{L^2}I_0^2 + \frac{4\pi^2}{L^2}I_0 + \frac{C_1}{L^2}I_1 \leqq \frac{C_2}{L^2}I_0^3 + \frac{C_3}{L^2}I_{-1}.$$

Hence we show (5).

Next we consider the behavior of I_ℓ for $\ell \in \mathbb{N}$. We can show

$$\partial_t \partial_s^k \tilde{\kappa} = \partial_s^{k+2}\tilde{\kappa} + \sum_{n=1}^{3} L^{-(3-n)} P_n^k(\tilde{\kappa}) + \sum_{n=1}^{2} L^{-(2-n)} P_n^k(\tilde{\kappa}) \int_0^L \tilde{\kappa}^2 ds \tag{4.15}$$

by induction on k. Indeed, we have

$$\begin{aligned}
\partial_t \kappa &= \partial_s^2 \kappa + \kappa^3 - \frac{\kappa^2}{2\pi}\int_0^L \kappa^2 ds \\
&= \partial_s^2 \kappa + \kappa^3 - \frac{\kappa^2}{2\pi}\int_0^L \tilde{\kappa}^2 ds + \frac{4\pi\kappa^2}{L} \\
&= \partial_s^2 \tilde{\kappa} + \sum_{n=1}^{3} L^{-(3-n)} P_n^0(\tilde{\kappa}) + \sum_{n=1}^{2} L^{-(2-n)} P_n^0(\tilde{\kappa}) \int_0^L \tilde{\kappa}^2 ds.
\end{aligned}$$

Hence (4.15) holds when $k = 0$. Next we assume that

$$\partial_t \partial_s^k \tilde{\kappa} = \partial_s^{k+2}\tilde{\kappa} + \sum_{n=1}^{3} L^{-(3-n)} P_n^k(\tilde{\kappa}) + \sum_{n=1}^{2} L^{-(2-n)} P_n^k(\tilde{\kappa}) \int_0^L \tilde{\kappa}^2 ds$$

holds. Since

$$\partial_t \partial_s = \partial_s \partial_t + \left(\kappa^2 - \frac{\kappa}{2\pi}\int_0^L \kappa^2 ds \right) \partial_s,$$

we show

$$\begin{aligned}
\partial_t \partial_s^{k+1} \kappa =& \partial_s \partial_t \partial_s^k \kappa + \left(\kappa^2 - \frac{\kappa}{2\pi}\int_0^L \kappa^2 ds \right) \partial_s^{k+1} \kappa \\
=& \partial_s^{k+2}\tilde{\kappa} + \sum_{n=1}^{3} L^{-(3-n)} P_n^{k+1}(\tilde{\kappa}) + \sum_{n=1}^{2} L^{-(2-n)} P_n^{k+1}(\tilde{\kappa}) \int_0^L \tilde{\kappa}^2 ds \\
&+ \sum_{n=1}^{3} L^{-(3-n)} P_n^{k+1}(\tilde{\kappa}) + \sum_{n=1}^{2} L^{-(2-n)} P_n^{k+1}(\tilde{\kappa}) \int_0^L \tilde{\kappa}^2 ds \\
=& \partial_s^{k+2}\tilde{\kappa} + \sum_{n=1}^{3} L^{-(3-n)} P_n^{k+1}(\tilde{\kappa}) + \sum_{n=1}^{2} L^{-(2-n)} P_n^{k+1}(\tilde{\kappa}) \int_0^L \tilde{\kappa}^2 ds.
\end{aligned}$$

Hence (4.15) holds for $k \geqq 0$. Hence we have

$$
\begin{aligned}
\frac{d}{dt} I_\ell = {} & 2L^{2\ell+1} \int_0^L \partial_s^\ell \tilde{\kappa} \partial_t \partial_s^\ell \tilde{\kappa}\, ds + L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 \partial_t (ds) \\
= {} & -2L^{2\ell+1} \int_0^L (\partial_s^{\ell+1} \tilde{\kappa})^2 ds + 2L^{2\ell+1} \int_0^L \partial_s^\ell \tilde{\kappa} \sum_{n=1}^{3} L^{-(3-n)} P_n^\ell(\tilde{\kappa}) ds \\
& + 2L^{2\ell+1} \int_0^L \partial_s^\ell \tilde{\kappa} \sum_{n=1}^{2} L^{-(2-n)} P_n^\ell(\tilde{\kappa}) ds \int_0^L \tilde{\kappa}^2 ds \\
& - L^{2\ell+1} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 \tilde{\kappa} \kappa\, ds + \frac{L^{2\ell+1}}{2\pi} \int_0^L (\partial_s^\ell \tilde{\kappa})^2 \kappa\, ds \int_0^L \tilde{\kappa}^2 ds \\
= {} & -\frac{2}{L^2} I_{\ell+1} + 2L^{2\ell+1} \int_0^L \partial_s^\ell \tilde{\kappa} \sum_{n=1}^{3} L^{-(3-n)} P_n^\ell(\tilde{\kappa}) ds \\
& + 2L^{2\ell+1} \int_0^L \partial_s^\ell \tilde{\kappa} \sum_{n=1}^{2} L^{-(2-n)} P_n^\ell(\tilde{\kappa}) ds \int_0^L \tilde{\kappa}^2 ds.
\end{aligned}
$$

