Bifurcation Analysis of Euler Buckling Problem from the viewpoint of Singularity Theory

(特異点論の観点からのオイラー座屈問題の分岐解析)

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Graduate School of Science and Engineering
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Declaration of Authorship

I, Atia Afroz, declare that this thesis titled, “Bifurcation Analysis of Euler Buckling Problem from the viewpoint of Singularity Theory” and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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Abstract

Doctor of Philosophy

Bifurcation Analysis of Euler Buckling Problem from the viewpoint of Singularity Theory

by Atia Afroz

We consider buckling of rod which is subjected to compressive force λ. In 1757, L. Euler's found the critical load of the system, and this problem is often called Euler buckling problem. This is actually a celebrated example of pitchfork bifurcation. M. Golubitsky and D. Schaeffer considered a modified version of Euler buckling problem in the variational formulation using strain energy and potential energy, and they show that this modified problem is a versal unfolding of the original problem. We consider a rather more general problem in the context of variational set-up and discuss smoothness of the problem, which is not discussed by M. Golubitsky and D. Schaeffer. This is important to apply Lyapunov-Schmidt reduction. We also describe 3-jets of the equations which define the bifurcation set B and the hysteresis set H, which enable us to draw figures of B and H approximately under suitable set-up.

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Dedicated to my loving daughter Tanisha

Chapter 1

Introduction

1.1 Introduction

One attractive field of application of singularity theory is bifurcation of solutions of partial differential equations or variational problems. M. Golubitsky and D. Schaeffer [Golubitsky and Schaeffer, 1979, Golubitsky and Schaeffer, 1985,Golubitsky and Schaeffer, 1988] showed how singularity theory works to investigate the bifurcation of modified problem of euler buckling problem.

We consider buckling of a rod which is subjected to compressed force λ which is known as the Euler buckling problem. The mathematical formulation of the Euler buckling problem is the problem of minimizing the energy

$$E = S + \lambda T, \quad S = \frac{1}{2}\int_0^l \left[\frac{u''}{(1-u'^2)^{1/2}}\right]^2 ds, \quad T = \int_0^l \sqrt{1-(u')^2}ds$$

on $U = \{u \in X : \|u'\|_\infty < 1-\epsilon\}, 0 < \epsilon \ll 1$, where X is the Sobolev space

$$X = \{u \in H^2[0,l] : u(0) = u(l) = 0\}.$$

Here S is the strain energy given by the integral of the square of curvature (remark that the curvature of the curve $s \mapsto (x(s), u(s))$ is $\frac{u''}{\sqrt{1-u'^2}}$, if s is an arc-length parameter), and T is the potential energy (the distance between two ends of the rod). This describes buckling of the rods with pinned ends. To the knowledge of the authors, this formulation first appeared in [Thompson and Hunt, 1973, pages 27-29] without using Sobolev space, and the formulation using the Sobolev space appeared in [Golubitsky and Schaeffer, 1979, page 76].

We are interested in the bifurcation of the zero set of the directional derivatives:

$$(D_\phi E)_u = \lim_{t\to 0}\frac{E|_{u+t\phi} - E|_u}{t} = \int_0^l \Big[\frac{u''\phi''}{1-(u')^2} + \Big(\frac{u'(u'')^2}{(1-(u')^2)^2} - \frac{\lambda u'}{(1-(u')^2)^{1/2}}\Big)\phi'\Big]ds \tag{1.1.1}$$

of directions $\phi \in X$ which may attain extreme of the total energy E. Clearly the function, which is identically zero, is a solution, and we often refer it as trivial solution. We are going to discuss the bifurcation from the trivial solution.

Differentiating (1.1.1) with the direction v and evaluating at $u = 0$, we obtain

$$X \to X', \quad v \mapsto \int_0^l (v'' + \lambda v)\phi'' ds \tag{1.1.2}$$

where X' denotes the (topological) dual space of X. The map (1.1.2) is continuously differentiable and bijective (see discussions in 6.0.1). So the inverse function theorem implies there are no other solution near the trivial solution whenever $v'' + \lambda v \neq 0$, that is, $\lambda \neq \pi^2 n^2/l^2$, $n \in \mathbb{Z}$. If $\lambda = \pi^2 n^2/l^2$, we apply Lyapunov-Schmidt reduction, and reduce the equation to a finite-dimensional set-up.

M. Golubitsky and D. Schaeffer have also considered a modified version of this problem in [Golubitsky and Schaeffer, 1979, (6.1)], namely, the problem of minimizing the modified energy

$$\frac{1}{2}\int_0^l \left[\frac{u''}{(1-u'^2)^{1/2}} - \alpha_1\right]^2 ds + \lambda \int_0^l \sqrt{1-(u')^2} ds + \alpha_2 u(\tfrac{l}{2}) \tag{1.1.3}$$

on U with $l = \pi$ where the first term is a modified strain energy with minimum when curvature is constant α_1, i.e., the rod is a circular arc, and the third term represents a central load of size α_2.

In actual situation, noise from various sources may cause some small perturbation of the idealized problem, and M. Golubitsky and D. Schaeffer call such perturbation as imperfect bifurcation or imperfection. Thus it becomes important to show whether a perturbation gives a p-$\mathcal{K}$-versal unfolding or not, since a p-$\mathcal{K}$-versal unfolding contains all nearby bifurcation of the idealized problem. See Definition 7.1 for the definition of p-$\mathcal{K}$-versality. They showed that this modified problem represents a p-$\mathcal{K}$-versal unfolding of the bifurcation equation of the original problem. To apply their criterion of versality [Golubitsky and Schaeffer, 1979, Lemma 4.3], we need to ensure the equation describing the problem is smooth (C^∞). Since we are in the context of a variational problem, it is not a priori clear, and proof was not discussed *loc. cit.*.

We actually consider a rather more general problem, the variational problem minimizing the energy (A.2.1), obtained by replacing α_1 by $\alpha_1\kappa$ in (1.1.3) where κ is defined in (A.2.2), since we do not have any reason to assume that a circular arc minimizes the strain energy. This problem also has a term for modified strain energy, which has minimum at a curve with given curvature $\alpha_1\kappa$. After stating the first variational formula (Lemma 3.4.1), the problem becomes to describe the change of bifurcation of zeros of the function

$$\Phi : U \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow X', \Phi(u, \lambda, \alpha_1, \alpha_2) = [\phi \mapsto (\Psi - \lambda\Lambda - \alpha_1 K)_u \cdot \phi + \alpha_2\phi(\tfrac{l}{2})],$$

where

$$U = \{u \in X : \|u'\|_\infty < 1 - \epsilon\}, 0 < \epsilon \ll 1.$$

See Lemma 3.4.1 for the definitions of Ψ, Λ, K.

We first show that

Theorem 1.1.1. *The function Φ is smooth.*

This theorem allows us to apply Lyapunov-Schmidt reduction to reduce the bifurcation problem to that of a finite-dimensional set-up. This theorem enables us to discuss the values of higher order differentials of Φ and we are going to apply the criterion of bifurcation type. In this thesis, we compute Taylor coefficients of F, which is defined in (6.0.3), and which describes the bifurcation of $\Phi = 0$. This is an unfolding of pitchfork bifurcation near $(x, \lambda, \alpha) = (0, \lambda^*, 0)$, $\lambda^* = \pi^2 n^2 / l^2$, and we show

Theorem 1.1.2. *If n is odd, then F is p-$\mathcal{K}$-versal.*

Roughly speaking, this implies all nearby bifurcation of a pitchfork bifurcation can be realized by Φ near $(0, \lambda^*, 0)$. See Definition 8.1.1 for the precise definition on p-$\mathcal{K}$-versality. Remark that M. Golubitsky and D. Schaeffer showed this theorem when $n = 1$.

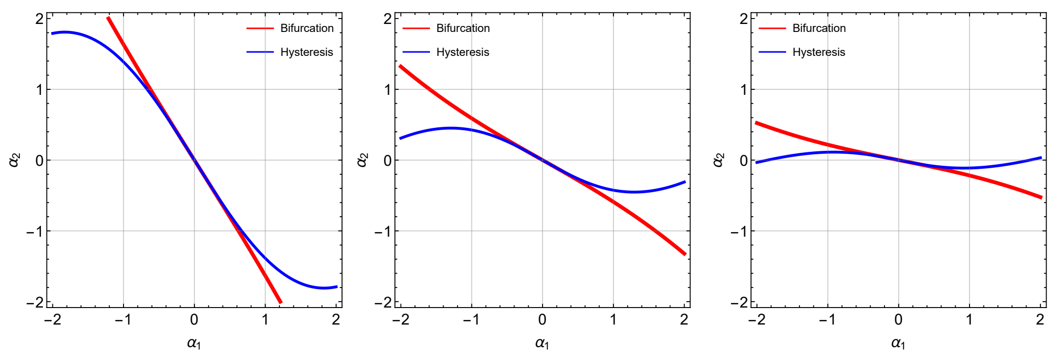
To describe how the pitchfork bifurcation changes nearby the origin, we recall the bifurcation set B and hysteresis set H, which are defined by

$$B = \{\alpha : \exists (x, \lambda),\ F(x, \lambda, \alpha) = 0, F_x(x, \lambda, \alpha) = F_\lambda(x, \lambda, \alpha) = 0\}, \tag{1.1.4}$$
$$H = \{\alpha : \exists (x, \lambda),\ F(x, \lambda, \alpha) = 0, F_x(x, \lambda, \alpha) = F_{xx}(x, \lambda, \alpha) = 0\} \tag{1.1.5}$$

in our situation. If n is odd, these sets are zeros of certain functions with the following 1-jet:

$$\left(\frac{4\pi n^2}{l^2} \sum_{i=0}^{\infty} \frac{n a_i}{n^2 - 4i^2}\right)\alpha_1 + \left((-1)^{\frac{n-1}{2}} \sqrt{\frac{2}{l}}\right)\alpha_2 \ (= \bar{F}_1 \alpha_1 + \bar{F}_2 \alpha_2, (\text{Chapter}7)).$$

In Proposition 9.1.1, we describe their 3-jets as (9.1.3) and (9.1.4), respectively, which enables us to draw B and H approximately near the origin. For example, $\kappa = 1/\sqrt{\pi/2}$, $n = 1$, the zeros of these 3-jets look like:



(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 1.1: Approximations of B and H ($a_0 = 1, a_{i \geq 1} = 0, b_i = 0$).

The thesis is organized as follows
In chapter 2, we discuss some preliminaries such as we recall the definition of Sobolev space with some related lemmas such as Hölder's inequality in

section 2.1. Section 2.2 is devoted to the study of versal unfolding with the definition, example and versal unfolding theorem. Section 2.3 is presented the $\mathcal{P}$-$\mathcal{K}$-equivalent, section 2.4 is presented the basic bifurcation theorem, section 2.5 is devoted to present the bifurcation set and hysteresis set.

In chapter 3, we present the basic idea of buckling rod with pinned ends in section 3.1, and we recall the definition of curvature in section 3.2. In section 3.3 and 3.4, we formulate the energy of Euler buckling probelm and minimize the problem by calculus of variations respectively.

In chapter 4, we investigate the differntiability of Φ in section 4.1 and section 4.2 is devoted to proving that "Φ is in C^1" by lemma 4.2.1 and section 4.3 is devoted to proving that "Φ is C^∞". In section 4.3 we prove our main theorem which is "Φ is smooth".

In chapter 5, we derive the Taylor coefficients of Φ up-to third order derivatives. In section 5.1 we present the few terms of Taylor expansion and section 5.2, 5.4, and 5.5 are devoted to derive the first, second and third derivatives of $(L)_u$ respectively. In section 5.3, we find the derivatives of the coeeficient of α_1 up-to order 3.

In chapter 6, we discuss the Lyapunov-Schmidt reduction and we define W. In section 6.1 we find all the first derivatives of W and in section 6.2 we find all the second derivatives of W.

In chapter 7, we discuss the bifurcation equation and derive the related Taylor coefficients up-to order 3. We define $F(x, \lambda, \alpha) = 0$, and in section 7.1, 7.2, 7.3 we find the first, second, third derivatives of $F(x, \lambda, \alpha)$ respectively . We evaluate all the derivatives at $(0, \lambda^*, 0)$.

In chapter 8, we discuss about the versality, and section 8.1 is devoted to present the definition of $\mathcal{P}$-$\mathcal{K}$-versality with an example 8.1.2 and a figure of bifurcation set and hysteresis set which showing all kind of perturbation in Figure 8.1.

In chapter 9, we derive the equation of bifurcation set and hysteresis set in section 9.1 and we present several numerical results which help us to describe the figures of the zeros of bifurcation set and hysteresis set. In section 9.2, we present several figures of zeros of the 3-jets of bifurcation set and hysteresis set.

Chapter 2

Preliminaries

In this chapter we recall some basic definitions and lemmas of classics such as Sobolev space, Hölder's inequality lemma, versal unfolding, $\mathcal{P}$-$\mathcal{K}$ equivalent, basic bifurcation theorem, bifurcation set and hystersis set, etc.

2.1 Sobolev Space

[Brezis, 2010, Marsden and Hughes, 1994] Let X^*, Y^*, Z^*, $\ldots$ be the dual spaces of the Banach spaces X, Y, Z, $\ldots$, respectively. We denote by X', Y', $\ldots$ the topological dual spaces of X, Y, $\ldots$, respectively. A map $f : X \to Y$ is continuous, where X and Y are Banach spaces, if for any $\varepsilon > 0$ there is $\delta > 0$ so that $\|f(x) - f(x')\|_Y < \varepsilon$ whenever $\|x - x'\|_X < \delta$.
A linear map $\psi : X \to Y$ is continuous if

$$\|\psi(x)\|_Y \lesssim \|x\|_X \quad \text{for any } x \in X.$$

A multi-linear map $\psi : X \times \cdots \times X \to Y$ is continuous, if

$$\|\psi(x_1, \ldots, x_k)\|_Y \lesssim \|x_1\|_X \cdots \|x_k\|_X \quad \text{for any } x_1, \ldots, x_k \in X.$$

Similarly, a multi-linear map

$$\psi : X \times \cdots \times X \longrightarrow Y^*, \quad (x_1, \ldots, x_k) \mapsto (y \mapsto \psi[x_1, \ldots, x_k] \cdot y)$$

is continuous and the image is in Y', if there is a positive constant C such that

$$|\psi[x_1, \ldots, x_k] \cdot y| \leq C\|x_1\|_X \cdots \|x_k\|_X \|y\|_Y \quad \text{for any } x_1, \ldots, x_k \in X, \text{ and } y \in Y.$$

By notational convention we denote this inequality by
$|\psi[x_1, \ldots, x_k] \cdot y| \lesssim \|x_1\|_X \cdots \|x_k\|_X \|y\|_Y$.

Let $L(X \times \cdots \times X, Y')$ denote the set of linear maps. A map $Z \to L(X \times \cdots \times X, Y')$, $z \mapsto \psi_z$, is continuous such that

$$|(\psi_{z_1}[x_1, \ldots, x_k] - \psi_{z_2}[x_1, \ldots, x_k]) \cdot y| \leq C\|z_1 - z_2\|_Z \|x_1\|_X \cdots \|x_k\|_X \|y\|_Y,$$

where C is the positive constant, for any $x_1, \ldots, x_k \in X$, $y \in Y$, and $z_1, z_2 \in Z$.

Let the set of functions $\mathcal{F}[0, l] \to \mathbb{R}$ modulo the equivalence relation $\underset{\text{a.e.}}{\sim}$. Here the equivalence relation of f and g i.e., $f \underset{\text{a.e.}}{\sim} g$ means, the functions

$f, g : [0, l] \to \mathbb{R}$ coincide except on a measure zero set. Let the Sobolev space $W^{k,p}[0, l] = \{u \in \mathcal{F}[0, l] : \|u\|_{k,p} < \infty\}$ equipped with the Sobolev norm

$$\|u\|_{k,p} = \Big(\sum_{i=0}^{k} \binom{k}{i} \|D^i u\|_p{}^p\Big)^{\frac{1}{p}}, \quad \|u\|_p = \begin{cases} \big(\int_0^l |u|^p ds\big)^{1/p}, & 1 \le p < \infty, \\ \max\{|u(s)| : s \in [0, l]\}, & p = \infty, \end{cases}$$

where $D^i u$ denote the ith order distributional derivatives of u. We denote by $L^p[0, l]$ the set $\{u : [0, l] \to \mathbb{R} : \|u\|_p < \infty\}$. We write the Hilbert space $H^k[0, l]$ for the sobolev space $W^{k,2}[0, l]$, which is the inner product with a Hilbert space

$$\langle u_m, u_j \rangle_k = \begin{cases} (1 + \frac{\pi^2 m^2}{l^2})^k, & \text{if } m = j; \\ 0, & \text{if } m \neq j, \end{cases}$$

where $u_m = \sqrt{2/l}\sin(m\pi s/l)$.

- If $u(s) = s^{-1/4} \in L^2(0, 1)$, then $\|u\|_2 = 2$, $\|u^2\|_2 = \infty$. So L^2 is not closed by product.
- If $u(s) = s^{-1/5} \in L^2(0, 1)$, then $\|u\|_2 = 5/3$, $\|u^2\|_2 = 5$. So we do not have $\|u^2\|_2 \le \|u\|_2{}^2$, in general.
- If $u = s^{-1/2}\chi_{(a,1)}(s)$, then $\|u\|_2{}^2 = -\log a$, $\|u^2\|_2 = (\frac{1}{a} - 1)^{1/2}$, and no constant C with $\|u^2\|_2 \le C\|u\|_2{}^2$ exist.

Remark that $f \in L^p \iff f^2 \in L^{p/2}$ also.

If $1 \le p < \infty$, then

- If $k < 1/p$ and $p \le q \le p/(1 - kp)$, then $W^{k,p} \subset L^q$.
- If $k = 1/p$ and $p \le q < \infty$, then $W^{k,p} \subset L^q$.
- If $k > 1/p$, then $W^{k,p} \subset C^{0,\alpha_2}$ where $\alpha = \min\{k - 1/p, 1\}$, $k \neq 1 + 1/p$; any α with $0 \le \alpha < 1$, $k = 1 + 1/p$.

Lemma 2.1.1 (Hölder's inequality). *[Brezis, 2010] If $1/p + 1/q = 1$, $p > 1$, then we have*

$$\int_0^l |uv|\, ds \le \Big(\int_0^l |u|^p ds\Big)^{\frac{1}{p}} \Big(\int_0^l |v|^q ds\Big)^{\frac{1}{q}}. \tag{2.1.1}$$

Proof. Set $\phi(t) = \frac{t^p}{p} + \frac{1}{q} - \alpha_1$, which has the minimal 0 at $t = 1$. Since

$$0 \le \phi\big((|u|/a)(|v|/b)^{-\frac{q}{p}}\big) = \tfrac{1}{p}(|u|/a)^p(|v|/b)^{-q} + \tfrac{1}{q} - (|u|/a)(|v|/b)^{-\frac{q}{p}}$$

for positive a, b, we obtain

$$\frac{|uv|}{ab} \le \frac{|u|^p}{pa^p} + \frac{|v|^q}{qb^q}, \quad \text{and thus} \quad \tfrac{1}{ab}\int_0^l |uv|\, ds \le \tfrac{1}{pa^p}\int_0^l |u|^p\, ds + \tfrac{1}{ab^q}\int_0^l |v|^q\, ds.$$

Setting $a^p = \int_0^l |u|^p ds$, $b^q = \int_0^l |v|^q ds$, the right hand side is 1, and the result follows. □

Lemma 2.1.2. (i) If $1 \le p \le q < \infty$, then $\|u\|_p \le l^{\frac{1}{p}-\frac{1}{q}}\|u\|_q$ for $u \in L^q[0,l]$.

(ii) $\lim\limits_{p\to\infty} \|u\|_p = \|u\|_\infty$.

(iii) We have $\|u\|_\infty \le C_k\|u\|_{k,2}$ for $u \in L^p[0,l]$ with $\|u\|_\infty < \infty$ where $C_k = \left(\frac{l}{2}\right)^{-\frac{1}{2}}\left(\sum_{m=1}^\infty (1+\pi^2m^2/l^2)^{-k}\right)^{\frac{1}{2}}$.

(iv) $\|v_1\cdots v_k\|_2 \le l^{\frac{1}{2}}(C_p)^k\|v_1\|_{1,p}\cdots\|v_k\|_{1,p}$.