Therefore we obtain (6). □

Theorem 4.8 *Let* $\boldsymbol{f}$ *be as in Theorem 4.7, and we set* $f, \boldsymbol{c}, r, \sigma, \tilde{\boldsymbol{f}}$ *same as Theorem 4.2. Then we show that there exists* $\boldsymbol{c}_\infty \in \mathbb{R}^2$ *such that*

$$
\|\boldsymbol{c}(t) - \boldsymbol{c}_\infty\| \leqq C e^{-\gamma t}.
$$

Proof. The time-derivative of $\boldsymbol{c}$ is

$$
\begin{aligned}
\frac{d}{dt}\boldsymbol{c} = {} & \frac{1}{L} \int_0^L \boldsymbol{f} \partial_t (ds) - \frac{1}{L^2} \frac{dL}{dt} \int_0^L \boldsymbol{f}\, ds \\
= {} & -\frac{1}{L} \int_0^L \left\{ (\partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa}) - \frac{1}{L} \int_0^L (\partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa})\, ds \right\} \boldsymbol{f}\, ds \\
= {} & -\frac{1}{L} \int_0^L \left\{ (\partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa}) - \frac{1}{L} \int_0^L (\partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa})\, ds \right\} \left(\boldsymbol{f} - \frac{1}{L} \int_0^L \boldsymbol{f}\, ds \right) ds.
\end{aligned}
$$

Since

$$
\partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa} = \kappa^2 - \frac{\kappa}{2\pi} \int_0^L \kappa^2 ds = \tilde{\kappa}^2 + \frac{2\pi}{L} \tilde{\kappa} - \frac{1}{L^2} \left(\frac{L}{2\pi} + 1 \right) I_0
$$

decays exponentially as $t \to \infty$, we find that $\boldsymbol{c}$ converges exponentially to a vector, say $\boldsymbol{c}_\infty$, as $t \to \infty$ in the same way as Theorem 4.4. Consequently $\mathrm{Im}\boldsymbol{f}$ converges to a circle with center at $\boldsymbol{c}_\infty$. □

From Theorem 4.7 and Theorem 4.8, the claims (A)–(F) in Theorem 4.2 hold also for global solution of (1.3). Hence it converges exponentially to a circle with center $\boldsymbol{c}_\infty$ and radius $\dfrac{L_\infty}{2\pi}$.

5 Higher order curvature flow

In this section, we consider the higher order curvature flows (1.8). This is the H^{-m}-gradient flow of length. Indeed, for any $\varphi \in H^{-m}$, we have

$$\begin{aligned}
\frac{d}{d\epsilon}L(\boldsymbol{f}+\epsilon\varphi\boldsymbol{\nu})\bigg|_{\epsilon=0} &= -\int_0^L \tilde{\kappa}\varphi\,ds = -\int_0^L \{(-1)^m\partial_s^m\tilde{\kappa}\}\left(\partial_s^{-m}\varphi\right)ds \\
&= -\int_0^L \left[\partial_s^{-m}\left\{(-1)^m\partial_s^{2m}\tilde{\kappa}\right\}\right]\left(\partial_s^{-m}\varphi\right)ds \\
&= -\left\langle(-1)^m\partial_s^{2m}\tilde{\kappa},\varphi\right\rangle_{H^{-m}}.
\end{aligned}$$

Since (1.8) is a parabolic equation, $\boldsymbol{f}$ is smooth for $t>0$ as long as the solution exists. Hence by shifting the initial time, we may assume the initial data is smooth. Then we have the following theorem.

Theorem 5.1 *Assume that $\boldsymbol{f}$ is a global solution of* (1.8) *such that the initial rotation number is* 1 *and the initial* (*signed*) *area is positive. Then for each $\ell \in \mathbb{N}\cup\{-1,0\}$, there exist $C_\ell>0$ and $\lambda_\ell>0$ such that*

$$I_\ell(t) \leqq C_\ell e^{-\lambda_\ell t}.$$

Proof. We have

$$\begin{aligned}
\frac{dL}{dt} &= -\int_0^L \partial_t\boldsymbol{f}\cdot\boldsymbol{\kappa}\,ds = (-1)^{m+1}\int_0^L \left(\partial_s^{2m}\tilde{\kappa}\right)\tilde{\kappa}\,ds = -\int_0^L \left(\partial_s^m\tilde{\kappa}\right)^2 ds \\
&= -\frac{1}{L^{2m+1}}I_m, \\
\frac{dA}{dt} &= -\int_0^L \partial_t\boldsymbol{f}\cdot\boldsymbol{\nu}\,ds = (-1)^{m+1}\int_0^L \left(\partial_s^{2m}\tilde{\kappa}\right)ds = 0.
\end{aligned}$$

When $\ell=-1$, we have

$$\frac{d}{dt}I_{-1} = \frac{d}{dt}\left(-\frac{4\pi A}{L^2}\right) = \frac{8\pi A}{L^3}\frac{dL}{dt} = -\frac{8\pi A}{L^{2(m+2)}}I_m \leqq -\lambda_{-1}I_{-1},$$

where λ_{-1} is a positive constant. Hence, the exponential decay of I_{-1} follows.

Next we consider the behavior of I_0. Since

$$\partial_t \kappa = (-1)^m \partial_s^{2m+2} \tilde{\kappa} + (-1)^m \kappa^2 \partial_s^{2m} \tilde{\kappa},$$

we have

$$\begin{aligned}
\frac{d}{dt} I_0 &= \frac{dL}{dt} \int_0^L \tilde{\kappa}^2 ds + L \int_0^L 2\tilde{\kappa} \partial_t \tilde{\kappa} ds + L \int_0^L \tilde{\kappa}^2 \partial_t (ds) \\
&= -\frac{I_m}{L^{2m+1}} \frac{I_0}{L} + 2L \int_0^L \tilde{\kappa} \left\{ (-1)^m \partial_s^{2m+2} \tilde{\kappa} + (-1)^m \kappa^2 \partial_s^{2m} \tilde{\kappa} + \frac{2\pi}{L^2} \frac{dL}{dt} \right\} ds \\
&\quad + (-1)^{m+1} L \int_0^L \tilde{\kappa}^2 \kappa \partial_s^{2m} \tilde{\kappa} ds \\
&= -\frac{I_0 I_m}{L^{2(m+1)}} - 2L \int_0^L \left(\partial_s^{m+1} \tilde{\kappa} \right)^2 ds + (-1)^m L \int_0^L \tilde{\kappa} \kappa \left(2\kappa - \tilde{\kappa} \right) \partial_s^{2m} \tilde{\kappa} \, ds \\
&= -\frac{I_0 I_m}{L^{2(m+1)}} - \frac{1}{L^{2(m+1)}} I_{m+1} \\
&\quad + (-1)^m L \int_0^L \tilde{\kappa} \left(\tilde{\kappa} + \frac{2\pi}{L} \right) \left\{ 2 \left(\tilde{\kappa} + \frac{2\pi}{L} \right) - \tilde{\kappa} \right\} \partial_s^{2m} \tilde{\kappa} \, ds.
\end{aligned}$$

Hence we find

(5.1)
$$\frac{d}{dt} I_0 + \frac{I_0 I_m}{L^{2(m+1)}} + \frac{2 I_{m+1}}{L^{2(m+1)}} = (-1)^m L \int_0^L \left(\tilde{\kappa}^3 + \frac{6\pi}{L} \tilde{\kappa}^2 + \frac{8\pi^2}{L^2} \tilde{\kappa} \right) \partial_s^{2m} \tilde{\kappa} \, ds.$$

Then terms on the right-hand side of (5.1) are

$$\begin{aligned}
(-1)^m L \int_0^L \tilde{\kappa}^3 \partial_s^{2m} \tilde{\kappa} \, ds &= L \int_0^L P_3^m(\tilde{\kappa}) \partial_s^m \tilde{\kappa} \, ds, \\
(-1)^m 6\pi \int_0^L \tilde{\kappa}^2 \partial_s^{2m} \tilde{\kappa} \, ds &= 6\pi \int_0^L P_2^m(\tilde{\kappa}) \partial_s^m \tilde{\kappa} \, ds, \\
(-1)^m \frac{8\pi^2}{L} \int_0^L \tilde{\kappa} \partial_s^{2m} \tilde{\kappa} \, ds &= \frac{8\pi^2}{L} \int_0^L \left(\partial_s^m \tilde{\kappa} \right)^2 ds = \frac{8\pi^2}{L^{2(m+1)}} I_m,
\end{aligned}$$