Proof. (i): Since $q/p \ge 1$, $1 - p/q = (q-p)/q$, we obtain that, by Hölder's inequality,

$$\int_0^l |u^p|\,ds \le \left(\int_0^l |u^p|^{\frac{q}{p}}\,ds\right)^{\frac{p}{q}}\left(\int_0^l 1ds\right)^{1-\frac{p}{q}}.$$

Taking p-th roots of both sides, we obtain the result.
(ii): If $0 \le t \le \|u\|_\infty$, then

$$\|u\|_p \ge \left(\int_{|u|\ge\alpha_1} |u|^p ds\right)^{\frac{1}{p}} \ge \left(\int_{|u|\ge\alpha_1} t^p ds\right)^{\frac{1}{p}} \ge \left(\int_0^l |t|^p ds\right)^{\frac{1}{p}} = tl^{\frac{1}{p}} \to t \qquad (p\to\infty)$$

and we conclude that $\varliminf\limits_{p\to\infty} \|u\|_p \ge \|u\|_\infty$. If $1 \le p < \infty$, then

$$\|u\|_p = \left(\int_0^l |u|^p ds\right)^{\frac{1}{p}} \le \left(\int_0^l \|u\|_\infty^p ds\right)^{\frac{1}{p}} = l^{\frac{1}{p}}\|u\|_\infty$$

and we obtain that $\varlimsup\limits_{p\to\infty} \|u\|_p \le \varlimsup\limits_{p\to\infty} l^{\frac{1}{p}}\|u\|_\infty = \|u\|_\infty$.
(iii): Set $V = \langle u_m : m = 1,2,\dots\rangle_{\mathbb{R}}$ denote the vector space generated by u_1, u_2, ... The closure of V contains $C^\infty[0,l]$ in $W^{k,p}[0,l]$, and is $W^{k,p}[0,l]$, because $C^\infty[0,l]$ is dense in $W^{k,p}[0,l]$.

For $u = \sum_{m=1}^\infty y_m u_m$, we have

$$\left|\sum_{m=1}^\infty y_m u_m\right| \le \left(\tfrac{l}{2}\right)^{-\frac{1}{2}}\sum_{m=1}^\infty |y_m| = \left(\tfrac{l}{2}\right)^{-\frac{1}{2}}\sum_{m=1}^\infty (1+\pi^2m^2/l^2)^{-\frac{k}{2}}(1+\pi^2m^2/l^2)^{\frac{k}{2}}|y_m|$$
$$\le \left(\tfrac{l}{2}\right)^{-\frac{1}{2}}\left(\sum_{m=1}^\infty (1+\pi^2m^2/l^2)^{-k}\right)^{\frac{1}{2}}\left(\sum_{m=1}^\infty (1+\pi^2m^2/l^2)^{k}|y_m|^2\right)^{\frac{1}{2}} \le C_k\|u\|_{k,2}.$$

(iii): $\|v_1\cdots v_k\|_2 \le l^{\frac{1}{2}}\|v_1\cdots v_k\|_\infty \le l^{\frac{1}{2}}\|v_1\|_\infty\cdots\|v_k\|_\infty \le l^{\frac{1}{2}}(C_p)^k\|v_1\|_{1,p}\cdots\|v_k\|_{1,p}$.
□

Let $C^k[0,l]$ be the set of C^k-functions defined on $[0,l]$.

Remark 2.1.3. *(i) We have $H^k[0,l] \subset C^{k-1}[0,l]$, by the Sobolev embedding theorem. In above all, we can choose a C^{k-1}-representative to express an element of $H^k[0,l]$. This implies that*

$$H^k[0,l] = \{f \in C^{k-1}[0,l] : \|f\|_{k,2} < \infty\}/\underset{\text{a.e.}}{\sim}$$

which means for any $u \in W^{k,2}[0,l]$ there is $\hat{u} \in C^{k-1}[0,l]$ so that $\|u - \hat{u}\|_{k,2} = 0$.

(ii) We have a natural embedding $C^{k+1}[0,l] \subset H^k[0,l]$.

(iii) If $u = \sum_{m=0}^{\infty} y_m u_m \in C^k[0,l]$ where $u_m = \frac{1}{\sqrt{l/2}} \sin \frac{m\pi s}{l}$, then $|y_m| \leq M_k/m^k$ where $M_k = \sup\{|u^{(k)}(s)| : s \in [0,l]\}$.

(iv) If u is of C^2-class, then we have $\|u\|_\infty = \sup\{|u(s)|\} \leq \sum_{m=1}^{\infty} \frac{|y_m|}{\sqrt{l/2}} \leq \sum_{m=1}^{\infty} \frac{M_2}{m^2\sqrt{l/2}} < \infty$.

2.2 Versal Unfolding

[Golubitsky and Schaeffer, 1985]

Definition 2.2.1. An unfolding G of g is versal if every other unfolding of g factors through G. A versal unfolding of g depending on the minimum number of parameters is often called miniversal. That minimum number is called the codimension of g.

Example 2.2.2. *$G_\alpha(x) = x^3 - \lambda + \alpha x$ is a versal unfolding of the pitchfork. Here the diagram of versal imperfections is a line.*

FIGURE 2.1: Versal imperfections diagram for $G_\alpha(x)$

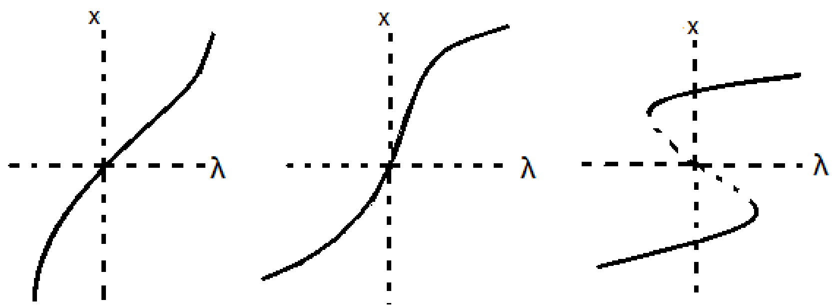


FIGURE 2.2: Associated diagrams of (2.1).

Theorem 2.2.3 (Golubitsky and Schaeffer, 1985). *(Versal Unfolding Theorem) Let $\mathcal{E}_{x,\lambda}$ denote the space of all functions and let g be a germ in $\mathcal{E}_{x,\lambda}$, and let G be a k-parameter unfolding of g. Then G is a versal unfolding of g if and only if*

$$\mathcal{E}_{x,\lambda} = T(g) + \left\langle \frac{\partial G}{\partial \alpha_1}(x,\lambda,0), \cdots, \frac{\partial G}{\partial \alpha_k}(x,\lambda,0) \right\rangle_{\mathbb{R}},$$

where $T(g)$ is tangent space to a germ g in $\mathcal{E}_{x,\lambda}$, consists of all germs of the form $ag + bg_x + cg_\lambda$, $a, b \in \mathcal{E}_{x,\lambda}$ and $c \in \mathcal{E}_\lambda$.

Corollary 1. *A versal unfolding G of a germ g is versal if and only if the number of parameters in G equals the codimension of $T(g)$.*

2.3 $\mathcal{P}$-$\mathcal{K}$ equivalent

[Marsden and Hughes, 1994, (Definition:1.12, p:442)]. Let f_1 and f_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent (contact equivalent) at $(0,0)$ if there is a (local) diffeomorphism of $\mathbb{R}^n \times \mathbb{R}$ to itself of the form $(x,\lambda) \mapsto (\varphi(x,\lambda), \Lambda(\lambda))$ such that $\varphi(0,0) = 0$, $\Lambda(0) = 0$, and a (smooth, local) map$(x,\lambda) \mapsto T(x,\lambda)$ from $\mathbb{R}^n \times \mathbb{R}$ to the invertible $m \times m$ matrices such that $f_1(x,\lambda) = T(x,\lambda) \cdot f_2(\varphi(x,\lambda), \Lambda(\lambda))$.

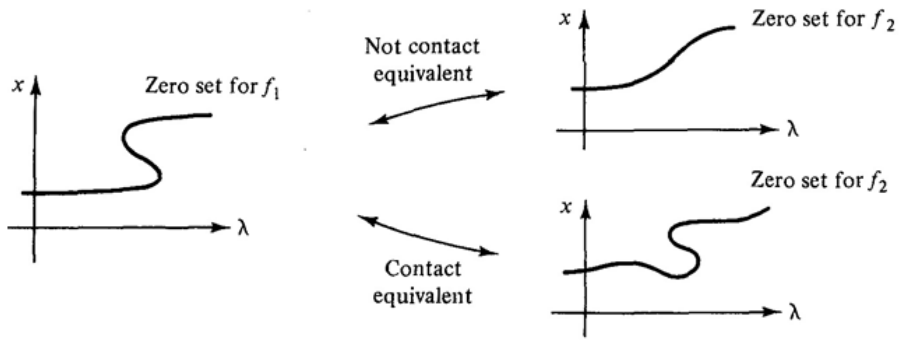


FIGURE 2.3: The zero set of f_1 and f_2

Proposition 2.3.1. *Let $H(x,\lambda)$ be a bifurcation problem [Golubitsky and Schaeffer, 1979, (Proposition 4.1, p:46)] satisfying $H(x,0) = x^m + \cdots$. Then H is $\mathcal{P}$-$\mathcal{K}$-equivalent to*
(I) $x^m \pm \lambda, m \geq 2$, if and only if $\frac{\partial H(0)}{\partial \lambda} \neq 0$, or to
(II) $x^m \pm \lambda x, m \geq 2$ if and only if $\frac{\partial H(0)}{\partial \lambda} = 0$, rank $(d^2H)(0) = 2$, and index $(d^2H)(0) = 1$.

2.4 Basic Bifurcation theorem

[Marsden and Hughes, 1994, (Theorem 1.5, p.434)]. Let $(0,\lambda_0)$ is a bifurcation point of the equation $f(x,\lambda) = 0$ if every neighborhood of $(0,\lambda_0)$ contains a solution (x,λ) with $x \neq 0$.

Here we discuss a basic bifurcation theorem for $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The following theorem interests for $x = 0$ is a trivial solution [$f(0,\lambda) = 0$ for all λ , so $(\partial f/\partial \lambda)(0,\lambda_0) = 0$], where f has some symmetry such as $f(x,\lambda) = -f(-x,\lambda)$, which forces $f_{xx}(0,\lambda) = 0$ and concerns the simplest case in which $(0,\lambda_0)$ could be a bifurcation point [so $(\partial f/\partial x)(0,\lambda_0)$ must vanish].

Theorem 2.4.1. *If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth mapping and satisfy the following conditions:*

$$(i)\ \ f(x_0,\lambda_0) = 0,\ \ f_x(x_0,\lambda_0) = 0,\ \ f_\lambda(x_0,\lambda_0) = 0,\ \ and\ \ f_{xx}(x_0,\lambda_0) = 0,\ \ and$$
$$(ii)\ \ f_{xxx}(x_0,\lambda_0) \neq 0,\ \ and\ \ f_{x\lambda}(x_0,\lambda_0) \neq 0.$$

Then (x_0, λ_0) is a bifurcation point. In fact, there is a smooth change of coordinates in a neighborhood of (x_0, λ_0) of the form

$$x = \phi(\overline{x}, \lambda) \;\; with \;\; \phi(0, \lambda_0) = x_0$$

and a smooth now here zero function $T(\overline{x}, \lambda)$ with $T(0, \lambda_0) = +1$ such that

$$T(\overline{x}, \lambda) f(\phi(\overline{x}, \lambda), \lambda) = \overline{x}^3 \pm \lambda \overline{x}$$

with $\pm$ depending on the sign $[f_{x\lambda}(x_0, \lambda_0) \cdot f_{xxx}(x_0, \lambda_0)]$. Here $T(\overline{x}, \lambda) f(\phi(\overline{x}, \lambda), \lambda)$ and $\overline{x}^3 \pm \lambda \overline{x}$ are $\mathcal{P} - \mathcal{K}$-equivallent. In Figure (2.4) we see that sub critical branch in (a) is unstable, while the supercritical branch in (b) is stable.

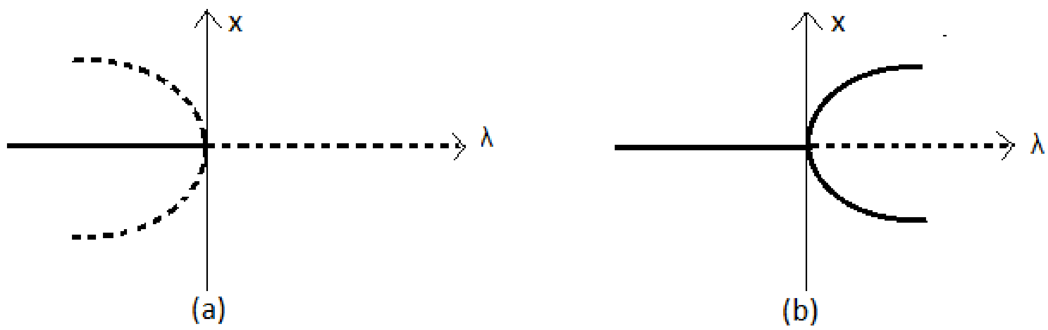


FIGURE 2.4: (a) Subcritical and (b) Supercritical bifurcation

2.5 Bifurcation Set and Hysteresis Set

Let $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ be versal unfolding of a germ $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.
A bifurcation point is a singular point of the bifurcation diagram.

$$B = \{\alpha | f_\alpha \;\text{ has a bifurcation point}\}$$

$$= \{\alpha \in \mathbb{R}^k : \exists (x, \lambda) \in \mathbb{R} \times \mathbb{R}, F(x, \lambda, \alpha) = 0, F_x(x, \lambda, \alpha) = 0, F_\lambda(x, \lambda, \alpha) = 0\}.$$

A hysteresis point is a point of the bifurcation diagram with vertical tangent.

$$H = \{\alpha | f_\alpha \;\text{ has a hysteresis point}\}$$

$$= \{\alpha \in \mathbb{R}^k : \exists (x, \lambda) \in \mathbb{R} \times \mathbb{R}, F(x, \lambda, \alpha) = 0, F_x(x, \lambda, \alpha) = 0, F_{xx}(x, \lambda, \alpha) = 0\}.$$

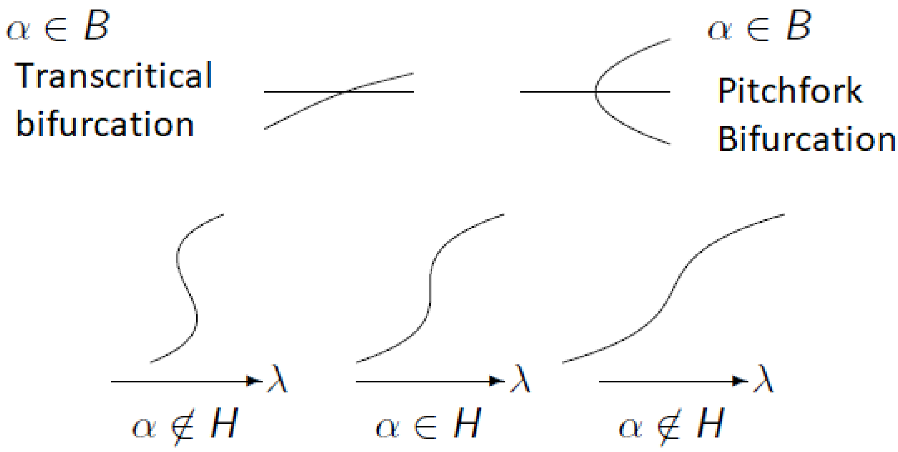


FIGURE 2.5: Bifurcation set and hysteresis set.

Chapter 3

Euler Buckling Problem

In this chapter, we discuss some basic topics of Euler buckling problem with the energy formulation and calculus of variations.

3.1 Buckling of the rod with pinned ends

In 1757, L. Euler found the critical load of this system, and it is often called the Euler buckling problem. We consider buckling of a rod which is subjected to compressed force λ.

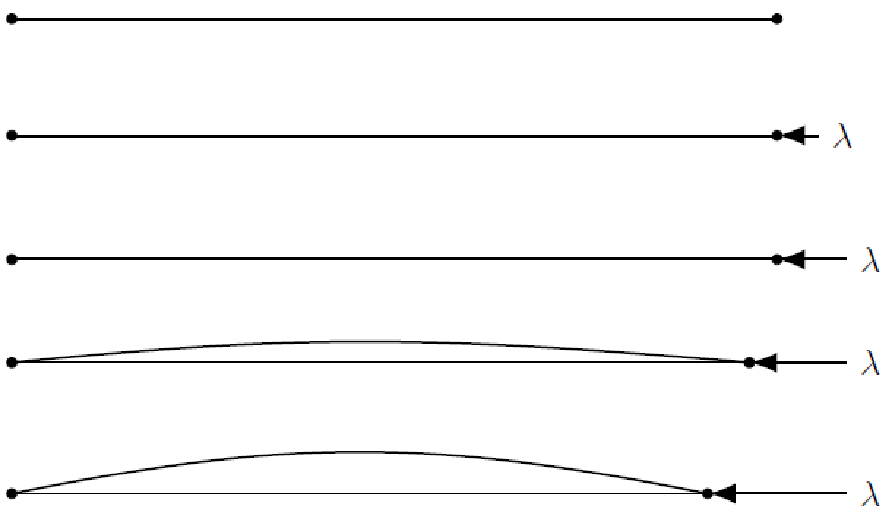


FIGURE 3.1: Buckling of the rod with pinned ends.

L. Euler found the critical load λ^* of the system, where l is the length of the rod. When $\lambda < \lambda^*$, nothing happen, when $\lambda > \lambda^*$ we have buckling. At $\lambda = \lambda^*$, we have bifurcation which is a famous example of pitchfork bifurcation.
We consider a model which neglects the compressibility of the beam and retains only its bending rigidity. If the length of the rod is l, these hypotheses lead to a variational problem posed in the Sobolev space

$$\chi = \{u \in H_2(0,l) : u(0) = u(l) = 0\}.$$

Here $H_2(0,l)$ consists of those functions in $L^2(0,l)$ whose second-order distributional derivatives also belong to $L^2(0,l)$.

3.2 Curvature

Curvature describes the shape of a curve. The curvature of the curve is the rate of change in the angle of direction of the tangent lines with respect to the arc length.

Lemma 3.2.1. *The Curvature κ of the curve $s \mapsto (x(s), u(s))$ is given by $\kappa = \frac{u''}{\sqrt{1-u'^2}}$.*

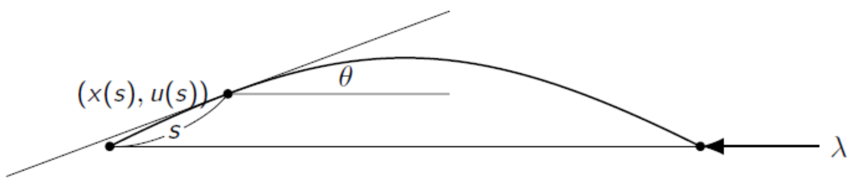


FIGURE 3.2: Curvature of the curve

Proof. Remember that the arc length of the curve isgiven by

$$\int ds \text{ where } ds = \sqrt{1 + \left(\frac{du}{dx}\right)^2} dx.$$

Since

$$u' = \frac{du}{ds} = \frac{\frac{du}{dx}}{\sqrt{1+\left(\frac{du}{dx}\right)^2}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \sin\theta,$$

and

$$\left(1-u'^2\right)^{\frac{1}{2}} = \left[1-\left(\frac{du}{ds}\right)^2\right]^{\frac{1}{2}} = \left[1-\sin^2\theta\right]^{\frac{1}{2}} = \cos\theta,$$

we obtain

$$\kappa = \frac{d\theta}{ds} = \left(\sin^{-1} u'\right)' = \frac{u''}{(1-u'^2)^{1/2}}.$$

□

3.3 Energy formulation of Euler buckling problem

Let $u(s)$ the deflection of the beam which is perpendicular to a reference line as a function of arc length along the beam.
The curvature of an element is given by

$$\kappa = \frac{d}{ds}\sin^{-1} u' = u''(1 - u'^2)^{-\frac{1}{2}}.$$

We have the strain energy

$$S = \frac{1}{2}\int_0^l \kappa^2 ds = \frac{1}{2}\int_0^l u''^2(1 - u'^2)^{-1} ds.$$

Then the potential energy of the system is given by

$$T = \int_0^l (1 - u'^2)^{1/2} ds.$$

We have the total energy E is

$$E = S + \lambda T = \frac{1}{2}\int_0^l \left[\frac{u''}{(1 - u'^2)^{1/2}}\right]^2 ds + \lambda \int_0^l \sqrt{1 - (u')^2} ds$$

on $U = \{u \in X : \|u'\|_\infty \leq 1 - \epsilon\}$, $0 < \epsilon \ll 1$, where X is the Sobolev space

$$X = \{u \in H^2[0, l] : u(0) = u(l) = 0\}.$$

Now we have the strain energy functional for our considered problem where α_1 represents a (constant) initial curvature of the beam

$$S = \frac{1}{2}\int_0^l (\kappa - \alpha_1)^2 ds = \frac{1}{2}\int_0^l \left[\frac{u''}{(1 - u'^2)^{1/2}} - \alpha_1\right]^2 ds.$$

Here λ is the compressive force and the deflection of the beam is the potential energy of the beam

We consider α_1 is a (constant) initial curvature of the beam and α_2 is a central load for the perturbed energy functional of this idealized problem. Here we set

$$V = u(\frac{l}{2}).$$

Now the total potential energy of the Euler buckling problem

$$E = S + \lambda T + \alpha_2 V = \frac{1}{2}\int_0^l (\kappa - \alpha_1)^2 ds + \lambda \int_0^l (1 - u'^2)^{1/2} ds + \alpha_2 u(\frac{l}{2}).$$

Therefore, the perturbed energy functional becomes

$$E(u,\lambda,\alpha) = \frac{1}{2}\int_0^l \left[\frac{u''}{(1-u'^2)^{\frac{1}{2}}} - \alpha_1\right]^2 ds + \lambda\int_0^l \left(1-u'^2\right)^{\frac{1}{2}} ds + \alpha_2 u\left(\frac{l}{2}\right). \tag{3.3.1}$$

Which is the modified Euler buckling problem of M. Golubitsky and D. Schaeffer's treatment. When the rod is perfectly straight in its unstressed position for $\alpha_1 = \alpha_2 = 0$ and not subjected to any external force other than the compressive force λ appearing in the second term in (3.3.1).