by use of integration by parts. Hence we show, by using Young's inequality,

$$
\begin{aligned}
&\left| L \int_0^L P_3^m(\tilde{\kappa}) \partial_s^m \tilde{\kappa} \, ds \right| \\
&\quad \leqq \frac{C}{L^{2(m+1)}} \sum_{\ell=0}^{m} \sum_{k+j=\ell} J_{m,4} J_{k,4} J_{j,4} J_{m-k-j,4} \\
&\quad \leqq \frac{C}{L^{2(m+1)}} \sum_{\ell=0}^{m} \sum_{k+j=\ell} J_{m+1,2}^{\frac{4m+1}{4(m+1)}} J_{0,2}^{\frac{3}{4(m+1)}} J_{m+1,2}^{\frac{4k+1}{4(m+1)}} J_{0,2}^{\frac{4(m-k)+3}{4(m+1)}} J_{m+1,2}^{\frac{4j+1}{4(m+1)}} J_{0,2}^{\frac{4(m-j)+3}{4(m+1)}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times J_{m+1,2}^{\frac{4(m-k-j)+1}{4(m+1)}} J_{0,2}^{\frac{4(k+j)+3}{4(m+1)}} \\
&\quad \leqq \frac{C}{L^{2(m+1)}} J_{m+1,2}^{\frac{2m+1}{m+1}} J_{0,2}^{\frac{2m+3}{m+1}} \\
&\quad = \frac{C}{L^{2(m+1)}} I_{m+1}^{\frac{2m+1}{2(m+1)}} I_0^{\frac{2m+3}{2(m+1)}} \\
&\quad \leqq \frac{C}{L^{2(m+1)}} \left(\epsilon I_{m+1} + C_\epsilon I_0^{2m+3} \right)
\end{aligned}
$$

for any $\epsilon > 0$ and appropriate constant C_ϵ. Similarly we also have

$$
\begin{aligned}
&\left| 6\pi \int_0^L P_2^m(\tilde{\kappa}) \partial_s^m \tilde{\kappa} \, ds \right| \\
&\quad \leqq \frac{C}{L^{2(m+1)}} \sum_{k=0}^{m} J_{m,3} J_{k,3} J_{m-k,3} \\
&\quad \leqq \frac{C}{L^{2(m+1)}} \sum_{k=0}^{m} J_{m+1,2}^{\frac{6m+1}{6(m+1)}} J_{0,2}^{\frac{5}{6(m+1)}} J_{m+1,2}^{\frac{6k+1}{6(m+1)}} J_{0,2}^{\frac{6(m-k)+5}{6(m+1)}} J_{m+1,2}^{\frac{6(m-k)+1}{6(m+1)}} J_{0,2}^{\frac{6k+5}{6(m+1)}} \\
&\quad \leqq \frac{C}{L^{2(m+1)}} J_{m+1,2}^{\frac{4m+1}{2(m+1)}} J_{0,2}^{\frac{2m+5}{2(m+1)}} \\
&\quad = \frac{C}{L^{2(m+1)}} I_{m+1}^{\frac{4m+1}{4(m+1)}} I_0^{\frac{2m+5}{4(m+1)}} \\
&\quad \leqq \frac{C}{L^{2(m+1)}} \left(\epsilon I_{m+1} + C_\epsilon I_0^{\frac{2m+5}{3}} \right).
\end{aligned}
$$

Therefore we have

$$
\frac{d}{dt} I_0 + \frac{I_0 I_m}{L^{2(m+1)}} + \frac{2 I_{m+1}}{L^{2(m+1)}} \leqq \frac{C}{L^{2(m+1)}} \left(\epsilon I_{m+1} + C_\epsilon I_0^{2m+3} + C_\epsilon I_0^{\frac{2m+5}{3}} + I_m \right).
$$

By using Young's inequality and Theorem 3.1, we obtain

$$
\begin{aligned}
I_0^{\frac{2m+5}{3}} &= I_0^{\frac{2m+3}{3}} I_0^{\frac{2}{3}} \leqq \epsilon I_0 + C_\epsilon I_0 \leqq \epsilon I_0 + C_\epsilon \left(I_{-1}^{\frac{m+1}{2}} I_{m+1} + I_{-1}^{\frac{m+1}{m+2}} I_{m+1}^{\frac{1}{m+2}} \right) \\
&\leqq \epsilon I_0 + C_\epsilon \left(I_{-1}^{\frac{m+1}{2}} + \epsilon' \right) I_{m+1} + C_{\epsilon,\epsilon'} I_{-1}
\end{aligned}
$$

where $\epsilon, \epsilon' > 0$ and C_ϵ and $C_{\epsilon,\epsilon'}$ are appropriate constants. Similarly, for $m \geqq 1$, we have

$$I_m \leqq C\left(I_{-1}^{\frac{1}{2}} I_{m+1} + I_{-1}^{\frac{1}{m+2}} I_{m+1}^{\frac{m+1}{m+2}}\right) \leqq C\left(I_{-1}^{\frac{1}{2}} + \epsilon\right) I_{m+1} + C_\epsilon I_{-1}.$$