For $\alpha = 0$ exhibits a supercritical bifurcation at $\lambda = n^2\pi^2/l^2$ from the trivial solution $u = 0$ by the minimizing idealized problem. We want to prove that the two parameters α_1 and α_2 provide a universal unfolding of the idealized problem.
We consider the initial curvature is not constant for minimizing the strain energy we consider the strain energy as follows:

$$S = \frac{1}{2}\int_0^l \Big(\frac{u''}{(1-(u')^2)^{1/2}} - \alpha_1\kappa\Big)^2 ds,$$

where κ is a function defined by

$$\kappa = \frac{1}{\sqrt{l/2}}\Big[a_0 + \sum_{i=1}^{\infty}\Big(a_i\cos\frac{2i\pi s}{l} + b_i\sin\frac{2i\pi s}{l}\Big)\Big].$$

3.4 Calculus of variations

We consider the problem of minimizing the functional

$$E : U \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad u \mapsto E(u,\lambda,\alpha), \quad \alpha = (\alpha_1,\alpha_2)$$

defined by

$$E(u,\lambda,\alpha) = \frac{1}{2}\int_0^l \Big(\frac{u''}{(1-(u')^2)^{1/2}} - \alpha_1\kappa\Big)^2 ds + \lambda\int_0^l \sqrt{1-(u')^2} ds + \alpha_2 u(l/2), \tag{3.4.1}$$

where $U = \{u \in X : \|u'\|_\infty < 1-\epsilon\}$, $0 < \epsilon \ll 1$, and X is the Sobolev space

$$X = \{u \in H^2[0,l] : u(0) = u(l) = 0\}.$$

Where κ is a function defined by

$$\kappa = \frac{1}{\sqrt{l/2}}\Big[a_0 + \sum_{i=1}^{\infty}\Big(a_i\cos\frac{2i\pi s}{l} + b_i\sin\frac{2i\pi s}{l}\Big)\Big] \tag{3.4.2}$$

with $\|\kappa_0\|_\infty < \infty$. Since $u \in H^2[0,l]$, we can choose that u is C^1, and there is a constant ε_1, $0 < \varepsilon_1 << 1$, so that $|u'(s)| < 1 - \varepsilon_1$.

We consider the directional derivatives of E at u defined by

$$\Phi_{(u,\lambda,\alpha)} \cdot \phi = (dE)_{(u,\lambda,\alpha)} \cdot \phi = \lim_{t\to 0} \tfrac{1}{t}(E(u+t\phi,\lambda,\alpha) - E(u,\lambda,\alpha)).$$

Lemma 3.4.1.

$$\Phi(u,\lambda,\alpha) \cdot \phi = (dE)_{(u,\lambda,\alpha)} \cdot \phi = ((\Psi)_u - \lambda(\Lambda)_u) \cdot \phi - \alpha_1 (K)_u \cdot \phi + \alpha_2 \phi(\tfrac{l}{2})$$

where

$$\begin{aligned}
(\Psi)_u \cdot \phi &= \int_0^l \Big(\frac{u''\phi''}{(1-(u')^2)} + \frac{u'(u'')^2\phi'}{(1-(u')^2)^2} \Big) ds, \\
(\Lambda)_u \cdot \phi &= \int_0^l \frac{u'\phi'}{(1-(u')^2)^{\frac{1}{2}}} ds, \\
(K)_u \cdot \phi &= \int_0^l \kappa \Big(\frac{\phi''}{(1-(u')^2)^{1/2}} + \frac{u'u''\phi'}{(1-(u')^2)^{3/2}} \Big) ds.
\end{aligned}$$

Proof. $(dE)_{(u,\lambda,\alpha)} \cdot \phi$

$$\begin{aligned}
&= \int_0^l \Big(\frac{u''}{(1-(u')^2)^{1/2}} - \alpha_1 \kappa \Big) \Big(\frac{\phi'}{(1-(u')^2)^{1/2}} \Big)' ds - \lambda \int_0^l \frac{u'\phi'}{(1-(u')^2)^{\frac{1}{2}}} ds + \alpha_2 \phi(\frac{l}{2}) \\
&= \int_0^l \Big(\frac{u''}{(1-(u')^2)^{1/2}} - \alpha_1 \kappa \Big) \Big(\frac{\phi''}{(1-(u')^2)^{1/2}} + \frac{u'u''\phi'}{(1-(u')^2)^{3/2}} \Big) ds \\
&\quad - \lambda \int_0^l \frac{u'\phi'}{(1-(u')^2)^{\frac{1}{2}}} ds + \alpha_2 \phi(\frac{l}{2}) \\
&= \int_0^l \Big(\frac{u''\phi''}{1-(u')^2} + \frac{u'(u'')^2\phi'}{(1-(u')^2)^2} \Big) ds - \lambda \int_0^l \frac{u'\phi'}{(1-(u')^2)^{\frac{1}{2}}} ds \\
&\quad - \alpha_1 \int_0^l \kappa \Big(\frac{\phi''}{(1-(u')^2)^{1/2}} + \frac{u'u''\phi'}{(1-(u')^2)^{3/2}} \Big) ds + \alpha_2 \phi(\tfrac{l}{2}) \\
&= ((\Psi)_u - \lambda(\Lambda)_u) \cdot \phi - \alpha_1 (K)_u \cdot \phi + \alpha_2 \phi(\tfrac{l}{2}).
\end{aligned}$$

□

Consider the map $\Phi = dE : X \times \mathbb{R} \times \mathbb{R}^2 \to X'$ defined by

$$(u,\lambda,\alpha) \mapsto [\phi \mapsto (dE)_{(u,\lambda,\alpha)} \cdot \phi].$$

Chapter 4

Smoothness of Φ

4.1 Differentiability of Φ

In this chapter, we prove that Φ is continuously differentiable. We know a function is locally Lipschitz continuous if it is C^1. So the result in the following section implies Φ is locally Lipschitz continuous.
We have the modified Euler buckling problem with the first variational formulation

$$\Phi(u,\lambda,\alpha)\cdot\phi = (dE)_{(u,\lambda,\alpha)}\cdot\phi = ((\Psi)_u - \lambda(\Lambda)_u)\cdot\phi - \alpha_1(K)_u\cdot\phi + \alpha_2\phi(\tfrac{l}{2}),$$

where $\Phi = dE : X\times\mathbb{R}\times\mathbb{R}^2 \to X'$. We first show

Lemma 4.1.1. *The image of Φ is in X'.*

Proof. Since

$$\begin{aligned}
|(\Psi)_u\cdot\phi| &\leq \int_0^l \Big(\big|\tfrac{u''\phi''}{1-(u')^2}\big| + \big|\tfrac{u'(u'')^2\phi'}{(1-(u')^2)^2}\big|\Big)ds\\
&\leq \big\|\tfrac{1}{1-(u')^2}\big\|_\infty \|u''\phi''\|_1 + \big\|\tfrac{u'\phi'}{(1-(u')^2)^2}\big\|_\infty \|(u'')^2\|_1\\
&\leq \big\|\tfrac{1}{1-(u')^2}\big\|_\infty \|u''\|_2\|\phi''\|_2 + \big\|\tfrac{u'}{(1-(u')^2)^2}\big\|_\infty \|u''\|_2^2\|\phi'\|_\infty, \qquad (4.1.1)\\
|(\Lambda)_u\cdot\phi| &\leq \int_0^l \big|\tfrac{u'\phi'}{(1-(u')^2)^{1/2}}\big| ds \leq \big\|\tfrac{u'}{(1-(u')^2)^{1/2}}\big\|_2 \|\phi'\|_2, \qquad (4.1.2)\\
|(K)_u\cdot\phi| &\leq \int_0^l \Big(\big|\tfrac{\kappa\phi''}{(1-(u')^2)^{1/2}}\big| + \big|\tfrac{\kappa u'u''\phi'}{(1-(u')^2)^{3/2}}\big|\Big)ds\\
&\leq \big\|\tfrac{\kappa}{(1-(u')^2)^{1/2}}\big\|_\infty \|\phi''\|_1 + \big\|\tfrac{\kappa u'\phi'}{(1-(u')^2)^{3/2}}\big\|_2 \|u''\|_2\\
&\leq l^{1/2}\big\|\tfrac{\kappa}{(1-(u')^2)^{1/2}}\big\|_\infty \|\phi''\|_2 + \big\|\tfrac{\kappa u'}{(1-(u')^2)^{3/2}}\big\|_\infty \|\phi'\|_2\|u''\|_2, \qquad (4.1.3)\\
|\phi(l/2)| &\leq \|\phi\|_\infty \leq C_2\|\phi\|_{1,2} \quad \text{(since } \phi \text{ can be choosen continuously)}, \qquad (4.1.4)
\end{aligned}$$

there is a positive constant C (may depend on u) such that

$$|\Phi_{(u,\lambda,\alpha)}\cdot\phi| \leq C\|\phi\|_{2,2}. \qquad \square$$

Next we observe that

$$((\Psi)_{u_1}-(\Psi)_{u_2})\cdot\phi=\int_0^l\begin{pmatrix}[\frac{w_2'w_2''}{2(1-(w_2'/2)^2)}w_1'+\frac{w_1''}{1-(w_2'/2)^2}w_1''+o(w_1')]\phi''\\+[\frac{(1+\frac{3}{4}w_2')((w_1'')^2+(w_2'')^2)}{4(1-(w_2'/2)^2)}w_1'+\frac{w_2'w_2''}{1-(w_2'/2)^2}w_1''+o(w_1')]\phi'\end{pmatrix}ds$$

where $w_1=u_1-u_2$ and $w_2=u_1+u_2$. Remark that $|w_2'|\le|u_1'|+|u_2'|<2(1-\varepsilon_1)$. Then there is a positive constant C_Ψ such that

$$\begin{aligned}|((\Psi)_{u_1}-(\Psi)_{u_2})\cdot\phi|\le&\int_0^l C_\Psi(|w_1'\phi''|+|w_1''\phi''|+|w_1'\phi'|+|w_1''\phi'|)ds\\\le&C_\Psi(\|w_1'\|_2+\|w_1''\|_2)\|\phi''\|_2+(\|w_1'\|_2+\|w_1''\|_2)\|\phi'\|_2\\\le&C_\Psi(\|w_1'\|_2+\|w_1''\|_2)(\|\phi'\|_2+\|\phi''\|_2).\end{aligned}\tag{4.1.5}$$

Similarly, we have

$$\begin{aligned}[(\Lambda)_{u_1}-(\Lambda)_{u_2}]\cdot\phi=&\int_0^l\left[\frac{u_1'}{(1-(u_1')^2)^{1/2}}-\frac{u_2'}{(1-(u_2')^2)^{1/2}}\right]\phi'ds\\=&\int_0^l\Big((1-(w_2'/2)^2)^{-\frac{3}{2}}+O((w_1')^2)\Big)w_1'\phi'ds,\end{aligned}$$

and thus conclude that there is a positive constant C_Λ so that

$$|[(\Lambda)_{u_1}-(\Lambda)_{u_2}]\cdot\phi|\le\int_0^l C_\Lambda|w_1'\phi'|ds\le C_\Lambda\|w_1'\|_2\|\phi'\|_2\tag{4.1.6}$$

Similarly we obtain that $(K)_{u_1}-(K)_{u_2}$

$$=\int_0^l\kappa\begin{pmatrix}[w_2'(1-(w_2'/2)^2)^{-\frac{3}{2}}+O(w_1')]w_1'\phi''\\+[w_2'(1-(\frac{w_2'}{2})^2)^{-\frac{3}{2}}w_1''+(\frac{1}{2}+(\frac{w_2'}{2})^2)(1-(\frac{w_2'}{2})^2)^{-\frac{5}{2}}w_2''w_1'+O(w_1')^2]\phi'\end{pmatrix}ds$$

and thus there is a positive constant C_K so that

$$|[(K)_{u_1}-(K)_{u_2}]\cdot\phi|\le\int_0^l C_K(|w_1'\phi''|+|w_1''\phi'|)ds\le C_K(\|w_1'\|_2\|\phi''\|_2+\|w_1''\|_2\|\phi'\|_2).\tag{4.1.7}$$

We thus conclude that

$$\begin{aligned}&|[(\Phi)_{(u_1,\lambda_1,\alpha)}-(\Phi)_{(u_2,\lambda_2,\beta)}]\cdot\phi|\\\le&|[(\Phi)_{(u_1,\lambda_1,\alpha)}-(\Phi)_{(u_2,\lambda_1,\beta)}]\cdot\phi|+|[(\Phi)_{(u_2,\lambda_1,\beta)}-(\Phi)_{(u_2,\lambda_2,\beta)}]\cdot\phi|\\\le&|[(\Psi)_{u_1}-(\Psi)_{u_2}]\cdot\phi|+|\lambda_1||[(\Lambda)_{u_1}-(\Lambda)_{u_2}]\cdot\phi|+|t_1||[(K)_{u_1}-(K)_{u_2}]\cdot\phi|\\&+|\lambda_1-\lambda_2||(\Lambda)_{u_2}\cdot\phi|+|\alpha_1-\beta_1||(K)_{u_2}\cdot\phi|+|\alpha_2-\beta_2||\phi(l/2)|\\\le&(C_\Psi+C_\Lambda+C_K)\|u_1-u_2\|_{2,2}\|\phi\|_{2,2}+|\lambda_1-\lambda_2|\Big\|\frac{u'}{(1-(u')^2)^{1/2}}\Big\|_\infty\|\phi'\|_1\\&+|\alpha_1-\beta_1|\Big(\Big\|\frac{\kappa}{(1-(u')^2)^{1/2}}\Big\|_\infty\|\phi''\|_1+\Big\|\frac{\kappa u'u''}{(1-(u')^2)^{3/2}}\Big\|_\infty\|\phi'\|_1\Big)+|\alpha_2-\beta_2|C_2\|\phi\|_{1,2}.\end{aligned}$$

This complete the proof.

4.2 Φ is C^1

Lemma 4.2.1. *Φ is C^1.*

Proof. By (4.1.6), (4.1.7), the following partial derivatives are continuous on u:

$$(\tfrac{\partial}{\partial\lambda}\Phi)_{(u,\lambda,\alpha)} = (\Lambda)_u : \mathbb{R} \to X', \qquad (\tfrac{\partial}{\partial\alpha_1}\Phi)_{(u,\lambda,\alpha)} = (K)_u : \mathbb{R} \to X'.$$

Since $(\frac{\partial}{\partial\alpha_2}\Phi)_{(u,\lambda,\alpha)} = \delta_{l/2} : \mathbb{R} \to X'$, it is continuous (constant) on u.

We consider the derivative of Ψ by $v \in X$:

$$X \longrightarrow X', \qquad v \mapsto (D_v\Phi)_{(u,\lambda,\alpha)} = \lim_{t\to 0}\frac{1}{t}[(\Phi)_{(u+tv,\lambda,\alpha)} - (\Phi)_{(u,\lambda,\alpha)}].$$

Using the first derivatives of Ψ, Λ and K, we can express this map by $v \mapsto (\Psi_1)_u[v] - \lambda(\Lambda_1)_u[v] - \alpha_1(K_1)_u[v]$. The first order differentials of Ψ, Λ, K are expressed as

$$\begin{aligned}
(\Psi_1)_u[v]\cdot\phi &= \int_0^l \big(\tfrac{2u'u''}{(1-(u')^2)^2}v'\phi'' + \tfrac{1}{1-(u')^2}v''\phi'' + \tfrac{(1+3(u')^2)(u'')^2}{(1-(u')^2)^3}v'\phi' + \tfrac{2u'u''}{(1-(u')^2)^2}v''\phi'\big)ds,\\
(\Lambda_1)_u[v]\cdot\phi &= \int_0^l \frac{v'}{(1-(u')^2)^{\frac{3}{2}}}\phi' ds,\\
(K_1)_u[v]\cdot\phi &= \int_0^l \kappa\Big(\tfrac{v'\phi''}{(1-(u')^2)^{3/2}} + \big(\tfrac{1+2(u')^2}{(1-(u')^2)^{5/2}}u''v' + \tfrac{u'v''}{(1-(u')^2)3/2}\big)\phi'\Big)ds,
\end{aligned}$$

respectively. We thus obtain that

$$\begin{aligned}
|(\Psi_1)_u[v]\cdot\phi| \leq & \|\tfrac{2u'u''}{(1-(u')^2)^2}\|_\infty\|v'\|_2\|\phi''\|_2 + \|\tfrac{1}{1-(u')^2}\|_\infty\|v''\|_2\|\phi''\|_2\\
& + \|\tfrac{(1+3(u')^2)(u'')^2}{(1-(u')^2)^3}\|_\infty\|v'\|_2\|\phi'\|_2 + \|\tfrac{2u'u''}{(1-(u')^2)^2}\|_\infty\|v''\|_2\|\phi'\|_2,\\
|(\Lambda_1)_u[v]\cdot\phi| \leq & \|\frac{1}{(1-(u')^2)^{\frac{1}{2}}}\|_\infty\|v'\|_2\|\phi'\|_2,\\
|(K_1)_u[v]\cdot\phi| \leq & \|\tfrac{\kappa}{(1-(u')^2)^{3/2}}\|_\infty\|v'\|_2\|\phi''\|_2 + \|\tfrac{\kappa(1+2(u')^2)}{(1-(u')^2)^{5/2}}\|_\infty\|v'\|_2\|\phi'\|_2\\
& + \|\tfrac{\kappa u'}{(1-(u')^2)3/2}\|_\infty\|v''\|_2\|\phi'\|_2,
\end{aligned}$$

Setting $w_1 = u_1 - u_2$ and $w_2 = u_1 + u_2$, we have

$$
\begin{aligned}
((\Psi_1)_{u_1}[v] - (\Psi_1)_{u_2}[v]) \cdot \phi &= \int_0^l \Big(\tfrac{(1+\frac{3}{4}(w_2')^2)w_2''}{(1-(w_2'/2)^2)^3} w_1' + \tfrac{w_2' w_1''}{(1-(w_2'/2)^2)^2} + O(w_1')^2 \Big) v' \phi'' ds, \\
&= \int_0^l \Big(\tfrac{w_2' w_1'}{(1-(w_2'/2)^2)^2} + O(w_1')^2 \Big) v'' \phi'' ds, \\
&= \int_0^l \Big(\tfrac{3}{2} \tfrac{w_2'(1+(w_1'/2)^2)((w_1'')^2+(w_2'')^2)}{(1-(w_2'/2)^2)^4} w_1' + \tfrac{(1+3(w_2'/2)^2)w_2'}{(1-(w_2'/2)^2)^3} w_1'' + O(w_1')^2 \Big) v' \phi' ds, \\
&= \int_0^l \Big(\tfrac{3}{2} \tfrac{(1+3(w_1'/2)^2)w_2''}{(1-(w_2'/2)^2)^3} w_1' + \tfrac{w_2'}{(1-(w_2'/2)^2)^2} w_1'' + O(w_1')^2 \Big) v'' \phi' ds, \\
((\Lambda_1)_{u_1}[v] - (\Lambda_1)_{u_2}[v]) \cdot \phi &= \int_0^l \tfrac{3}{2} \frac{v' w_1' w_2'}{(1-(w_1'/2)^2)^{\frac{5}{2}}} \phi' ds, \\
((K_1)_{u}[v] - (K_1)_{u_2}[v]) \cdot \phi &= \int_0^l \kappa \Big(\tfrac{1+(w_2')^2/2}{(1-(w_2'/2)^2)^{5/2}} w_1' + O(w_1')^2 \Big) v' \phi'' ds \\
+ \int_0^l \kappa \Big[\Big(\tfrac{3}{8} \tfrac{(6+(w_2')^2)w_2' w_2''}{(1-(w_2'/2)^2)^{7/2}} w_1' &+ \tfrac{1+(w_2')^2/2}{(1-(w_2'/2)^2)^{5/2}} w_1'' + O(w_1')^2 \Big) v' + \Big(\tfrac{(1+(w_2')/2)}{(1-(w_2'/2)^2)^{5/2}} w_1' + O(w_1')^2 \Big) v'' \Big] \phi' ds.
\end{aligned}
$$

Since each parénthesis in the integrands is continuous, for any positive number ε, there is a positive number δ such that

$$|(\text{each parénthesis in the expressions above})| \le \varepsilon \quad \text{whenever} \quad \|w_1\|_{2,2} < \delta.$$

We conclude that, if $\|w_1\|_{2,2} < \delta$, then

$$
\begin{aligned}
|((\Psi_1)_{u_1} - (\Psi_1)_{u_2})[v] \cdot \phi| &\le \varepsilon (\|v'\|_2 \|\phi''\|_2 + \|v''\|_2 \|\phi''\|_2 + \|v'\|_2 \|\phi'\|_2 + \|v''\|_2 \|\phi'\|_2), \\
|((\Lambda_1)_{u_1} - (\Lambda_1)_{u_2})[v] \cdot \phi| &\le \varepsilon \|v'\|_2 \|\phi'\|_2, \\
|((K_1)_{u_1} - (K_1)_{u_2})[v] \cdot \phi| &\le \varepsilon (\|v'\|_2 \|\phi''\|_2 + \|v'\|_2 \|\phi'\|_2 + \|v'\|_2 \|\phi'\|_2 + \|v''\|_2 \|\phi'\|_2),
\end{aligned}
$$

which show the assertion. □

Set $Z = \{u \in H^3[0, l] : u(0) = u(l) = 0\}$. We consider the restriction of Φ:

$$\Phi|_Z : (U \cap Z) \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow X',\ \Phi(u, \lambda, \alpha) = [\phi \mapsto (\Psi - \lambda\Lambda - \alpha_1 K)_u \cdot \phi + \alpha_2 \phi(\tfrac{l}{2})].$$

4.3 Φ is C^∞

Lemma 4.3.1. $\Phi|_Z$ *is* C^∞.

Proof. It is enough to show that Φ is C^k for any k. The k-th order differential of Ψ is of the form

$$\phi \mapsto (\Psi_k)_u[v_1, \dots, v_k] \cdot \phi = \int_0^l \sum_{i_1, \dots, i_k, j = 1}^{2} A^{\Psi}_{i_1, \dots, i_k, j}(u) v_1^{(i_1)} \cdots v_k^{(i_k)} \phi^{(j)} ds$$

where $A^{\Psi}_{i_1,\dots,i_k,j}(u)$ are rational functions of u' and u'' whose denominators are certain powers of $1-(u')^2$. Thus we obtain that

$$
\begin{aligned}
|(\Psi_k)_u[v_1,\dots,v_k]\cdot\phi| &\le \sum_{i_0,i_1,\dots,i_k=1}^{2} \|A^{\Psi}_{i_1,\cdots,i_k,j}(u)\|_\infty \int_0^l |v_1^{(i_1)}\cdots v_k^{(i_k)}\phi^{(j)}|ds \\
\le L_k^{\Psi}\sum_{i_1,\dots,i_k=1}^{2}\|v_1^{(i_1)}\cdots v_k^{(i_k)}\|_2\|\phi^{(j)}\|_2 &\le L_k^{\Psi} l^{\frac{1}{2}}(C_2)^k \sum_{i_1,\dots,i_k=1}^{2}\|v_1^{(i_1)}\|_{1,2}\cdots\|v_k^{(i_k)}\|_{1,2}\|\phi^{(j)}\|_2 \\
&\le L_k^{\Psi} l^{\frac{1}{2}}(C_2)^k\|v_1\|_{3,2}\cdots\|v_k\|_{3,2}\|\phi\|_{2,2}
\end{aligned}
$$

where $L_k^{\Psi}=\sup\{\|A^{\Psi}_{i_1,\dots,i_k,j}\|_\infty : i_1,\dots,i_k,j=1,2\}$.

Similarly we have

$$
\begin{aligned}
&\big((\Psi_k)_{u_1}[v_1,\dots,v_k]-(\Psi_k)_{u_2}[v_1,\dots,v_k]\big)\cdot\phi \\
&=\int_0^l \sum_{i_0,i_1,\dots,i_k,j=1}^{2} B^{\Psi}_{i_0,i_1,\dots,i_k,j}(u)(u_1-u_2)^{(i_0)}v_1^{(i_1)}\cdots v_k^{(i_k)}\phi^{(j)}ds
\end{aligned}
$$

where $B^{\Psi}_{i_0,i_1,\dots,i_k,j}(u)$ are meromorphic functions of u' and u'' whose denominators are certain powers of $1-(u')^2$. Therefore we conclude that

$$
\begin{aligned}
&\big|\big((\Psi_k)_{u_1}[v_1,\dots,v_k]-(\Psi_k)_{u_2}[v_1,\dots,v_k]\big)\cdot\phi\big| \\
&\le \sum_{i_0,i_1,\dots,i_k,j=1}^{2}\|B^{\Psi}_{i_0,i_1,\cdots,i_k,j}(u)\|_\infty\int_0^l |(u_1-u_2)^{(i_0)}v_1^{(i_1)}\cdots v_k^{(i_k)}\phi^{(j)}|ds \\
&\le M_k^{\Phi}\sum_{i_0,i_1,\dots,i_k,j=1}^{2}\|(u_1^{(i_0)}-u_2^{(i_0)})v_1^{(i_1)}\cdots v_k^{(i_k)}\|_2\|\phi^{(j)}\|_2 \\
&\le M_k^{\Phi} l^{\frac{1}{2}}(C_2)^{k+1}\sum_{i_0,i_1,\dots,i_k,j=1}^{2}\|u_1^{(i_0)}-u_2^{(i_0)}\|_{1,2}\|v_1^{(i_1)}\|_{1,2}\cdots\|v_k^{(i_k)}\|_{1,2}\|\phi^{(j)}\|_2 \\
&\le M_k^{\Phi} l^{\frac{1}{2}}(C_2)^{k+1}\|u_1-u_2\|_{3,2}\|v_1\|_{3,2}\cdots\|v_k\|_{3,2}\|\phi^{(j)}\|_{2,2}
\end{aligned}
$$

where $M_k^{\Psi}=\sup\{\|B^{\Psi}_{i_0,i_1,\dots,i_k,j}\|_\infty : i_0,i_1,\dots,i_k,j=1,2\}$.

The continuity of the higher order differentials of Φ containing differentiations by one of λ, α_1, α_2 can be shown similarly and we omit the details. □

Here we discuss several estimates before the proof of the theorem (4.3.7). Let

$$A(x)=(1-x^2)^{-\frac{d}{2}}\sum_{i=0}^{n}a_i x^i,\quad \text{where } a_i\in\mathbb{R}. \tag{4.3.1}$$

We also consider $|A|(x)=(1-x^2)^{-\frac{d}{2}}\sum_{i=0}^{n}|a_i|x^i$.

$$A(u')=(1-(u')^2)^{-\frac{d}{2}}\sum_{i=0}^{n}a_i(u')^i,\quad \text{where } a_i\in\mathbb{R}. \tag{4.3.2}$$

Lemma 4.3.2. *If $\|u'\|_\infty\le\varepsilon<1$, then $\|A(u')\|_\infty\le|A|(\varepsilon)$.*

Proof. The estimate is obtained by

$$\|A(u')\|_\infty \leq \left\|\frac{1}{1-(u')^2}\right\|_\infty^{\frac{d}{2}} \sum_{i=1}^{n} |a_i| \|u'\|_\infty^i \leq \frac{\sum_{i=0}^{n} |a_i| \varepsilon^i}{(1-\varepsilon^2)^{\frac{d}{2}}} = |A|(\varepsilon). \qquad \square$$

Lemma 4.3.3. *If* $\|u_i'\|_\infty \leq \varepsilon < 1$ *for* $i = 1, 2$, *then* $\|A(u_1') - A(u_2')\|_\infty \leq C(A, \varepsilon)\|u_1' - u_2'\|_\infty$ *where*

$$C(A,\varepsilon) = \frac{|A|(\varepsilon)}{(1-\varepsilon^2)^{\frac{d}{2}}}\left[\sum_{s:2s<\max\{i,k\}} \left|\binom{\frac{d}{2}}{s}\right| |2s-i| \varepsilon^{2s-1} + \frac{2}{(1-\varepsilon^2)^2}\right].$$

Proof. Since $\|A(u_1') - A(u_2')\|_\infty$

$$\leq \sum_{i=1}^{n} |a_i| \|(1-(u_2')^2)^{\frac{d}{2}}(u_1')^i - (1-(u_1')^2)^{\frac{d}{2}}(u_2')^i\|_\infty \left\|\frac{1}{1-(u_1')^2}\right\|_\infty^{\frac{d}{2}} \left\|\frac{1}{1-(u_2')^2}\right\|_\infty^{\frac{d}{2}}$$

$$\leq \sum_{i=1}^{n} |a_i| \left\|\sum_{s=0}^{\infty} \binom{\frac{d}{2}}{s} [(u_1')^i (u_2')^{2s} - (u_1')^{2s}(u_2')^i]\right\|_\infty \frac{1}{(1-\varepsilon^2)^d},$$

the following estimate gives the result.

$$\left\|\sum_{s=0}^{\infty} \binom{\frac{d}{2}}{s} [(u_1')^i (u_2')^{2s} - (u_1')^{2s}(u_2')^i]\right\|_\infty \leq \sum_{s=0}^{\infty} \left|\binom{\frac{d}{2}}{s}\right| \|(u_1')^i (u_2')^{2s} - (u_1')^{2s}(u_2')^i\|_\infty$$

$$= \sum_{2s<i} \left|\binom{\frac{d}{2}}{s}\right| \|(u_1')^i (u_2')^{2s} - (u_1')^{2s}(u_2')^i\|_\infty + \sum_{2s>i} \left|\binom{\frac{d}{2}}{s}\right| \|(u_1')^i (u_2')^{2s} - (u_1')^{2s}(u_2')^i\|_\infty$$

$$= \sum_{2s<i} \left|\binom{\frac{d}{2}}{s}\right| \|u_1' u_2'\|_\infty^{2s} \|(u_1')^{i-2s} - (u_2')^{i-2s}\|_\infty + \|u_1' u_2'\|_\infty^i \sum_{2s>i} \left|\binom{\frac{d}{2}}{s}\right| \|(u_1')^{2s-i} - (u_2')^{2s-i}\|_\infty$$

$$\begin{aligned} =& \|u_1' - u_2'\|_\infty \Bigg[\sum_{2s<i} \left|\binom{\frac{d}{2}}{s}\right| \|u_1'\|_\infty^{2s} \|u_2'\|_\infty^{2s} \sum_{p+q=i-2s-1} \|u_1'\|_\infty^p \|u_2'\|_\infty^q \\ &+ \|u_1'\|_\infty^i \|u_2'\|_\infty^i \sum_{2s>i} \left|\binom{\frac{d}{2}}{s}\right| \sum_{p+q=2s-i-1} \|u_1'\|_\infty^p \|u_2'\|_\infty^q \Bigg] \end{aligned}$$

$$= \|u_1' - u_2'\|_\infty \left[\sum_{2s<i} \left|\binom{\frac{d}{2}}{s}\right| \varepsilon^{4s} (i-2s) \varepsilon^{i-2s-1} + \varepsilon^{2i} \sum_{2s>i} \left|\binom{\frac{d}{2}}{s}\right| (2s-i) \varepsilon^{2s-i-1}\right]$$

$$\le \|u_1' - u_2'\|_\infty \varepsilon^i \left[\sum_{2s<i} \left| \binom{\frac{d}{2}}{s} \right| (i-2s)\varepsilon^{2s-1} + \sum_{2s>i} \left| \binom{\frac{d}{2}}{s} \right| (2s-i)\varepsilon^{2s-1} \right]$$
$$\le \|u_1' - u_2'\|_\infty \varepsilon^i \left[\sum_{2s<i} \left| \binom{\frac{d}{2}}{s} \right| (i-2s)\varepsilon^{2s-1} + \sum_{s:i<2s<d} \left| \binom{\frac{d}{2}}{s} \right| (2s-i)\varepsilon^{2s-1} + \sum_{2s>i, 2s\ge d} (2s-i)\varepsilon^{2s-1} \right]$$
$$\le \|u_1' - u_2'\|_\infty \varepsilon^i \left[\sum_{s:2s<\max\{i,d\}} \left| \binom{\frac{d}{2}}{s} \right| |2s-i|\varepsilon^{2s-1} + \frac{2}{(1-\varepsilon^2)^2} \right].$$

For the last inequality, we use the following inequality:

$$\sum_{2s>i, 2s\ge d} (2s-i)\varepsilon^{2s-1} = \left\{ \begin{matrix} \frac{\varepsilon^i(1+\varepsilon^2)}{(1-\varepsilon^2)^2} & (i:\text{odd}) \\ \frac{2\varepsilon^{i+1}}{(1-\varepsilon^2)^2} & (i:\text{even}) \end{matrix} \right\} \le \frac{2\varepsilon^i}{(1-\varepsilon^2)^2}. \qquad \square$$

We condsider a k-linear form $X \times \cdots \times X \to \mathbb{R}$, $(v_1, \dots, v_k) \mapsto I(u)[v_1, \dots, v_k]$, defined by

$$I(u)[v_1, \dots, v_k] = \int_0^l A(u')\,(u'')^j v_1^{(i_1)} \cdots v_k^{(i_k)} ds$$

where $A(u')$ is given by (4.3.2), and $i_1, \dots, i_k = 1, 2$.

Lemma 4.3.4. *If $j + i_1 + \cdots + i_k \le k+2$, then there is a positive constant C such that*

$$|I(u)[v_1, \dots, v_k]| \le C\|A(u')\|_\infty \|u\|_{2,2}^j \|v_1\|_{2,2} \cdots \|v_k\|_{2,2}. \tag{4.3.3}$$

Proof. If $i_1, i_2 = 1, 2$, then

$$\begin{aligned} \left| \int_0^l A(u') f_1^{(i_1)} f_2^{(i_2)} f_3' \cdots f_k'\, ds \right| &\le \|f_1^{(i_1)}\|_2 \|A(u') f_2^{(i_2)} f_3' \cdots f_k'\|_2 \\ &\le \|f_1^{(i_1)}\|_2 \|f_2^{(i_2)}\|_2 \|A(u') f_3' \cdots f_k'\|_\infty \\ &\le \|f_1^{(i_1)}\|_2 \|f_2^{(i_2)}\|_2 \|A(u')\|_\infty \|f_3'\|_\infty \cdots \|f_k'\|_\infty \\ &\le C_2^{k-2} \|f_1^{(i_1)}\|_2 \|f_2^{(i_2)}\|_2 \|A(u')\|_\infty \|f_3'\|_{1,2} \cdots \|f_k'\|_{1,2} \\ &\le C_2^{k-2} \|A(u')\|_\infty \|f_1\|_{2,2} \cdots \|f_k\|_{2,2}. \end{aligned}$$

If $j + i_1 + \cdots + i_k \le k+2$, then the number $\#\{a : i_a = 2\}$ is at most $2-j$, and we complete the proof by the estimate above. $\square$

Then we have

$$|I(u)[v_1, \dots, v_k]| \lesssim \|A(u')\|_\infty \|u\|_{3,2}^j \|v_1\|_{3,2} \cdots \|v_k\|_{3,2},$$

where $j + i_1 + \cdots + i_k \geq k + 2$. We obtain that

$$\begin{aligned}\left|\int_0^l A(u', u'') f_1^{(i_1)} \cdots f_k^{(i_k)}\, ds\right| \leq& \|A(u', u'')\|_\infty \|f_1^{(i_1)} \cdots f_k^{(i_k)}\|_2 \\ \lesssim& \|A(u', u'')\|_\infty \|f_1^{(i_1)} \cdots f_k^{(i_k)}\|_\infty \\ \leq& \|A(u', u'')\|_\infty \|f_1^{(i_1)}\|_\infty \cdots \|f_k^{(i_k)}\|_\infty \\ \leq& \|A(u', u'')\|_\infty \|f_1^{(i_1)}\|_{1,2} \cdots \|f_k^{(i_k)}\|_{1,2} \\ \leq& \|A(u', u'')\|_\infty \|f_1\|_{3,2} \cdots \|f_k\|_{3,2},\end{aligned}$$

Remark 4.3.5. *Set $B(x) = x^j$. Since $D_v(A(u')B(u'')) = A'_u[v']B(u'') + A(u')B'_u[v'']$, where D_v denotes the directional derivative with direction v,*

$$\begin{aligned}&\lim_{t \to 0} \frac{1}{t}(A(u' + tv')B(u'' + tu'') - A(u')B(u'') \\ =&\lim_{t \to 0} \frac{1}{t}((A(u' + tv') - A(u'))B(u'' + tu'') + A(u')(B(u'' + tv'') - B(u''))) \\ =&A'_u[v']B(u'') + A(u')B'_u[v''].\end{aligned}$$

We obtain that

$$I_1(u)[v_1, \ldots, v_k, v] = \lim_{t \to 0} \frac{1}{t}(I(u + tv)[v_1, \ldots, v_k] - I(u)[v_1, \ldots, v_k])$$

is a $(k + 1)$-linear form, which is a linear combination of integrals of the type

$$\int_0^l A(u')(u'')^{j'} v_1^{(i'_1)} \cdots v_k^{(i'_k)} v^{(i'_{k+1})} ds \quad (j', i'_1, \ldots, i'_{k+1} = 1, 2)$$

with $j' + i'_1 + \cdots + i'_k + i'_{k+1} \leq k + 3$, whenever $j + i_1 + \cdots + i_k \leq k + 2$. We thus obtain that

$$|I_1(u)[v_1, \ldots, v_k, v]| \leq C\|v_1\|_{2,2} \cdots \|v_k\|_{2,2}\|v\|_{2,2}$$

where C is a constant (depending on only u), by Lemma 4.3.4.

Lemma 4.3.6. *If $j + i_1 + \cdots + i_k \leq k + 2$, then*

$$|I(u_1)[v_1, \ldots, v_k] - I(u_2)[v_1, \ldots, v_k]| \leq C\|u_1 - u_2\|_{2,2}\|v_1\|_{2,2} \cdots \|v_k\|_{2,2}$$

for some constant C.

Proof. If $j = 2$, then we can assume that $i_1 = \cdots = i_k = 1$, and we obtain

$$\begin{aligned}\text{LHS} =&\left|\int_0^l ([A(u'_1) - A(u'_2)](u''_2)^2 + A(u'_1)[(u''_1)^2 - (u''_2)^2])v'_1 \cdots v'_k ds\right| \\ \leq& \int_0^l \left|([A(u'_1) - A(u'_2)](u''_2)^2 + A(u'_1)(u''_1 - u''_2)(u''_1 + u''_2))v'_1 \cdots v'_k\right| ds \\ \leq& C_2^k(\|A(u'_1) - A(u'_2)\|_\infty \|u_2\|_{2,2}^2 + \|A(u'_1)\|_\infty \|u_1 - u_2\|_{2,2}\|u_1 + u_2\|_{2,2})\|v_1\|_{2,2} \cdots \|v_k\|_{2,2},\end{aligned}$$

by Lemma 4.3.4. So the result follows by Lemmas 4.3.2 and 4.3.3.

If $j = 1$, then we can assume that $i_2 = \cdots = i_k = 1$, and we obtain

$$\begin{aligned}\mathrm{LHS} =& \left|\int_0^l ([A(u_1') - A(u_2')]u_2'' + A(u_1')[u_1'' - u_2''])v_1^{(i_1)}v_2' \cdots v_k' ds\right| \\ \leq& C_2^{k-1}(\|A(u_1') - A(u_2')\|_\infty \|u_2\|_{2,2} + \|A(u_1')\|_\infty \|u_1 - u_2\|_{2,2})\|v_1\|_{2,2} \cdots \|v_k\|_{2,2},\end{aligned}$$

by Lemma 4.3.4. So the result follows by Lemmas 4.3.2 and 4.3.3.