Taking ϵ, ϵ' sufficiently small, we have

$$\frac{d}{dt} I_0 + \frac{I_0 I_m}{L^{2(m+1)}} + \frac{C_1 I_{m+1}}{L^{2(m+1)}} \leqq \frac{C_2}{L^{2(m+1)}} I_0^{2m+3} + \frac{C_3}{L^{2(m+1)}} e^{-\lambda_{-1} t} \tag{5.2}$$

for sufficiently large t. Since

$$\frac{dL}{dt} + \frac{I_m}{L^{2m+1}} = 0,$$

we have

$$\int_0^\infty I_m dt < \infty.$$

From Wirtinger's inequlity, we obtain

$$\int_0^\infty I_\ell dt < \infty$$

for $\ell \in \{0, \ldots, m\}$. From (5.2) and $\int_0^\infty I_0 dt < \infty$, we can show

$$\frac{I_0 I_m}{L^{2(m+1)}} > \frac{C_2}{L^{2(m+1)}} I_0^{2m+3}$$

for sufficiently large t. Hence we have

$$I_0 \leqq C_0 e^{-\lambda_0 t}.$$

Next we consider the behavior of I_ℓ for $\ell \geqq 1$. By direct calculations, we have

$$\begin{aligned}\frac{d}{dt} I_\ell = &(2\ell+1) L^{2\ell} \frac{dL}{dt} \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right)^2 ds + L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right)^2 \partial_t(ds) \\ &+ 2L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right)\left(\partial_t \partial_s^\ell \tilde{\kappa}\right) ds\end{aligned}$$

and

$$\begin{aligned}(2\ell+1) L^{2\ell} \frac{dL}{dt} \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right)^2 ds &= -(2\ell+1) L^{2\ell} \int_0^L \left(\partial_s^m \tilde{\kappa}\right)^2 ds \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right)^2 ds \\ &= -\frac{2\ell+1}{L^{2(m+1)}} I_m I_\ell, \\ L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right)^2 \partial_t(ds) &= (-1)^{m+1} L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right)^2 \kappa \partial_s^{2m} \tilde{\kappa}\, ds \\ &= (-1)^m L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right) \sum_{n=0}^{1} L^{-(1-n)} P_{n+2}^{2m+\ell}(\tilde{\kappa})\, ds.\end{aligned}$$

We can show

$$\partial_t \partial_s^\ell \tilde{\kappa} = (-1)^m \partial_s^{2m+\ell+2} \tilde{\kappa} + (-1)^m \sum_{n=0}^{2} L^{-(2-n)} P_{n+1}^{2m+\ell}(\tilde{\kappa}) \tag{5.3}$$

by induction on ℓ. Indeed, since

$$\partial_t \kappa = (-1)^m \partial_s^{2m+2} \tilde{\kappa} + (-1)^m \kappa^2 \partial_s^{2m} \tilde{\kappa},$$

we have

$$\begin{aligned}
\partial_t \partial_s \tilde{\kappa} &= \partial_t \partial_s \kappa = \partial_s \partial_t \kappa + (\partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa}) \partial_s \kappa \\
&= \partial_s \left\{ (-1)^m \partial_s^{2m+2} \tilde{\kappa} + (-1)^m \kappa^2 \partial_s^{2m} \tilde{\kappa} \right\} + (-1)^m \kappa \left(\partial_s^{2m} \tilde{\kappa} \right) (\partial_s \tilde{\kappa}) \\
&= (-1)^m \partial_s^{2m+3} \tilde{\kappa} + (-1)^m 2\kappa \left(\partial_s \kappa \right) \left(\partial_s^{2m} \tilde{\kappa} \right) + (-1)^m \kappa^2 \partial_s^{2m+1} \tilde{\kappa} \\
&\quad + (-1)^m \kappa \left(\partial_s^{2m} \tilde{\kappa} \right) (\partial_s \tilde{\kappa}) \\
&= (-1)^m \partial_s^{2m+3} \tilde{\kappa} + (-1)^m \sum_{n=0}^{2} L^{2-n} P_{n+1}^{2m+1}(\tilde{\kappa}).
\end{aligned}$$