If $j = 0$, then we can assume that $i_3 = \cdots = i_k = 1$, and we obtain

$$\mathrm{LHS} = \left|\int_0^l (A(u_1') - A(u_2'))v_1^{(i_1)} \cdots v_k^{(i_k)} ds\right| \leq C_2^{k-2}\|A(u_1') - A(u_2')\|_\infty \|v_1\|_{2,2} \cdots \|v_k\|_{2,2},$$

by Lemma 4.3.4. So the result follows by Lemma 4.3.3. □

Our main theorem 4.3.7 is as follows:

Theorem 4.3.7. *Φ is C^∞.*

Proof. As in Remark 4.3.5, we see that the k-th order differential of Φ by u is of the form

$$\phi \mapsto (\Phi_k)_u[v_1, \ldots, v_k] \cdot \phi = \int_0^l \sum_{j=0}^{2} \sum_{i_1,\ldots,i_{k+1}=1}^{2} A^\Phi_{j,i_1,\ldots,i_{k+1}}(u')(u'')^j v_1^{(i_1)} \cdots v_k^{(i_k)} \phi^{(i_{k+1})} ds$$

where $A^\Phi_{j,i_1,\ldots,i_k,i_{k+1}}(u)$ are of the form (4.3.1) so that $j + i_1 + \cdots + i_k + i_{k+1} \leq k + 3$. As Lemma 4.3.6, this is continuous.

The continuity of the higher order differentials of Φ containing differentiations by one of λ, α_1, α_2 can be shown similarly and we omit the details. □

Remark 4.3.8. *In Lemma 4.3.4, it is important to assume that $j + i_1 + \cdots + i_k \leq k + 2$. We do not know that (4.3.3) holds true when $j + i_1 + \cdots + i_k > k + 2$, despite of the fact that the inequality changing $\|\cdot\|_{2,2}$ to $\|\cdot\|_{3,2}$ holds true.*

Chapter 5

Taylor coefficients of Φ

In this chapter we write the Taylor expansion of Φ then we compute some derivatives of $(L)_u$ up-to third orders and derivatives of the coefficients of the initial curvature.

5.1 Taylor expansion of Φ

We present the first few terms of Taylor expansions of $(\Psi)_u$, $(\Lambda)_u$, $(K)_u$ at $u = 0$. They look like

$$
\begin{aligned}
&\tfrac{\pi^4}{l^4}(y_1u_1^* + 2^4y_2u_2^* + 3^4y_3u_3^* + 4^4y_4u_4^* + \cdots)\\
&+\tfrac{\pi^6}{l^7}[y_1^3 + 20y_1y_2^2 + 9y_3(y_1^2 + 6y_2^2 + 10y_1y_3) + 8y_4(11y_1y_2 + 45y_2y_3 + 34y_1y_4) + \cdots]u_1^*\\
&+\tfrac{\pi^6}{l^7}[4y_2(5y_1^2 + 16y_2^2) + 36y_2y_3(3y_1 + 13y_3) + 4y_4(11y_1^2 + 90y_1y_3 + 171y_3^2 + 320y_2y_4) + \cdots]u_2^*\\
&+\tfrac{\pi^6}{l^7}[3y_1^3 + 54y_1y_2^2 + 9y_3(10y_1^2 + 52y_2^2 + 81y_3^2) + 72y_4(5y_1y_2 + 19y_2y_3 + 50y_3y_4) + \cdots]u_3^*\\
&+\tfrac{\pi^6}{l^7}[44y_1^2y_2 + 36y_2y_3(10y_1 + 19y_3) + 16y_4(17y_1^2 + 80y_2^2 + 225y_3^2 + 256y_4^2) + \cdots]u_4^* + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
&\tfrac{\pi^2}{l^2}(y_1u_1^* + 2^2y_2u_2^* + 3^2y_3u_3^* + 4^2y_4u_4^* + \cdots)\\
&+\tfrac{\pi^4}{l^5}[\tfrac{3}{4}y_1^3 + 6y_1y_2^2 + \tfrac{9}{4}y_3(y_1^2 + 4y_2^2 + 6y_1y_3) + 12y_4(y_1y_2 + 3y_2y_3 + 2y_1y_4) + \cdots]u_1^*\\
&+\tfrac{\pi^4}{l^5}[6y_2(y_1^2 + 2y_2^2) + 18y_2y_3(y_1 + 3y_3) + 6y_4(y_1^2 + 6y_1y_3 + 9y_3^2 + 16y_2y_4) + \cdots]u_2^*\\
&+\tfrac{\pi^4}{l^5}[\tfrac{3}{4}y_1^3 + 9y_1y_2^2 + \tfrac{27}{4}y_3(2y_1^2 + 8y_2^2 + 9y_3^2) + 36y_4(y_1y_2 + 3y_2y_3 + 6y_3y_4) + \cdots]u_3^*\\
&+\tfrac{\pi^4}{l^5}[6y_1^2y_2 + 18y_2y_3(2y_1 + 3y_3) + 24y_4(y_1^2 + 4y_2^2 + 9y_3^2 + 8y_4^2) + \cdots]u_4^* + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{l^2}{\pi}\begin{pmatrix}(-2u_1^* - 6u_3^* - 10u_5^* + \cdots)\sqrt{2}a_0 + (\tfrac{4}{3}u_1^* - \tfrac{108}{5}u_3^* - \tfrac{500}{21}u_5^* + \cdots)a_1\\ +(\tfrac{4}{15}u_1^* + \tfrac{108}{7}u_3^* - \tfrac{500}{9}u_5^* + \cdots)a_2 + \cdots\end{pmatrix}\\
&+\frac{\sqrt{2}\pi^3}{l^5}\begin{pmatrix}(-2y_1^2 - 8y_2^2 - 6y_3(2y_1 + 3y_3) - 32y_4(y_2 + y_4) + \cdots)u_1^*\\ +(-16y_1y_2 - 48y_2y_3 - 32(y_1 + 3y_3)y_4 + \cdots)u_2^*\\ +(-6y_1^2 - 24y_2^2 - 18y_3(2y_1 + 3y_3) - 96y_4(y_2 + y_4) + \cdots)u_3^*\\ +(-32y_1y_2 - 96y_2y_3 - 64(y_1 + 3y_3)y_4 + \cdots)u_4^* + \cdots\end{pmatrix}a_0
\end{aligned}
$$

$$+\frac{\pi^3}{l^5}\begin{pmatrix} (-\frac{4}{5}y_1^2-\frac{976}{105}y_2^2-\frac{4}{35}y_3(234y_1+127y_3)-\frac{64y_4}{3465}(3245y_2+1277y_4)+\cdots)u_1^* \\ +(-\frac{1952}{105}y_1y_2-\frac{1568}{15}y_2y_3-\frac{64}{3465}(3245y_1+9639y_3)y_4+\cdots)u_2^* \\ +(-\frac{468}{35}y_1^2-\frac{784}{15}y_2^2-\frac{4}{385}y_3(2794y_1+16767y_3)-\frac{64}{15015}y_4(41769y_2+65593y_4)+\cdots)u_3^* \\ +(-\frac{3776}{63}y_1y_2-\frac{9792}{55}y_2y_3-\frac{128}{45045}(16601y_1+196779y_3)y_4+\cdots)u_4^*+\cdots \end{pmatrix}a_1$$
$$+\frac{\pi^3}{l^5}\begin{pmatrix} (\frac{52}{35}y_1^2-\frac{1712}{315}y_2^2-\frac{4}{1155}y_3(242y_1+10731y_3)-\frac{64}{45045}y_4(32773y_2+29253y_4)+\cdots)u_1^* \\ +(-\frac{3424}{315}y_1y_2+\frac{10656}{385}y_2y_3-\frac{64}{45045}(32773y_1+90255y_3)y_4+\cdots)u_2^* \\ +(-\frac{44}{105}y_1^2+\frac{5328}{385}y_2^2-\frac{4}{5005}(93002y_1-88209y_3)y_3-\frac{64}{1365}y_4(2735y_2+1391y_4)+\cdots)u_3^* \\ +(-\frac{161344}{3465}y_1y_2-\frac{35008}{273}y_2y_3-\frac{128}{15015}(9751y_1+15301y_3)y_4+\cdots)u_4^*+\cdots \end{pmatrix}a_2+\cdots,$$

respectively.
Now we denote by $(\Phi_k)_u$ the k-th order differential coefficient of $(\Phi)_u$ at u, and by Φ_k the k-th order differential of Φ at $u=0$. We denote by $(L_k)_u$ the k-th order differential coefficient of $(L)_u$ at u, and by L_k the k-th order differential of L at $u=0$, and so on.

5.2 First order derivative of $(L)_u$

Lemma 5.2.1. *Set $(L)_u=(\Psi)_u-\lambda(\Lambda)_u$. The first derivative of $(L)_u$ at $u=0$ is given by*

$$L_1[v]\cdot\phi=\int_0^l(v''+\lambda v)\phi''ds.$$

Proof.

$$\begin{aligned}L_1[v]\cdot\phi=&(\Psi_1[v]-\lambda\Lambda_1[v])\cdot\phi=\int_0^l v''\phi''ds-\lambda\int_0^l v'\phi'ds\\ =&\int_0^l v''\phi''ds-\Big[v\phi'\Big]_0^l+\lambda\int_0^l v\phi''ds=\int_0^l(v''+\lambda v)\phi''ds.\end{aligned}$$ □

Set $u_m=\sqrt{2/l}\sin(m\pi s/l)$ and u_m^* is an element in X' defined by $u_m^*(u_j)=\delta_{m,j}$. If $\lambda^*=n^2\pi^2/l^2$, then

$$L_1[u_m]=\frac{\pi^4}{l^4}m^2(m^2-n^2)u_m^*,$$
$$L_1^{-1}[u_m^*]=\frac{l^4}{\pi^4}\frac{1}{m^2(m^2-n^2)}u_m^*\quad(m\neq n).$$

5.3 Derivatives of the coefficient of α_1

Lemma 5.3.1. *When* $\kappa = \frac{1}{\sqrt{l/2}}[a_0 + \sum_{i=1}^{\infty}\big(a_i\cos(2i\pi s/l) + b_i\sin(2i\pi s/l)\big)]$, *we have*

$$K_0 = -\frac{\pi^2}{l^2}\Big[\sum_{m:\,odd}\frac{4}{\pi}\sum_{i=0}^{\infty}\frac{m^3 a_i}{m^2-4i^2}u_m^* + \sum_{m:\,even} m^2 b_{m/2}u_m^*\Big], \quad K_1[u_a] = 0 \quad \textit{and}$$

$$K_2[u_a,u_b] = -\frac{ab\pi^3}{l^5}\sum_{i=0}^{\infty}\Big(a_i\sum_{m\not\equiv a+b(2)} m\Big(\sum_{\varepsilon_1,\varepsilon_2=\pm1}\frac{(\varepsilon_1 a+\varepsilon_2 b+m)^2}{(\varepsilon_1 a+\varepsilon_2 b+m)^2-4i^2}\Big)u_m^* + b_i\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm1}\varepsilon_3 i(\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i)u^*_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i}\Big).$$

Proof. For K_0, we have

$$K_0 = \sum_{m=1}^{\infty}\Big(\int_0^l \kappa u_m'' ds\Big)u_m^* = \sum_m\Big(\int_0^l \frac{a_0+\sum_{i=1}^{\infty}[a_i\cos(2i\pi s/l)+b_i\sin(2i\pi s/l)]}{\sqrt{l/2}}u_m''ds\Big)u_m^*$$
$$= -\frac{\pi^2}{l^2}\Big[\sum_{m\equiv1(2)}\frac{4}{\pi}\sum_{i=0}^{\infty}\frac{m^3a_i}{m^2-4i^2}u_m^* + \sum_{m:\,\mathrm{even}} m^2 b_{m/2}u_m^*\Big].$$

We remark that the second order differential of K at $u=0$ is given by $K_2[v_1,v_2]:\phi\mapsto\int_0^l\kappa(v_1'v_2'\phi''+(v_1'v_2''+v_1''v_2')\phi')ds$, that is,

$$K_2[v_1,v_2] = \sum_m\Big(\int_0^l\kappa(v_1'v_2'u_m''+(v_1'v_2''+v_1''v_2')u_m')ds\Big)u_m^*.$$

This implies that, if $\kappa=\frac{1}{\sqrt{l/2}}$, then $K_2[u_a,u_b] = -\frac{8ab\pi^3}{l^5}\sum_{m\not\equiv a+b(2)} mu_m^*$; if $\kappa = \frac{\cos\frac{2is\pi}{l}}{\sqrt{l/2}}$, then

$$K_2[u_a,u_b] = \sum_m\Big(\int_0^l\frac{\cos\frac{2is\pi}{l}}{\sqrt{l/2}}(u_a'u_b'u_m''+(u_a'u_b''+u_a''u_b')u_m')ds\Big)u_m^*$$
$$= \frac{ab\pi^3}{2l^5}\Big(\sum_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i+m\neq0}((-1)^{a+b+m}-1)m\frac{\varepsilon_1 a+\varepsilon_2 b+m}{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i+m}\Big)u_m^*$$
$$= -\frac{ab\pi^3}{l^5}\sum_{m\not\equiv a+b(2)} m\Big(\sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm1}\frac{\varepsilon_1 a+\varepsilon_2 b+m}{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i+m}\Big)u_m^*$$
$$= -\frac{ab\pi^3}{l^5}\sum_{m\not\equiv a+b(2)} m\Big(\sum_{\varepsilon_1,\varepsilon_2=\pm1}\frac{(\varepsilon_1 a+\varepsilon_2 b+m)^2}{(\varepsilon_1 a+\varepsilon_2 b+m)^2-4i^2}\Big)u_m^*;$$

and if $\kappa = \frac{\sin\frac{2is\pi}{l}}{\sqrt{l/2}}$, then

$$\begin{aligned}
K_2[u_a,u_b] &= \sum_m \Big(\int_0^l \frac{\sin\frac{2is\pi}{l}}{\sqrt{l/2}} (u_a'u_b'u_m'' + (u_a'u_b'' + u_a''u_b')u_m')ds \Big) u_m^* \\
&= -\frac{ab\pi^3}{2l^5} \sum_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i+m=0} \varepsilon_3 m(\varepsilon_1 a+\varepsilon_2 b+m)u_m^* \\
&= -\frac{ab\pi^3}{2l^5} \sum_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i+m=0} \varepsilon_3(\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i)2iu^*_{-\varepsilon_1 a-\varepsilon_2 b-2\varepsilon_3 i} \\
&= -\frac{ab\pi^3}{l^5} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm 1} \varepsilon_3 i(\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i)u^*_{\varepsilon_1 a+\varepsilon_2 b+2\varepsilon_3 i}.
\end{aligned}$$

These equations follow from computing several definite integrals of products of trigonometric functions. One can prove them applying product-sum formulas repeatedly. One can also check them by using a computer algebra system like Mathematica. □

Since $K_2[u_n,u_n] = \sum_m u_m^* \int_0^l \kappa\,[u_n'u_n'u_m'' + (u_n'u_n'' + u_n''u_n')u_m']\,ds$, we have

$$\begin{aligned}
K_2[u_n,u_n]\cdot u_n &= \int_0^l \kappa \Big[u_n'u_n'u_n'' + (u_n'u_n'' + u_n''u_n')u_n'\Big]ds = 3\int_0^l \kappa(u_n')^2 u_n'' ds \\
&= \begin{cases} -\frac{3\pi^3}{l^5}\sum_{i=0}^\infty \frac{8n^5(3n^2-4i^2)}{(n^2-4i^2)(9n^2-4i^2)}a_i, & n: \text{odd}, \\ -\frac{3n^4\pi^4}{2l^5}(b_{n/2}+b_{3n/2}), & n: \text{even}. \end{cases}
\end{aligned}$$

5.4 Second order derivative of $(L)_u$

Lemma 5.4.1. *The second derivative of $(L)_u$ is*

$$(L_2)_u[v_1,v_2]\cdot\phi = \int_0^l [2u'v_2'v_1''\phi'' + 2u''\phi''v_2'v_1' + 2u'v_1'v_2''\phi'' + 2u''v_1''v_2'\phi' + 2v_2''v_1''\phi'u' + 2u''v_2''v_1'\phi']ds.$$

So, setting $u=0$, we obtain $L_2[v_1,v_2]\cdot\phi = 0$.

5.5 Third order derivative of $(L)_u$

For the third order differential coefficient L_3 of L at $u=0$, we have

$$L_3[v_1,v_2,v_3] = \sum_m \left[\int_0^l \begin{pmatrix} 2(v_1''v_2'v_3' + v_1'v_2''v_3' + v_1'v_2'v_3'')u_m'' \\ +(2(v_1''v_2''v_3' + v_1''v_2'v_3'' + v_1'v_2''v_3'') - 3\lambda v_1'v_2'v_3')u_m' \end{pmatrix} ds \right] u_m^*.$$

Lemma 5.5.1. *We have*

$$L_3[u_a,u_b,u_c]\cdot u_n = \frac{abcn^2\pi^5}{l^7} \sum_{\varepsilon_1 a+\varepsilon_2 b+\varepsilon_3 c=n} \left[1 - \frac{3n}{2} + abcn\varepsilon_1\varepsilon_2\varepsilon_3\Big(\frac{\varepsilon_1}{a}+\frac{\varepsilon_2}{b}+\frac{\varepsilon_3}{c}\Big)\right]$$

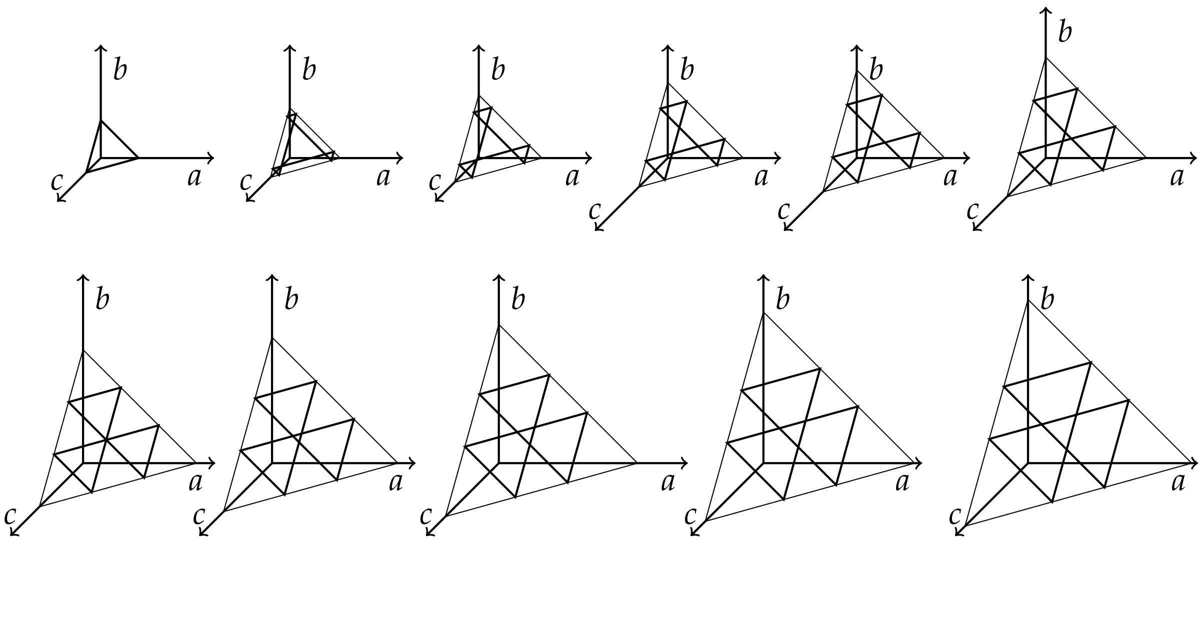
where $\varepsilon_i = \pm 1$, $i = 1, 2, 3$.

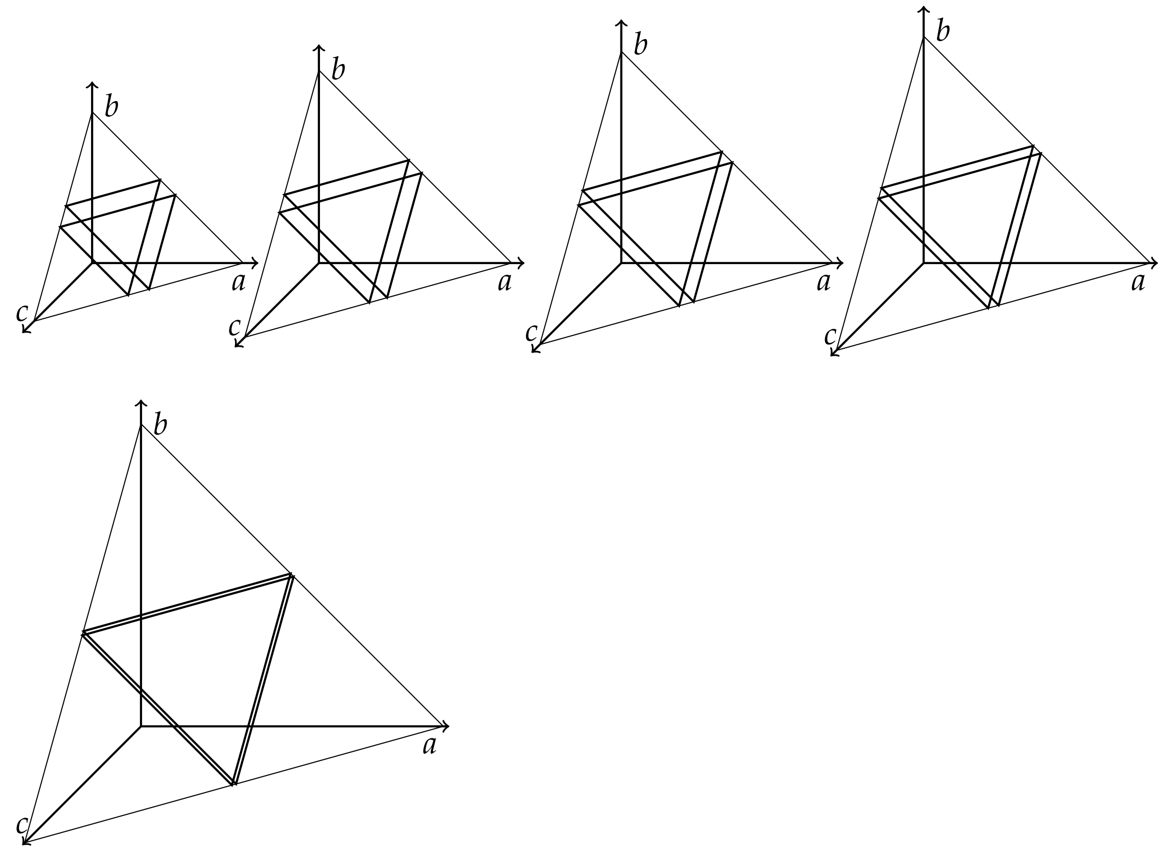
Proof. Computing several integrals, we have

$$\begin{aligned}
L_3[u_a, u_b, u_c] \cdot u_n =& 2\frac{abcn^2\pi^5}{2l^7} \#\{(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{-1, 1\}^3 : \varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n\} \\
&+ 2\frac{a^2b^2c^2n\pi^5}{2l^7} \sum_{(\varepsilon_1,\varepsilon_2,\varepsilon_3):\varepsilon_1 a+\varepsilon_2 b+\varepsilon_3 c=n} \varepsilon_1\varepsilon_2\varepsilon_3\Big(\frac{\varepsilon_1}{a} + \frac{\varepsilon_2}{b} + \frac{\varepsilon_3}{c}\Big) \\
&- 3\lambda^* \frac{abcn\pi^3}{2l^5} \#\{(\varepsilon_1, \varepsilon_2, \varepsilon_3) : \varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n\} \\
=& \frac{abcn^2\pi^5(2-3n)}{2l^7} \#\{(\varepsilon_1, \varepsilon_2, \varepsilon_3) : \varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n\} \\
&+ \frac{2a^2b^2c^2n^3\pi^5}{2l^7} \sum_{\varepsilon_1 a+\varepsilon_2 b+\varepsilon_3 c=n} \varepsilon_1\varepsilon_2\varepsilon_3\Big(\frac{\varepsilon_1}{a} + \frac{\varepsilon_2}{b} + \frac{\varepsilon_3}{c}\Big) \\
=& \frac{abcn^2\pi^5}{l^7} \sum_{\varepsilon_1 a+\varepsilon_2 b+\varepsilon_3 c=n} \Big[1 - \frac{3n}{2} + abcn\varepsilon_1\varepsilon_2\varepsilon_3\Big(\frac{\varepsilon_1}{a} + \frac{\varepsilon_2}{b} + \frac{\varepsilon_3}{c}\Big)\Big].
\end{aligned}$$

□

We present pictures of the sets
$$\Big\{(a,b,c) : \begin{matrix} a+b+c=m \\ a+b-c=\pm n \end{matrix}\Big\} = \{(\tfrac{m\pm n}{2}, 0, \tfrac{m\mp n}{2}), (\tfrac{m\pm n}{2}-1, 1, \tfrac{m\mp n}{2}), \dots, (0, \tfrac{m\pm n}{2}, \tfrac{m\mp n}{2})\}$$
below.





Lemma 5.5.2. *We have* $PL_3[u_n, u_n, v] = \frac{3n^6\pi^6}{2l^7}(u_n + 9u_{3n}) \cdot v.$

Proof. $PL_3[u_n, u_n, v]$

$$
\begin{aligned}
&= \int_0^l [2(u_n' u_n' v'' + 2u_n' u_n'' v')u_n'' + (2(u_n'' u_n'' v' + 2u_n' u_n'' v'') - 3\lambda^* u_n' u_n' v')u_n']ds \\
&=3 \int_0^l [2(u_n')^2 u_n'' v'' + (2u_n'(u_n'')^2 - \lambda^*(u_n')^3)v']ds \\
&=\frac{6n^6\pi^6}{2l^7}(u_n + 9u_{3n}) \cdot v + \frac{6n^6\pi^6}{2l^7}(u_n - 3u_{3n}) \cdot v - 3\lambda^* \frac{3n^4\pi^4}{2l^5}(u_n + u_{3n}) \cdot v \\
&=\frac{3n^6\pi^6}{2l^7}(u_n + 9u_{3n}) \cdot v.
\end{aligned}
$$
□

Since $u_m = (\pi^2 m^2 / l^2)\Delta^{-1} u_m$,

$$
L_3[\Delta^{-1}u_a, \Delta^{-1}u_b, \Delta^{-1}u_c] \cdot u_n = \frac{n^2}{abcl\pi} \sum_{\varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n} \left[1 - \frac{3n}{2} - abcn\varepsilon_1\varepsilon_2\varepsilon_3\left(\frac{\varepsilon_1}{a} + \frac{\varepsilon_2}{b} + \frac{\varepsilon_3}{c}\right)\right]
$$

$$
\Delta^{-1}L_3[\Delta^{-1}u_a, \Delta^{-1}u_b, \Delta^{-1}u_c] = -\frac{l}{\pi^3} \sum_m \sum_{\varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n} \left[\frac{3n-2}{2abc} + n\varepsilon_1\varepsilon_2\varepsilon_3\left(\frac{\varepsilon_1}{a} + \frac{\varepsilon_2}{b} + \frac{\varepsilon_3}{c}\right)\right] u_m^*
$$

If m is big, then we have
$\#\{(a, b, c) : \varepsilon_1 a + \varepsilon_2 b + \varepsilon_3 c = n\} \simeq 3m.$

Chapter 6

Lyapunov-Schmidt reduction

In this chapter we discuss about Lyapunov-Schmidt reduction and find the related derivatives up-to order 3. For discussing the bifurcation we need to consider the Lyapunov-Schmidt reduction. Lyapunov-Schmidt reduction is an effective procedure to reduce an infinite dimensional bifurcation problem to a finite dimensional problem.

The basic idea is decompose the given equation into two equivalent. When $\lambda^* = (n\pi/l)^2$, $u_n = \sqrt{2/l}\sin(n\pi s/l)$ is a non-zero function which generates the kernel of $L_1 = \Psi_1 - \lambda^*\Lambda_1$. Thus the orthogonal projection of X to $\ker L_1$ is

$$P : X \longrightarrow X, \quad u \mapsto \frac{\langle u, u_n\rangle_2}{\langle u_n, u_n\rangle_2} u_n, \quad \text{and} \quad Q : X \longrightarrow X, \quad u \mapsto u - P(u),$$

is the orthogonal projection of X to $(\ker L_1)^\perp$, the orthogonal complement to $\ker L_1$.

The equation $\Phi(u, \lambda, \alpha) = 0$ is equivalent that

$$P\Phi(u, \lambda, \alpha) = 0, \text{ and } Q\Phi(u, \lambda, \alpha) = 0.$$

Observe that the differential map $(\ker L_1)^\perp \to (\ker L_1)^\perp$, $v \mapsto D_v Q\Phi$, at $(0, \lambda^*, 0)$ is given by

$$v \mapsto \left[\phi \mapsto \int_0^l (v'' + \lambda^* v)\phi'' ds\right],$$

which is an isomorphism.

Lemma 6.0.1. *The differential map (or derivative) map (1.1.2):*

$$X \to X', \quad v \mapsto \left[\phi \mapsto \int_0^l (v'' + \lambda v)\phi'' ds\right], \tag{6.0.1}$$

induces an isomorphism from $(\ker L_1)^\perp$ *to the annihilator of* $\ker L_1$ *in* X'.

Proof. By the discussion in §2, the map (6.0.1) is C^1. If $\int_0^l (v'' + \lambda^* v)\phi'' ds = 0$ for any $\phi \in X$, then $v'' + \lambda^* v = 0$ and we obtain that $v = 0$ since λ^* is not an eigenvalue of the restriction of Laplacian to $(\text{Ker } L_1)^\perp$. This shows the map (**??**) is injective. We show it is surjective. Take $v^* \in X'$ arbitrary. By Riesz representation theorem, there is $w \in X$ so that $\langle w, \phi\rangle_2 = v^*(\phi)$ for any $\phi \in X$.

Setting $v = \sum_{m \neq n} a_m u_m$ where $\frac{\pi^2 m^2}{l^2}(\frac{\pi^2 m^2}{l^2} - \lambda^*) a_m = \langle w, u_m \rangle_2$, we have

$$\int_0^l (v'' + \lambda^* v)\phi''\, ds = \langle w, \phi \rangle_2 = v^*(\phi).$$

Since $\langle v, v \rangle_2 = \sum_{m \neq n} \frac{(1+\frac{\pi^2 m^2}{l^2})^2 \langle w, u_m \rangle_2{}^2}{\frac{\pi^4 m^4}{l^4}(\frac{\pi^2 m^2}{l^2} - \lambda^*)^2} < \infty$, we have $v \in X$. This completes the proof. □

By implicit function theorem [Ralph Abraham and Tudor S., 1988, p. 2.5.7], the later equation defines a function

$$W : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to (\ker L_1)^{\perp} \subset X, \qquad (x, \lambda, \alpha) \mapsto W(x, \lambda, \alpha)$$

by

$$Q\Phi(xu_n + W(x, \lambda, \alpha), \lambda, \alpha) = 0 \quad \text{near } (0, \lambda^*, 0). \tag{6.0.2}$$

Lyapunov-Schmidt reduction says that the bifurcation of zero of $\Phi(u, \lambda, \alpha)$ is described by the zero of $F(x, \lambda, \alpha)$ where

$$F(x, \lambda, \alpha) = P\Phi(xu_n + W(x, \lambda, \alpha), \lambda, \alpha). \tag{6.0.3}$$

6.1 The first order derivatives of W

Lemma 6.1.1. *The differential coefficients of W at $(0, \lambda^*, 0)$ are given as follows: $\bar{W}_x = 0$, $\bar{W}_\lambda = 0$,*

$$\bar{W}_1 = -\frac{l^2}{\pi^2}\Big[\frac{4}{\pi} \sum_{\substack{m:\, odd \\ m \neq n}} \frac{m}{m^2 - n^2} \sum_{i=0}^{\infty} \frac{a_i}{m^2 - 4i^2} u_m^* + \sum_{\substack{m:\, even \\ m \neq n}} \frac{b_{m/2}}{m^2 - n^2} u_m^*\Big], \quad \textit{and}$$

$$\bar{W}_2 = -\frac{l^4}{\pi^4} \frac{1}{\sqrt{l/2}} \sum_{\substack{m:\, odd \\ m \neq n}} \frac{1}{m^2(m^2 - n^2)} u_m^*.$$

Here we put a bar above a function to indicate evaluation at $(0, \lambda^*, 0)$. We also have that

$$\begin{aligned}\bar{W}_1 \alpha_1 + \bar{W}_2 \alpha_2 = &-\frac{l^2}{\pi^2} \sum_{\substack{m:\, odd \\ m \neq n}} \frac{1}{m^2 - n^2}\Big(\frac{4\alpha_1}{\pi} \sum_{i=0}^{\infty} \frac{m a_i}{m^2 - 4i^2} + \frac{l^2}{\pi^2} \frac{\alpha_2}{m^2 \sqrt{l/2}}\Big) u_m^* \\ &- \frac{l^2}{\pi^2} \sum_{\substack{m:\, even \\ m \neq n}} \frac{b_{m/2}}{m^2 - n^2} u_m^*.\end{aligned}$$

Proof. We remark that

$$Q\Phi(u, \lambda, \alpha) = Q(L)_u - \alpha_1 Q(K)_u + \alpha_2 Q\delta = 0, \quad Q(L)_u = Q(\Psi)_u - \lambda Q(\Lambda)_u \tag{6.1.1}$$

where $u = xu_n + W(x, \lambda, \alpha)$. Differentiating (6.0.2) with respect to x, λ, α_1, α_2, we obtain that

$$Q(L_1)_u[u_n + W_x] - \alpha_1 Q(K_1)_u[u_n + W_x] = 0 \tag{6.1.2}$$
$$Q(L_1)_u[W_\lambda] - \alpha_1 Q(K_1)_u[W_\lambda] - Q\Lambda(u) = 0 \tag{6.1.3}$$
$$Q(L_1)_u[W_1] - \alpha_1 Q(K_1)_u[W_1] - Q(K)_u = 0 \tag{6.1.4}$$
$$Q(L_1)_u[W_2] - \alpha_1 Q(K_1)_u[W_2] + Q\delta = 0 \tag{6.1.5}$$

where $u = xu_n + W(x, \lambda, \alpha)$. We denote W_i for W_{α_i}, for shortness. We evaluate them at $(0, \lambda^*, 0)$ and obtain

$$L_1[\bar{W}_x] = 0, \quad L_1[\bar{W}_\lambda] = 0, \quad L_1[\bar{W}_1] = QK_0, \quad L_1[\bar{W}_2] = -Q\delta.$$

Since $L_1 u_n = 0$. Thus we obtain that $\bar{W}_x = 0$, $\bar{W}_\lambda = 0$, $\bar{W}_1 = L_1^{-1} QK_0$, and $\bar{W}_2 = -L_1^{-1} Q\delta$, which conclude the results. □

6.2 The second order derivatives of W

By differentiating (6.1.2) with respect to x, λ, α_1, α_2, we obtain

$$\begin{aligned}
Q(L_2)_u[u_n + W_x, u_n + W_x] + Q(L_1)_u[W_{xx}] - Q(K_2)_u[u_n + W_x, u_n + W_x] \\
-Q(K_1)_u[W_{xx}] = 0, \\
Q(L_2)_u[u_n + W_x, W_\lambda] + Q(L_1)_u[W_{x\lambda}] - Q(K_2)_u[u_n + W_x, W_\lambda] - Q(K_1)_u[W_{x\lambda}] \\
-Q(\Lambda_1)_u[u_n + W_x] = 0, \\
Q(L_2)_u[u_n + W_x, W_1] + Q(L_1)_u[W_{x1}] - Q(K_2)_u[u_n + W_x, W_1] - Q(K_1)_u[W_{x1}] \\
-Q(K_1)_u[u_n + W_x] = 0, \\
Q(L_2)_u[u_n + W_x, W_2] + Q(L_1)_u[W_{x2}] - Q(K_2)_u[u_n + W_x, W_2] - Q(K_1)_u[W_{x2}] = 0,
\end{aligned}$$

and, by evaluating them at $(0, \lambda^*, 0)$, we conclude

$$QL_1[\bar{W}_{xx}] = 0, \quad QL_1[\bar{W}_{x\lambda}] = Q\Lambda_1[u_n] = 0, \quad QL_1[\bar{W}_{x1}] = 0, \quad QL_1[\bar{W}_{x2}] = 0.$$

By differentiating (6.1.3) with respect to λ, α_1, α_2, we obatin

$$\begin{aligned}
Q(L_2)_u[W_\lambda, W_\lambda] + Q(L_1)_u[W_{\lambda\lambda}] - Q(K_2)_u[W_\lambda, W_\lambda] - Q(K_1)_u[W_{\lambda\lambda}] = Q(\Lambda_1)_u[W_\lambda], \\
Q(L_2)_u[W_\lambda, W_1] + Q(L_1)_u[W_{\lambda 1}] - Q(K_1)_u[W_\lambda] - Q(K_2)_u[W_\lambda, W_{\lambda 1}] \\
-Q(K_1)_u[W_1] = Q(\Lambda_1)_u[W_1], \\
Q(L_2)_u[W_\lambda, W_2] + Q(L_1)_u[W_{\lambda 2}] - Q(K_2)_u[W_\lambda, W_2] - Q(K_1)_u[W_{\lambda 2}] = Q(\Lambda_1)_u[W_2],
\end{aligned}$$

and, by evaluating them at $(0, \lambda^*, 0)$, we conclude

$$QL_1[\bar{W}_{\lambda\lambda}] = 0, \quad QL_1[\bar{W}_{\lambda 1}] = Q\Lambda_1[\bar{W}_1] = Q\Lambda_1[\bar{W}_1], \quad QL_1[\bar{W}_{\lambda 2}] = Q\Lambda_1[\bar{W}_2] = Q\Lambda_1[\bar{W}_2].$$

By differentiating (6.1.4) and (6.1.5) with respect to α_1, α_2, we obtain

$$\begin{aligned}Q(L_2)_u[W_1,W_1]+Q(L_1)_u[W_{11}]-Q(K_2)_u[W_1,W_1]-Q(K_1)_u[W_1]-Q(K_1)_u[W_{11}]\\-Q(K_1)_u[W_1]=0,\\Q(L_2)_u[W_1,W_2]+Q(L_1)_u[W_{12}]-Q(K_2)_u[W_1,W_2]-Q(K_1)_u[W_{12}]-Q(K_1)_u[W_2]=0,\\Q(L_2)_u[W_2,W_2]+Q(L_1)_u[W_{22}]-Q(K_2)_u[W_2,W_2]-Q(K_1)_u[W_{22}]=0,\end{aligned}$$

and, by evaluating them at $(0,\lambda^*,0)$, we conclude

$$QL_1[\bar{W}_{11}]-Q(K_1)[\bar{W}_1]-Q(K_1)[\bar{W}_1]=0,\quad QL_1[\bar{W}_{12}]=0,\quad QL_1[\bar{W}_{22}]=0.$$

We evaluate them at $(0,\lambda^*,0)$ and obtain

$$\begin{aligned}L_1[\bar{W}_{xx}]&=0,\\L_1[\bar{W}_{x\lambda}]&=Q\Lambda_1[u_n],\\L_1[W_{x1}]&=0,\\L_1[W_{x2}]&=0,\\L_1[\bar{W}_{\lambda\lambda}]&=0,\\L_1[\bar{W}_{\lambda 1}]&=Q\Lambda_1[\bar{W}_1],\\L_1[\bar{W}_{\lambda 2}]&=Q\Lambda_1[\bar{W}_2],\\L_1[\bar{W}_{11}]&=0,\\L_1[\bar{W}_{12}]&=0,\\L_1[\bar{W}_{22}]&=0,\end{aligned}$$

and we thus conclude that

$$\begin{array}{lllll}\bar{W}_{xx}=0, & \bar{W}_{x\lambda}=0, & \bar{W}_{x1}=0, & \bar{W}_{x2}=0, & \bar{W}_{\lambda\lambda}=0,\\ \bar{W}_{\lambda 1}=L_1^{-1}Q\Lambda_1[\bar{W}_1], & & \bar{W}_{\lambda 2}=L_1^{-1}Q\Lambda_1[\bar{W}_2], & & \\ \bar{W}_{11}=0\ (\text{since } K_1=0), & & \bar{W}_{12}=0, & & \bar{W}_{22}=0.\end{array}$$

Set $k_m=\frac{4}{\pi}\sum_{i=0}^{\infty}\frac{ma_i}{m^2-4i^2}$, m odd; $b_{m/2}$, if m even. We look $\bar{W}_{\lambda 1}$ and $\bar{W}_{\lambda 2}$ closely and obtain

$$\bar{W}_{\lambda 1}=L_1^{-1}Q\Lambda_1[\bar{W}_1]=L_1^{-1}Q\Lambda_1[\bar{W}_1]=\frac{\sqrt{2l}}{\pi}\sum_{m\neq n}\frac{k_m}{m^2-n^2}L_1^{-1}\Lambda_1[u_m^*]$$

$$\begin{aligned}&=\frac{\sqrt{2l}}{\pi}\sum_{m\neq n}\frac{k_m}{m^2-n^2}L_1^{-1}(m\pi/l)^2u_m^*=\frac{\sqrt{2l}}{\pi}\sum_{m\neq n}\frac{k_m}{m^2-n^2}\frac{(l/\pi)^2}{n^2-m^2}(m\pi/l)^2u_m^*\\&=-\frac{\sqrt{2l}}{\pi}\sum_{m\neq n}\frac{m^2k_m}{(m^2-n^2)^2}u_m^*,\end{aligned}$$

$$\begin{aligned}\bar{W}_{\lambda 2} =& L_1^{-1} Q \Lambda_1[\bar{W}_2] = L_1^{-1} Q \Lambda_1[\bar{W}_2] = (l/\pi)^2 \sqrt{2/l} \sum_{m: m \neq n, m\text{: odd}} L_1^{-1} Q \Lambda_1[u_m^*] \\ =& \sqrt{2/l} \sum_{m: m \neq n, m\text{: odd}} L_1^{-1} Q m^2 u_m^* = \sqrt{2/l} \sum_{m: m \neq n, m\text{: odd}} \frac{m^2}{n^2 - m^2} u_m^*.\end{aligned}$$

Chapter 7

Bifurcation equation $F = 0$ and its Taylor coefficients

In this chapter, we derive the first, second, and third derivatives of

$$F(x,\lambda,\alpha) = P\Phi(xu_n + W(x,\lambda,\alpha),\lambda,\alpha) = P(L)_u - \alpha_1 P(K)_u + \alpha_2 P\delta \quad (7.0.1)$$

where $(L)_u = (\Psi)_u - \lambda(\Lambda)_u$, $u = xu_n + W(x,\lambda,\alpha)$. We denote F_i for F_{α_i}, F_{xi} for $F_{x\alpha_i}$, and so on. For discussing the bifurcation set and hysteresis set we need to find the derivatives of $F(x,\lambda,\alpha)$ up-to third order and we compute all the derivatives at $(0,\lambda^*,0)$.