Hence we obtain (5.3) when $\ell = 1$. If (5.3) holds for $\ell \geqq 1$, since

$$\begin{aligned}
\partial_t \partial_s^{\ell+1} \tilde{\kappa} &= \partial_s \partial_t \partial_s^\ell \tilde{\kappa} + (\partial_t \boldsymbol{f} \cdot \boldsymbol{\kappa}) \partial_s^\ell \tilde{\kappa} \\
&= \partial_s \left\{ (-1)^m \partial_s^{2m+\ell+2} \tilde{\kappa} + (-1)^m \sum_{n=0}^{2} L^{-(2-n)} P_{n+1}^{2m+\ell}(\tilde{\kappa}) \right\} \\
&\quad + (-1)^m \kappa \left(\partial_s^{2m} \tilde{\kappa} \right) \partial_s \left(\partial_s^\ell \tilde{\kappa} \right) \\
&= (-1)^m \partial_s^{2m+\ell+3} \tilde{\kappa} + (-1)^m \sum_{n=0}^{2} L^{-(2-n)} P_{n+1}^{2m+\ell+1}(\tilde{\kappa}),
\end{aligned}$$

we show (5.3) for $\ell + 1$. Hence we have

$$\begin{aligned}
&2L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa} \right) \left(\partial_t \partial_s^\ell \tilde{\kappa} \right) ds \\
&\quad = 2L^{2\ell+1} \int_0^L \left(\partial_s^{\ell+m+1} \tilde{\kappa} \right)^2 ds \\
&\qquad + (-1)^m 2L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa} \right) \sum_{n=0}^{2} L^{-(2-n)} P_{n+1}^{2m+\ell}(\tilde{\kappa}) \, ds \\
&\quad = -\frac{2}{L^{2(m+1)}} I_{m+\ell+1} + (-1)^m 2L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa} \right) \sum_{n=0}^{2} L^{-(2-n)} P_{n+1}^{2m+\ell}(\tilde{\kappa}) \, ds.
\end{aligned}$$

Therefore we have

$$
\begin{aligned}
\frac{d}{dt} I_\ell + & \frac{2\ell+1}{L^{2(m+1)}} I_m I_\ell + \frac{2}{L^{2(m+1)}} I_{m+\ell+1} \\
&= (-1)^m 2L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right) \sum_{n=0}^{2} L^{-(2-n)} P_{n+1}^{2m+\ell}(\tilde{\kappa})\, ds.
\end{aligned}
$$

When $n = 0$, after integration by parts m times, using Theorem 3.1 and Young's inequality, we have

$$
\begin{aligned}
(-1)^m & 2L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right) L^{-2} P_1^{2m+\ell}(\tilde{\kappa})\, ds \\
&= cL^{2\ell-1} \int_0^L \left(\partial_s^{m+\ell} \tilde{\kappa}\right)^2 ds = \frac{c}{L^{2(m+1)}} I_{m+\ell} \\
&\leqq \frac{C}{L^{2(m+1)}} \left(I_{-1}^{\frac{1}{2}} I_{m+\ell+1} + I_{-1}^{\frac{1}{m+\ell+2}} I_{m+\ell+1}^{\frac{m+\ell+1}{m+\ell+2}} \right) \\
&\leqq \frac{C}{L^{2(m+1)}} \left\{ \left(I_{-1}^{\frac{1}{2}} + \epsilon \right) I_{m+\ell+1} + C_\epsilon I_{-1} \right\}.
\end{aligned}
$$

When $n = 1$, we have

$$
\begin{aligned}
(-1)^m & 2L^{2\ell+1} \int_0^L \left(\partial_s^\ell \tilde{\kappa}\right) L^{-1} P_2^{2m+\ell}(\tilde{\kappa})\, ds \\
&= (-1)^m 2L^{2\ell} \int_0^L \sum_{k=0}^{2m+\ell} c_k \left(\partial_s^\ell \tilde{\kappa}\right) \left(\partial_s^k \tilde{\kappa}\right) \left(\partial_s^{2m+\ell-k} \tilde{\kappa}\right) ds.
\end{aligned}
$$

We set

$$
\begin{aligned}
K_1 = & \left\{ k \in \{0, \ldots, 2m+\ell\} \mid \max\{k, 2m+\ell-k\} > m+\ell \right\}, \\
K_2 = & \left\{ k \in \{0, \ldots, 2m+\ell\} \mid \max\{k, 2m+\ell-k\} \leqq m+\ell \right\}.
\end{aligned}
$$

If $\max\{k, 2m+\ell-k\} > m+\ell$, then $\min\{k, 2m+\ell-k\} < m+\ell$. When

$k \in K_1$, from integration by parts $\max\{k, 2m+\ell-k\}-m-\ell$ times, we have

$$
\begin{aligned}
&(-1)^m 2L^{2\ell} \int_0^L \sum_{k=0}^{2m+\ell} c_k \left(\partial_s^\ell \tilde{\kappa}\right) \left(\partial_s^k \tilde{\kappa}\right) \left(\partial_s^{2m+\ell-k} \tilde{\kappa}\right) ds \\
&\quad = 2L^{2\ell} \int_0^L \sum_{k \in K_1} (-1)^{\max\{k,2m+\ell-k\}-\ell} c_k \left(\partial_s^{m+\ell} \tilde{\kappa}\right) P_2^{m+\ell}(\tilde{\kappa})\, ds \\
&\qquad\quad + (-1)^m 2L^{2\ell} \int_0^L \sum_{k \in K_2} c_k \left(\partial_s^\ell \tilde{\kappa}\right) \left(\partial_s^k \tilde{\kappa}\right) \left(\partial_s^{2m+\ell-k} \tilde{\kappa}\right) ds \\
&\quad \leqq \frac{C}{L^{2(m+1)}} \sum_{k \in K_1} \sum_{k'=0}^{m+\ell} J_{k',3} J_{m+\ell-k',3} J_{m+\ell,3} + \frac{C}{L^{2(m+1)}} \sum_{k \in K_2} J_{\ell,3} J_{k,3} J_{2m+\ell-k,3} \\
&\quad \leqq \frac{C}{L^{2(m+1)}} \sum_{k \in K_1} \sum_{k'=0}^{m+\ell} J_{m+\ell+1,2}^{\frac{6k'+1}{6(m+\ell+1)}} J_{0,2}^{\frac{6(m+\ell-k')+5}{6(m+\ell+1)}} J_{m+\ell+1,2}^{\frac{6(m+\ell-k')+1}{6(m+\ell+1)}} J_{0,2}^{\frac{6k'+5}{6(m+\ell+1)}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times J_{m+\ell+1,2}^{\frac{6(m+\ell)+1}{6(m+\ell+1)}} J_{0,2}^{\frac{5}{6(m+\ell+1)}} \\
&\qquad\quad + \frac{C}{L^{2(m+1)}} \sum_{k \in K_2} J_{m+\ell+1,2}^{\frac{6\ell+1}{6(m+\ell+1)}} J_{0,2}^{\frac{6m+5}{6(m+\ell+1)}} J_{m+\ell+1,2}^{\frac{6k+1}{6(m+\ell+1)}} J_{0,2}^{\frac{6(m+\ell-k)+5}{6(m+\ell+1)}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times J_{m+\ell+1,2}^{\frac{6(2m+\ell-k)+1}{6(m+\ell+1)}} J_{0,2}^{\frac{-6(m-k)+5}{6(m+\ell+1)}} \\
&\quad \leqq \frac{C}{L^{2(m+1)}} J_{m+\ell+1,2}^{\frac{4(m+\ell)+1}{2(m+\ell+1)}} J_{0,2}^{\frac{2(m+\ell)+5}{2(m+\ell+1)}} \\
&\quad = \frac{C}{L^{2(m+1)}} I_{m+\ell+1}^{\frac{4(m+\ell)+1}{4(m+\ell+1)}} I_0^{\frac{2(m+\ell)+5}{4(m+\ell+1)}} \\
&\quad \leqq \frac{C}{L^{2(m+1)}} \left(\epsilon I_{m+\ell+1} + C_\epsilon I_0^{\frac{2(m+\ell)+5}{3}} \right).
\end{aligned}
$$

When $n = 2$, we have

$$
P_3^{2m+\ell}(\tilde{\kappa}) = \sum_{\alpha=0}^{2m+\ell} \sum_{k+j=\alpha} c_{k,j} \left(\partial_s^k \tilde{\kappa}\right) \left(\partial_s^j \tilde{\kappa}\right) \left(\partial_s^{2m+\ell-k-j} \tilde{\kappa}\right).
$$

We set

$$
\begin{aligned}
K_{\alpha,1} &= \{(k,j) \,|\, k+j=\alpha,\ \max\{k,j,2m+\ell-k-j\} > m+\ell\}, \\
K_{\alpha,2} &= \{(k,j) \,|\, k+j=\alpha,\ \max\{k,j,2m+\ell-k-j\} \leqq m+\ell\}.
\end{aligned}
$$

If $\max\{k,j,2m+\ell-k-j\} > m+\ell$, the other terms are less than $m+\ell$. When $k \in K_{\alpha,1}$, from integration by parts $\max\{k,j,2m+\ell-k-j\}-m-\ell$

times, we have

$$
\begin{aligned}
&(-1)^m 2L^{2\ell+1}\int_0^L \sum_{\alpha=0}^{2m+\ell}\sum_{k+j=\alpha} c_{kj}\left(\partial_s^k\tilde\kappa\right)\left(\partial_s^j\tilde\kappa\right)\left(\partial_s^{2m+\ell-k-j}\tilde\kappa\right)ds \\
&\quad = 2L^{2\ell+1}\int_0^L \sum_{\alpha=0}^{2m+\ell}\sum_{(k,j)\in K_{\alpha,1}} (-1)^{\max\{k,j,2m+\ell-k-j\}-\ell} c_{kj}\left(\partial_s^{m+\ell}\tilde\kappa\right) P_3^{m+\ell}(\tilde\kappa)\,ds \\
&\qquad + (-1)^m 2L^{2\ell+1}\int_0^L \sum_{\alpha=0}^{2m+\ell}\sum_{(k,j)\in K_{\alpha,2}} \left(\partial_s^k\tilde\kappa\right)\left(\partial_s^j\tilde\kappa\right)\left(\partial_s^{2m+\ell-k-j}\tilde\kappa\right)ds \\
&\quad \leqq \frac{C}{L^{2(m+1)}}\sum_{\alpha=0}^{2m+\ell}\sum_{(k,j)\in K_{\alpha,1}}\sum_{\beta=0}^{m+\ell}\sum_{k'+j'=\beta} J_{k',4}J_{j',4}J_{m+\ell-k'j',4}J_{m+\ell,4} \\
&\qquad + \sum_{\alpha=0}^{2m+\ell}\sum_{(k,j)\in K_{\alpha,2}} J_{\ell,4}J_{k,4}J_{j,4}J_{2m+\ell-k-j,4} \\
&\quad \leqq \frac{C}{L^{2(m+1)}}\sum_{\alpha=0}^{2m+\ell}\sum_{(k,j)\in K_{\alpha,1}}\sum_{\beta=0}^{m+\ell}\sum_{k'+j'=\beta} J_{m+\ell+1,2}^{\frac{4k'+1}{4(m+\ell+1)}} J_{0,2}^{\frac{4(m+\ell-k')+3}{4(m+\ell+1)}} J_{m+\ell+1,2}^{\frac{4j'+1}{4(m+\ell+1)}} J_{0,2}^{\frac{4(m+\ell-j')+3}{4(m+\ell+1)}} \\
&\qquad\qquad \times J_{m+\ell+1,2}^{\frac{4(m+\ell-k'-j')+1}{4(m+\ell+1)}} J_{0,2}^{\frac{4(k'+j')+3}{4(m+\ell+1)}} J_{m+\ell+1,2}^{\frac{4(m+\ell)+1}{4(m+\ell+1)}} J_{0,2}^{\frac{3}{4(m+\ell+1)}} \\
&\qquad + \frac{C}{L^{2(m+1)}}\sum_{\alpha=0}^{2m+\ell}\sum_{(k,j)\in K_{\alpha,2}} J_{m+\ell+1,2}^{\frac{4\ell+1}{4(m+\ell+1)}} J_{0,2}^{\frac{4m+3}{4(m+\ell+1)}} J_{m+\ell+1,2}^{\frac{4k+1}{4(m+\ell+1)}} J_{0,2}^{\frac{4(m+\ell-k)+3}{4(m+\ell+1)}} \\
&\qquad\qquad \times J_{m+\ell+1,2}^{\frac{4j+1}{4(m+\ell+1)}} J_{0,2}^{\frac{4(m+\ell-j)+3}{4(m+\ell+1)}} J_{m+\ell+1,2}^{\frac{4(2m+\ell-k-j)+1}{4(m+\ell+1)}} J_{0,2}^{\frac{-4(m-k-j)+3}{4(m+\ell+1)}} \\
&\quad = \frac{C}{L^{2(m+1)}} J_{m+\ell+1,2}^{\frac{2(m+\ell)+1}{m+\ell+1}} J_{0,2}^{\frac{2(m+\ell)+3}{m+\ell+1}} \\
&\quad = \frac{C}{L^{2(m+1)}} I_{m+\ell+1}^{\frac{2(m+\ell)+1}{2(m+\ell+1)}} I_0^{\frac{2(m+\ell)+3}{2(m+\ell+1)}} \\
&\quad \leqq \frac{C}{L^{2(m+1)}}\left(\epsilon I_{m+\ell+1} + C_\epsilon I_0^{2(m+\ell)+3}\right).
\end{aligned}
$$

Taking $\epsilon > 0$ sufficiently small, we obtain

$$
\frac{d}{dt}I_\ell + \frac{2\ell+1}{L^{2(m+1)}} I_m I_\ell + \frac{C_1}{L^{2(m+1)}} I_{m+\ell+1} \leqq \frac{C_2}{L^{2(m+1)}}\left(I_{-1} + I_0^{\frac{2(m+\ell)+5}{3}} + I_0^{2(m+\ell)+3}\right)
$$

for sufficiently large $t > 0$. Hence we obtain

$$
I_\ell \leqq C_\ell e^{-\lambda_\ell t}.
$$

□

From Theorem 5.1, the claims (A)–(F) in Theorem 4.2 hold also for global solution of (1.8).

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