7.1 The first order derivatives of F

Differentiating (7.0.1) by x, λ, α_1, α_2, we obtain that

$$F_x = P(L_1)_u[u_n + W_x] - \alpha_1 P(K_1)_u[u_n + W_x], \quad (7.1.1)$$
$$F_\lambda = P(L_1)_u[W_\lambda] - \alpha_1 P(K_1)_u[W_\lambda] - P(\Lambda)_u, \quad (7.1.2)$$
$$F_1 = P(L_1)_u[W_1] - \alpha_1 P(K_1)_u[W_1] - P(K)_u, \quad (7.1.3)$$
$$F_2 = P(L_1)_u[W_2] - \alpha_1 P(K_1)_u[W_2] + P\delta. \quad (7.1.4)$$

Evaluating them at $(0,\lambda^*,0)$, we have $\bar{F}_x = PL_1[u_n] = 0$, $\bar{F}_\lambda = 0$,

$$\bar{F}_1 = -PK_0 = \frac{\pi^2}{l^2}\Big[\sum_{m:\,\mathrm{odd}} \frac{4}{\pi}\sum_{i=0}^{\infty} \frac{m^3 a_i}{m^2 - 4i^2} u_m^* + \sum_{m:\,\mathrm{even}} m^2 b_{m/2} u_m^*\Big] \cdot u_n \quad \text{(by Lemma 5.3.1)} \quad (7.1.5)$$

$$\bar{F}_2 = P\delta = u_n(l/2) = \begin{cases} (-1)^{\frac{n-1}{2}}\sqrt{2/l}, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases} \quad (7.1.6)$$

Here put a bar above a function to indicate evaluation at $(0,\lambda^*,0)$.

7.2 The second order derivatives of F

Differentiating (7.1.1) by x, λ, α_1, α_2, we obtain that

$$\begin{aligned} F_{xx} =& P(L_2)_u[u_n + W_x, u_n + W_x] + P(L_1)_u[W_{xx}] \\ &- \alpha_1 P(K_2)_u[u_n + W_x, u_n + W_x] - \alpha_1 P(K_1)_u[W_{xx}], \end{aligned} \tag{7.2.1}$$

$$\begin{aligned} F_{x\lambda} =& P(L_2)_u[u_n + W_x, W_\lambda] + P(L_1)_u[W_{x\lambda}] - P(\Lambda_1)_u[u_n + W_x] \\ &- \alpha_1 P(K_2)_u[u_n + W_x, W_\lambda] - \alpha_1 P(K_1)_u[W_{x\lambda}], \end{aligned} \tag{7.2.2}$$

$$\begin{aligned} F_{x1} =& P(L_2)_u[u_n + W_x, W_1] + P(L_1)_u[W_{x1}] \\ &- P(K_1)_u[u_n + W_x] - \alpha_1 P(K_2)_u[u_n + W_x, W_1] - \alpha_1 P(K_1)_u[W_{x1}], \end{aligned} \tag{7.2.3}$$

$$F_{x2} = P(L_2)_u[u_n + W_x, W_2] + P(L_1)_u[W_{x2}] - \alpha_1 P(K_2)_u[u_n + W_x, W_2] - \alpha_1 P(K_1)_u[W_{x2}]. \tag{7.2.4}$$

Evaluating them at $(0, \lambda^*, 0)$, we have

$$\bar{F}_{xx} = 0, \qquad \bar{F}_{x\lambda} = -P\Lambda_1[u_n] = -\frac{n^2\pi^2}{l^2}, \qquad \bar{F}_{x1} = 0, \qquad \bar{F}_{x2} = 0.$$

Differentiating (7.1.2) by λ, κ, α_1, α_2, we obtain that

$$F_{\lambda\lambda} = P(L_2)_u[W_\lambda, W_\lambda] + P(L_1)_u[W_{\lambda\lambda}] - \alpha_1 P(K_1)_u[W_{\lambda\lambda}] - P(\Lambda_1)_u[W_\lambda], \tag{7.2.5}$$

$$\begin{aligned} F_{\lambda 1} =& P(L_2)_u[W_\lambda, W_\lambda] + P(L_1)_u[W_{\lambda\lambda}] - P(\Lambda_1)_u[W_1] \\ &- P(K_1)_u[W_\lambda] - \alpha_1 P(K_2)_u[W_\lambda, W_1] - \alpha_1 P(K_1)_u[W_{\lambda\alpha_1}], \end{aligned} \tag{7.2.6}$$

$$F_{\lambda 2} = P(L_2)_u[W_\lambda, W_2] + P(L_1)_u[W_{\lambda 2}] - \alpha_1 P(K_2)_u[W_\lambda, W_2] - \alpha_1 P(K_1)_u[W_{\lambda 2}] - P(\Lambda_1)_u[W_2]. \tag{7.2.7}$$

By evaluate them at $(0, \lambda^*, 0)$, we have

$$\bar{F}_{\lambda\lambda} = 0, \qquad \bar{F}_{\lambda 1} = -P\Lambda_1[\bar{W}_1] = 0, \qquad \bar{F}_{\lambda 2} = P\Lambda_1[\bar{W}_2] = 0.$$

Differentiating (7.1.2) and (7.1.3) by α_1, α_2, we obtain that

$$F_{11} = P(L_2)_u[W_1, W_1] + P(L_1)_u[W_{11}] - \alpha_1 P(K_2)_u[W_1, W_1] - \alpha_1 P(K_1)_u[W_{11}] - 2P(K_1)_u[W_1], \tag{7.2.8}$$

$$F_{12} = P(L_2)_u[W_1, W_2] + P(L_1)_u[W_{12}] - \alpha_1 P(K_2)_u[W_1, W_2] - \alpha_1 P(K_1)_u[W_{12}] - P(K_1)_u[W_2], \tag{7.2.9}$$

$$F_{22} = P(L_2)_u[W_2, W_2] + P(L_1)_u[W_{22}] - \alpha_1 P(K_2)_u[W_2, W_2] - \alpha_1 P(K_1)_u[W_{22}]. \tag{7.2.10}$$

Evaluating them at $(0, \lambda^*, 0)$, we have $\bar{F}_{11} = 0$, $\bar{F}_{12} = 0$, and $\bar{F}_{22} = 0$.

7.3 The third order derivatives of F

Lemma 7.3.1. $\bar{F}_{xxx} = \frac{3n^6\pi^6}{2l^7}$.

Proof. Differentiating (7.2.1) by x and evaluating them at $(0,\lambda^*,0)$, we obtain that

$$\begin{aligned}\bar{F}_{xxx} =&PL_3[u_n,u_n,u_n] = \int_0^l 6((u_n')^2u_n''u_n'' + (u_n'(u_n'')^2)u_n')ds - 3\lambda^*\int_0^l (u_n')^3u_n'ds\\ =&12\int_0^l (u_n'u_n'')^2ds - 3\lambda^*\int_0^l (u_n')^4ds = 12\frac{n^6\pi^6}{2l^7} - 3\lambda^*\frac{3n^4\pi^4}{2l^5} = \frac{3n^6\pi^6}{2l^7}. \qquad \square\end{aligned}$$

$$\bar{W}_1 = -\tfrac{l^2}{\pi^2}\textstyle\sum_{m\neq n}\frac{k_m}{m^2-n^2}u_m^*$$

$$PK_2[u_n,u_n] = \begin{cases} -\frac{\pi^3}{l^5}\sum_{i=0}^\infty \frac{8n^5(3n^2-4i^2)}{(n^2-4i^2)(9n^2-4i^2)}a_i, n: \text{odd}\\ -\frac{\pi^4}{l^5}\frac{n^4}{2}(b_{n/2}+b_{3n/2}), n: \text{even}\end{cases}$$

$$k_{3n} = \begin{cases} \frac{4}{\pi}\sum_{i=0}^\infty \frac{3na_i}{9n^2-4i^2}, & n: \text{odd},\\ b_{3n/2}, & n: \text{even}.\end{cases}$$

Similarly, we obtain that

$$\bar{F}_{xx\lambda}=0,\ \bar{F}_{x\lambda 1}=0,\ \bar{F}_{x\lambda 2}=0,\ \bar{F}_{\lambda\lambda\lambda}=0,\ \bar{F}_{\lambda 11}=0,\ \bar{F}_{\lambda 12}=0,\ \bar{F}_{\lambda 22}=0,$$

$$\bar{F}_{x\lambda\lambda} = P\Lambda_1[L_1^{-1}Q\Lambda_1[u_n]] = 0,\ \bar{F}_{\lambda\lambda 1} = -P\Lambda_1[L_1^{-1}Q\Lambda_1[QK_0]] = 0,\ \bar{F}_{\lambda\lambda 2} = P\Lambda_1[L_1^{-1}Q\Lambda_1[Q\delta]] = 0.$$

Lemma 7.3.2. *When n is odd, $\bar{F}_{xx1} = \frac{3n^5\pi^3}{4l^5}\sum_{i=0}^\infty \frac{69n^2-20i^2}{(9n^2-4i^2)(n^2-4i^2)}a_i$, and $\bar{F}_{xx2} = -\frac{3n^2\pi^2}{16l^3}\sqrt{\frac{2}{l}}$. When n is even, $\bar{F}_{xx1} = \frac{3n^4\pi^4}{2l^5}b_{n/2} - \frac{3n^4\pi^4}{16l^5}b_{3n/2}$, and $\bar{F}_{xx2} = 0$.*

Proof. Differentiating (7.2.1) by α_1 and evaluating them at $(0,\lambda^*,0)$, we obtain that

$$\bar{F}_{xx1} = PL_3[u_n,u_n,\bar{W}_1] - PK_2[u_n,u_n].$$

When n is odd,

$$\begin{aligned}\bar{F}_{xx1} =&PL_3[u_n,u_n,\bar{W}_1] - PK_2[u_n,u_n] \quad \text{(by Lemma 5.5.2)}\\ =&\frac{3n^6\pi^6}{2l^7}(\langle u_n,\bar{W}_1\rangle + 9\langle u_{3n},\bar{W}_1\rangle) + \frac{3\pi^3}{l^5}\sum_{i=0}^\infty \frac{8n^5(3n^2-4i^2)a_i}{(9n^2-4i^2)(n^2-4i^2)}\\ =&-\frac{27n^6\pi^6}{2l^7}\frac{l^2}{\pi^2}\sum_{m=3n}\frac{k_m}{m^2-n^2} + \frac{3\pi^3}{l^5}\sum_{i=0}^\infty \frac{8n^5(3n^2-4i^2)a_i}{(9n^2-4i^2)(n^2-4i^2)}\\ =&-\frac{27n^4\pi^4}{2l^5}\frac{k_{3n}}{8} + \frac{3n^4\pi^3}{l^5}\sum_{i=0}^\infty \frac{8n(3n^2-4i^2)a_i}{(9n^2-4i^2)(n^2-4i^2)}\\ =&\frac{3n^5\pi^3}{l^5}\left[-\frac{9}{4}\sum_{i=0}^\infty \frac{3a_i}{9n^2-4i^2} + 8\sum_{i=0}^\infty \frac{(3n^2-4i^2)a_i}{(9n^2-4i^2)(n^2-4i^2)}\right]\\ =&\frac{3n^5\pi^3}{4l^5}\sum_{i=0}^\infty \frac{69n^2-20i^2}{(9n^2-4i^2)(n^2-4i^2)}a_i.\end{aligned}$$

When n is even,

$$\begin{aligned}\bar{F}_{xx1} =&PL_3[u_n, u_n, \bar{W}_1] - PK_2[u_n, u_n] \quad \text{(by Lemma 5.5.2)}\\ =&\frac{3n^6\pi^6}{2l^7}(\langle u_n, \bar{W}_1\rangle + 9\langle u_{3n}, \bar{W}_1\rangle) + \frac{3n^4\pi^4}{2l^5}(b_{n/2} + b_{3n/2})\\ =&\frac{3n^6\pi^6}{2l^7}\left(-\frac{9l^2}{\pi^2}\frac{b_{3n/2}}{9n^2 - n^2}\right) + \frac{3n^4\pi^4}{2l^5}(b_{n/2} + b_{3n/2})\\ =&-\frac{27n^4\pi^4}{16l^5}b_{3n/2} + \frac{3n^4\pi^4}{2l^5}(b_{n/2} + b_{3n/2}) = \frac{3n^4\pi^4}{2l^5}b_{n/2} - \frac{3n^4\pi^4}{16}l^5 b_{3n/2}.\end{aligned}$$

Since $\bar{W}_2 = -\frac{l^4}{\pi^4}\sum_{\substack{m:\,\text{odd}\\ m\neq n}} \frac{\sqrt{2/l}}{m^2(m^2-n^2)}u_m^*$,

$$\bar{F}_{xx2} = PL_3[u_n, u_n, L_1^{-1}Q\delta] = PL_3[u_n, u_n, \bar{W}_2]$$

$$\begin{aligned}=&\frac{3n^6\pi^6}{2l^7}(\langle u_n, \bar{W}_2\rangle + 9\langle u_{3n}, \bar{W}_2\rangle) \quad \text{(by Lemma 5.5.2)}\\ =&-\frac{3n^6\pi^6}{2l^7}\frac{l^4}{\pi^4}\sum_{m=3n,\, m:\text{odd}} \frac{\sqrt{2/l}}{m^2(m^2-n^2)} = \begin{cases} -\frac{3n^2\pi^2}{16l^3}\sqrt{\frac{2}{l}}, & n : \text{odd};\\ \frac{n^2\pi^2}{48l^3}\sqrt{\frac{2}{l}}, & n : \text{odd};\\ 0, & n : \text{even}.\end{cases} \qquad \square\end{aligned}$$

Lemma 7.3.3. *If we set* $C(\alpha) = \frac{1}{6}(\bar{F}_{111}\alpha_1^3 + 3\bar{F}_{112}\alpha_1^2\alpha_2 + 3\bar{F}_{122}\alpha_1\alpha_2^2 + \bar{F}_{222}\alpha_2^3)$, *then*

$$\begin{aligned}C(\alpha) =&\frac{1}{6}PL_3[\alpha_1\bar{W}_1 + \alpha_2\bar{W}_2, \alpha_1\bar{W}_1 + \alpha_2\bar{W}_2, \alpha_1\bar{W}_1 + \alpha_2\bar{W}_2]\\ &-\frac{\alpha_1}{2}PK_2[\alpha_1\bar{W}_1 + \alpha_2\bar{W}_2, \alpha_1\bar{W}_1 + \alpha_2\bar{W}_2],\end{aligned}$$

or equivalently,

$$C(\alpha) = \left(\frac{1}{6}PL_3[u, u, u] - \frac{1}{2}\alpha_1 PK_2[u, u]\right)\Big|_{u=\alpha_1\bar{W}_1+\alpha_2\bar{W}_2}.$$

Proof. Differentiating (7.2.8), (7.2.9), (7.2.7) by α_1 and α_2 and evaluating them at $(0, \lambda^*, 0)$, we obtain that

$$\begin{aligned}&\bar{F}_{111} =PL_3[\bar{W}_1, \bar{W}_1, \bar{W}_1] - 3PK_2[\bar{W}_1, \bar{W}_1], \quad \bar{F}_{112} =PL_3[\bar{W}_1, \bar{W}_1, \bar{W}_2] - 2PK_2[\bar{W}_1, \bar{W}_2],\\ &\bar{F}_{122} =PL_3[\bar{W}_1, \bar{W}_2, \bar{W}_2] - PK_2[\bar{W}_2, \bar{W}_2], \quad \bar{F}_{222} =PL_3[\bar{W}_2, \bar{W}_2, \bar{W}_2]. \qquad \square\end{aligned}$$

Remark 7.3.4. *As we will see in the differential coefficients* $\bar{F}_{x11}$, $\bar{F}_{x12}$, $\bar{F}_{x22}$ *are not important to describe the equation of bifurcation set and hysteresis set up to order 3, and we will not investigate their exact values.*

Chapter 8

Versality

In this chapter we introduce the notion of versal unfolding of f which contains all nearby deformation of f. In actual situation, several noise may cause some small perturbation of the idealized problem, which M. Golubitsky and D. Schaeffer call imperfect bifurcation or imperfection. When bifurcation diagram is subjected to small perturbations or imperfecions then we need to discuss the perturbation of singularity theory.
Let $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^m$ be smooth and $f(0,0) = 0$. An l-parameter unfolding of f is a smooth map $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^l \mapsto \mathbb{R}^m$ such that $F(x, \lambda, 0) = f(x, \lambda)$ for all x, λ.

We recall a criterion of pitchfork bifurcation in [Golubitsky and Schaeffer, 1979] (or in [Marsden and Hughes, 1994, 1.5 Theorem]).

Lemma 8.0.1. *If*

$$\bar{f} = \bar{f}_x = \bar{f}_{xx} = \bar{f}_\lambda = 0, \quad \bar{f}_{xxx} \neq 0, \qquad \bar{f}_{x\lambda} \neq 0,$$

the bifurcation of $f(x, \lambda) = 0$ at $(0, \lambda^*)$, is a pitchfork.

8.1 $\mathcal{P}$-$\mathcal{K}$-versality

[Golubitsky and Schaeffer, 1979]

Definition 8.1.1. We say that an unfolding $F : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^k, (0, \lambda^*, 0)) \to (\mathbb{R}, 0)$, $(x, \lambda, \alpha) \mapsto F(x, \lambda, \alpha)$, of $f : (\mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$, $(x, \lambda) \mapsto f(x, \lambda)$, is **$\mathcal{P}$-$\mathcal{K}$-versality**, if

$$\mathcal{E}_{x,\lambda} F + \mathcal{E}_{x,\lambda} F_x + \mathcal{E}_\lambda F_\lambda + \langle F_i|_{(x,\lambda,\alpha)=(0,\lambda^*,0)} : i = 1, \ldots, k \rangle_{\mathbb{R}} = \mathcal{E}_{x,\lambda}.$$

Here $\mathcal{E}_{x,\lambda}$, $\mathcal{E}_\lambda$ denote the ring of C^∞-function germs on $(\mathbb{R}^2, (0, \lambda^*))$, $(\mathbb{R}, \lambda^*)$ with variables (x, λ), and variable λ, respectively.

Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are $\mathcal{P}$-$\mathcal{K}$- equivalent if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \to \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.
If $n = 0$, we simply say these germs are $\mathcal{K}$-equivalent.
Let $F : (\mathbb{R}^k \times \mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that F is a $\mathcal{K}$-versal deformation of

$f = F|\mathbb{R}^k \times \{0\}$ if $\mathcal{E}_k = T_{\mathcal{E}}(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{0\}, \cdots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{0\} \right\rangle_{\mathbb{R}}$, where $T_{\mathcal{E}}(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \cdots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}$.

M. Golubitsky and D. Schaefer used the term "a universal unfolding" for this definition. We prefer to use the word "$\mathcal{P}$-$\mathcal{K}$-versal", because it fits recent usage of terminologies in singularity theory.

All small perturbations of the bifurcation are embedded in its p-$\mathcal{K}$-versal unfolding. In the context of imperfections, it is important to decide whether the given perturbation defines a p-$\mathcal{K}$-versal unfolding, or not, since the imperfection parameters are embedded in the parameter space of this p-$\mathcal{K}$-versal unfolding.

Example 8.1.2. *When $F(x, \lambda, \alpha_1, \alpha_2) = x^3 - \lambda x + \alpha_1 x^2 + \alpha_2$, we have $B = \{\alpha_2 = 0\}$ and $H = \{\alpha_1^3 = 27\alpha_2\}$. The bifurcation diagrams of the zeros of $f_\alpha(x, \lambda) = F(x, \lambda, \alpha)$ are shown as follows:*

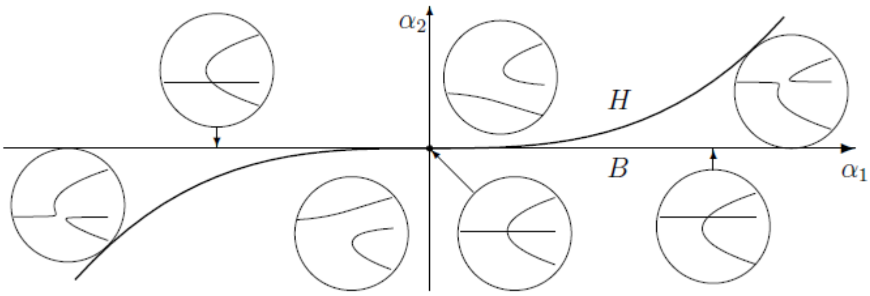


FIGURE 8.1: Bifurcation set B and hysteresis set H for Example (8.1.2)

This example shows all kind of perturbations of the problem which predicted by M. Golubitsky and D. Schaeffer. We find that two imperfection parameters are necessary to describe an arbitrary small perturbation of this problem, where the expilict notion of imeperfection parameters are unfolding parameters. Versality is important because by the versal unfolding parameters, we can discuss about the imperfections or possible perturbed bifurcation diagrams.

Lemma 8.1.3. *If n is odd, F is p-$\mathcal{K}$-versal.*

Proof. Since

$$\begin{vmatrix} \bar{F}_x & \bar{F}_{xx} & \bar{F}_{xxx} & \bar{F}_{x\lambda} \\ \bar{F}_\lambda & \bar{F}_{x\lambda} & \bar{F}_{xx\lambda} & \bar{F}_{\lambda\lambda} \\ \bar{F}_1 & \bar{F}_{x1} & \bar{F}_{xx1} & \bar{F}_{\lambda 1} \\ \bar{F}_2 & \bar{F}_{x2} & \bar{F}_{xx2} & \bar{F}_{\lambda 2} \end{vmatrix} = \begin{vmatrix} 0 & 0 & \bar{F}_{xxx} & \bar{F}_{x\lambda} \\ 0 & \bar{F}_{x\lambda} & \bar{F}_{xx\lambda} & \bar{F}_{\lambda\lambda} \\ \bar{F}_1 & 0 & \bar{F}_{xx1} & 0 \\ \bar{F}_2 & 0 & \bar{F}_{xx2} & 0 \end{vmatrix} = (\bar{F}_{x\lambda})^2 \begin{vmatrix} \bar{F}_1 & \bar{F}_{xx1} \\ \bar{F}_2 & \bar{F}_{xx2} \end{vmatrix} \neq 0,$$

F is $\mathcal{P}$-$\mathcal{K}$-versal, by [Golubitsky and Schaeffer, 1979, Lemma 4.3]. $\square$

Chapter 9

Bifurcation set and hysteresis set

In this chapter, we derive the equation of bifurcation set and hysteresis set and draw the figures of the bifurcation set and hysteresis set.
Now we consider the bifurcation set of the zero of

$$F = \frac{x^3}{6}\bar{F}_{xxx} + \bar{F}_{x\lambda}\lambda x + \bar{F}_1\alpha_1 + \bar{F}_2\alpha_2 + \frac{x^2}{2}\ell(\alpha) + xQ(\alpha) + C(\alpha) + O(4),$$

where $\ell(\alpha) = \bar{F}_{xx1}\alpha_1 + \bar{F}_{xx2}\alpha_2$, $Q(\alpha) = \frac{1}{2}(\bar{F}_{x11}\alpha_1^2 + 2\bar{F}_{x12}\alpha_1\alpha_2 + \bar{F}_{x22}\alpha_2^2)$, and $C(\alpha)$ is defined in Lemma 7.3.3.

9.1 Equation of B and H

To describe how the pitchfork bifurcation changes nearby the origin, we recall the bifurcation set B and hysteresis set H, which are defined by

$$B = \{\alpha : \exists(x,\lambda),\ F(x,\lambda,\alpha) = 0, F_x(x,\lambda,\alpha) = F_\lambda(x,\lambda,\alpha) = 0\}, \qquad (9.1.1)$$
$$H = \{\alpha : \exists(x,\lambda),\ F(x,\lambda,\alpha) = 0, F_x(x,\lambda,\alpha) = F_{xx}(x,\lambda,\alpha) = 0\}. \qquad (9.1.2)$$

From 7.1.5 and 7.1.6, we have the values of $\bar{F}_1$ and $\bar{F}_2$ and if n is odd, these sets are zeros of certain functions with the following 1-jet:

$$\Big(\frac{4\pi n^2}{l^2}\sum_{i=0}^{\infty}\frac{na_i}{n^2-4i^2}\Big)\alpha_1 + \Big((-1)^{\frac{n-1}{2}}\sqrt{\tfrac{2}{l}}\Big)\alpha_2 \ (= \bar{F}_1\alpha_1 + \bar{F}_2\alpha_2).$$

In the following Proposition 9.1.1, we describe their 3-jets as (9.1.3) and (9.1.4), respectively, which enables us to draw B and H approximately near the origin.

Proposition 9.1.1. *The bifurcation set B ((9.1.1)) and the hysteresis set H ((9.1.2)) are zeros of smooth functions with the following 3-jets*

$$\bar{F}_1\alpha_1 + \bar{F}_2\alpha_2 + C(\alpha), \text{ and} \qquad (9.1.3)$$
$$\bar{F}_1\alpha_1 + \bar{F}_2\alpha_2 + C(\alpha) - \frac{2l^{14}}{27n^{12}\pi^{12}}\ell(\alpha)^3, \qquad (9.1.4)$$

respectively.

Proof. Since $\bar{F}_{xxx} = \frac{3n^6\pi^6}{2l^7}$, we have

$$\begin{aligned} F_x =& \frac{3n^6\pi^6}{4l^7}x^2 - \frac{n^2\pi^2}{l^2}\lambda + x\ell(\alpha) + Q(\alpha) + O(3), \\ F_\lambda =& -\frac{n^2\pi^2}{l^2}x + O(3), \quad F_{xx} = \frac{3n^6\pi^6}{2l^7}x + \ell(\alpha) + O(3). \end{aligned}$$

$F_x = F_\lambda = 0$ defines (x, λ) as a function of α and we obtain that

$$x = O(3), \qquad \lambda = \frac{l^2}{n^2\pi^2}Q(\alpha) + O(3).$$

Since $F - xF_x = -\frac{n^6\pi^6}{2l^5}x^3 + \bar{F}_1\alpha_1 + \bar{F}_2\alpha_2 - \frac{x^2}{2}\ell(\alpha) + C(\alpha) + O(4)$, we obtain that the 3-jet of the equation for bifurcation set is (9.1.3).

Similarly $F_x = F_{xx} = 0$ defines (x, λ) as a function of α and we obtain that

$$x = -\frac{2l^7}{3n^6\pi^6}\ell(\alpha) + O(3), \qquad \lambda = O(2),$$

and thus the 3-jets of the equation for hysteresis set. (9.1.4). □

We present the data for $C(\alpha)$ (see Lemma 7.3.3) as follows: Set $k_m = \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{ma_i}{m^2-4i^2}$, m odd; $b_{m/2}$, if m even.

$$\begin{aligned}
PL_3[\bar{W}_1, \bar{W}_1, \bar{W}_1] &= -\frac{l^6}{\pi^6}\sum_{a,b,c\neq n}\frac{k_ak_bk_c}{(a^2-n^2)(b^2-n^2)(c^2-n^2)}PL_3[u_a, u_b, u_c], \\
PL_3[\bar{W}_1, \bar{W}_1, \bar{W}_2] &= -\frac{l^6}{\pi^6}\sum_{c:\mathrm{odd}, a,b,c\neq n}\frac{k_ak_b(l^2/\pi^2)\sqrt{2/l}}{c^2(a^2-n^2)(b^2-n^2)(c^2-n^2)}PL_3[u_a, u_b, u_c], \\
PL_3[\bar{W}_1, \bar{W}_2, \bar{W}_2] &= -\frac{l^6}{\pi^6}\sum_{b,c:\mathrm{odd}, a,b,c\neq n}\frac{k_a(l^4/\pi^4)2/l}{b^2c^2(a^2-n^2)(b^2-n^2)(c^2-n^2)}PL_3[u_a, u_b, u_c], \\
PL_3[\bar{W}_2, \bar{W}_2, \bar{W}_2] &= -\frac{l^6}{\pi^6}\sum_{a,b,c:\mathrm{odd}, a,b,c\neq n}\frac{(l^6/\pi^6)(2/l)^{3/2}}{a^2b^2c^2(a^2-n^2)(b^2-n^2)(c^2-n^2)}PL_3[u_a, u_b, u_c], \\
PK_2[\bar{W}_1, \bar{W}_1] &= \frac{l^4}{\pi^4}\sum_{a,b\neq n}\frac{k_ak_b}{(a^2-n^2)(b^2-n^2)}PK_2[u_a, u_b], \\
PK_2[\bar{W}_1, \bar{W}_2] &= \frac{l^6}{\pi^6}\sqrt{\frac{2}{l}}\sum_{a,b\neq n, b:\mathrm{odd}}\frac{k_a}{b^2(a^2-n^2)(b^2-n^2)}PK_2[u_a, u_b], \\
PK_2[\bar{W}_2, \bar{W}_2] &= \frac{l^8}{\pi^8}\frac{2}{l}\sum_{a,b\neq n, \mathrm{odd}}\frac{1}{a^2b^2(a^2-n^2)(b^2-n^2)}PK_2[u_a, u_b].
\end{aligned}$$

In th remaining part of this section, we describe numerical results on the data above to describe $C(\alpha)$ assuming $b_i = 0$ $(i \geq 1)$ and $n = 1$. Remark that $k_m = \frac{4m}{\pi}\sum_{i=0}^{\infty} a_i/(m^2-4i^2)$, if m is odd; 0, if m is even, and we have

$$\bar{W}_1 = -\frac{l^2}{\pi^2}\sum_{m\neq n}\frac{k_m}{m^2-n^2}u_m^* = -\frac{4l^2}{\pi^3}\sum_{m:\mathrm{odd},\neq n}\frac{m}{m^2-n^2}\sum_{i=0}^{\infty}\frac{a_i}{m^2-4i^2}u_m^*,$$

$$\bar{W}_1 = -\frac{l^2}{\pi^2}\sum_{m:\text{odd},\neq n}\frac{4/\pi}{m(m^2-n^2)}u_m^*, \quad \bar{W}_2 = -\frac{l^4}{\pi^4}\sum_{m:\text{odd},\neq n}\frac{\sqrt{2/l}}{m^2(m^2-n^2)}u_m^*.$$

We have

$$\begin{aligned}
PL_3[\bar{W}_1,\bar{W}_1,\bar{W}_1] =& \frac{c_0}{l\pi}\Big(\frac{4}{\pi}\Big)^3, \qquad PL_3[\bar{W}_1,\bar{W}_1,\bar{W}_2] = \frac{lc_1}{\pi^3}\Big(\frac{4}{\pi}\Big)^2\Big(\frac{2}{l}\Big)^{1/2},\\
PL_3[\bar{W}_1,\bar{W}_2,\bar{W}_2] =& \frac{l^3c_2}{\pi^5}\frac{4}{\pi}\frac{2}{l}, \qquad PL_3[\bar{W}_2,\bar{W}_2,\bar{W}_2] = \frac{l^5c_3}{\pi^7}\Big(\frac{2}{l}\Big)^{3/2},
\end{aligned}$$

where c_0, c_1, c_2, c_3 are constants. The approximate values of c_i are given by

$$\begin{aligned}
c_0 \simeq& 0.305307a_0^3 + 1.20457a_0^2a_1 + 0.556055a_0^2a_2 + 0.449847a_0^2a_3 + \cdots\\
&+ 1.5754a_0a_1^2 + 1.60049a_0a_1a_2 + 1.23451a_0a_1a_3 + \cdots\\
&+ 0.0536143a_0a_2^2 + 0.410507a_0a_2a_3 - 0.0983358a_0a_3^2 + \cdots\\
&+ 0.683785a_1^3 + 1.15217a_1^2a_2 + 0.821541a_1^2a_3 + \cdots\\
&- 0.121613a_1a_2^2 + 0.763853a_1a_2a_3 - 0.154765a_1a_3^2 + \cdots\\
&+ 0.0918374a_2^3 - 0.322925a_2^2a_3 + 0.0171554a_2a_3^2 + 0.0409826a_3^2 + \cdots\\
c_1 \simeq& (0.0560462a_0 + 0.147036a_1 + 0.078606a_2 + 0.0592183a_3 + \cdots)a_0\\
&+ (0.0965134a_1 + 0.112754a_2 + 0.0758876a_3 + \cdots)a_1\\
&+ (0.00853948a_2 + 0.0472655a_3 + \cdots)a_2 - 0.00887054a_3^2 + \cdots\\
c_2 \simeq& 0.0105423a_0 + 0.0141242a_1 + 0.00815088a_2 + 0.00496213a_3 + \cdots,\\
&- 0.00720015a_2^2 + \cdots\\
c_3 \simeq& 0.00218564.
\end{aligned}$$

We also remark that

$$PL_3[u_a,u_b,u_c]\cdot u_n = -\sum_{\varepsilon_1 a+\varepsilon_2 b+\varepsilon_3 c=1}\Big[\frac{1}{2} + abc\varepsilon_1\varepsilon_2\varepsilon_3\Big(\frac{\varepsilon_1}{a}+\frac{\varepsilon_2}{b}+\frac{\varepsilon_3}{c}\Big)\Big].$$

Since

$$\begin{aligned}
PL_3[\bar{W}_1,\bar{W}_1,\bar{W}_1] =& \frac{l^6}{\pi^6}\sum_{a,b,c:\text{odd},\neq 1}\frac{(4/\pi)^3}{abc(a^2-1)(b^2-1)(c^2-1)}PL_3[u_a,u_b,u_c],\\
PL_3[\bar{W}_1,\bar{W}_1,\bar{W}_2] =& \frac{l^8}{\pi^8}\sum_{a,b,c:\text{odd},\neq 1}\frac{(4/\pi)^2\sqrt{2/l}}{abc^2(a^2-1)(b^2-1)(c^2-1)}PL_3[u_a,u_b,u_c],\\
PL_3[\bar{W}_1,\bar{W}_2,\bar{W}_2] =& \frac{l^{10}}{\pi^{10}}\sum_{a,b,c:\text{odd},\neq 1}\frac{(4/\pi)2/l}{ab^2c^2(a^2-1)(b^2-1)(c^2-1)}PL_3[u_a,u_b,u_c],\\
PL_3[\bar{W}_2,\bar{W}_2,\bar{W}_2] =& \frac{l^{12}}{\pi^{12}}\sum_{a,b,c:\text{odd},\neq 1}\frac{(2/l)^{3/2}}{a^2b^2c^2(a^2-1)(b^2-1)(c^2-1)}PL_3[u_a,u_b,u_c],\\
PL_3[u_a,u_b,u_c] =& \frac{abc\pi^5}{l^7}\sum_{\varepsilon_1 a+\varepsilon_2 b+\varepsilon_3 c=1}\Big[-\frac{1}{2} - abc\varepsilon_1\varepsilon_2\varepsilon_3\Big(\frac{\varepsilon_1}{a}+\frac{\varepsilon_2}{b}+\frac{\varepsilon_3}{c}\Big)\Big].
\end{aligned}$$

This numerical result follows computing the summations above with $m \leq 500$. We remark that the convergence of c_0 is very slow, and we are not sure how many digits are correct for this approximate value.

Since

$$\begin{aligned} PK_2[u_a, u_b] &= -\frac{abn\pi^3}{l^5}\sum_{i=0}^{\infty} a_i \sum_{a+b\not\equiv n(2)} \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \frac{(\varepsilon_1 a+\varepsilon_2 b+n)^2}{(\varepsilon_1 a+\varepsilon_2 b+n)^2-4i^2} \\ &= -\frac{4abn\pi^3}{l^5}\sum_{i=0}^{\infty} a_i \sum_{a+b\not\equiv n(2)} \Big(1+\sum_{\varepsilon_1,\varepsilon_2=\pm 1} \frac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+n)^2-4i^2}\Big), \end{aligned}$$

we have, if n is odd,

$$\begin{aligned} PK_2[\bar W_1, \bar W_1] =& \frac{l^4}{\pi^4}\sum_{a,b:\text{odd},\neq n} \frac{(4/\pi)^2 ab}{(a^2-n^2)(b^2-n^2)} \sum_{i_1,i_2=0}^{\infty} \frac{a_{i_1}a_{i_2}}{(a^2-4i_1^2)(b^2-4{i_2}^2)} PK_2[u_a,u_b], \\ =& -\frac{64n}{\pi^3 l}\sum_{a,b:\text{odd},\neq n} \frac{a^2b^2}{(a^2-n^2)(b^2-n^2)} \sum_{i,i_1,i_2=0}^{\infty} \frac{a_i a_{i_1}a_{i_2}}{(a^2-4i_1^2)(b^2-4{i_2}^2)} \\ & \times \Big(1+\sum_{\varepsilon_1,\varepsilon_2=\pm 1} \frac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+n)^2-4i^2}\Big), \\ PK_2[\bar W_1, \bar W_2] =& \frac{l^6}{\pi^6}\sqrt{\frac{2}{l}}\sum_{a,b:\text{odd},\neq n} \frac{(4/\pi)a}{b^2(a^2-n^2)(b^2-n^2)} \sum_{i_1=0}^{\infty} \frac{a_{i_1}}{a^2-4i_1^2} PK_2[u_a,u_b] \\ =& -\frac{16nl}{\pi^4}\sqrt{\tfrac{2}{l}}\sum_{a,b:\text{odd},\neq n} \frac{a^2}{b(a^2-n^2)(b^2-n^2)} \sum_{i,i_1=0}^{\infty} \frac{a_{i_1}a_i}{a^2-4i_1^2}\Big(1+\sum_{\varepsilon_1,\varepsilon_2=\pm 1} \tfrac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+n)^2-4i^2}\Big), \\ PK_2[\bar W_2, \bar W_2] =& \frac{l^8}{\pi^8}\frac{2}{l}\sum_{a,b\neq 1,\text{odd}} \frac{1}{a^2b^2(a^2-n^2)(b^2-n^2)} PK_2[u_a,u_b] \\ =& -\frac{8nl^2}{\pi^5}\sum_{i=0}^{\infty} a_i \sum_{a,b\neq 1,\text{odd}} \frac{1}{ab(a^2-n^2)(b^2-n^2)}\Big(1+\sum_{\varepsilon_1,\varepsilon_2=\pm 1} \frac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+n)^2-4i^2}\Big). \end{aligned}$$

Assume that $n=1$. Since

$$\sum_{\substack{a:\,\text{odd}\\ a\neq 1}} \frac{a^2}{(a^2-1)(a^2-4i^2)} = \frac{12i^2+1}{4(4i^2-1)^2}, \qquad \sum_{a:\,\text{odd},>1} \frac{1}{a(a^2-1)} = \frac{3}{4}-\log 2.$$

we obtain

$$
\begin{aligned}
PK_2[\bar{W}_1,\bar{W}_1] &= -\frac{4}{\pi^3 l}\sum_{i,i_1,i_2=0}^{\infty} a_i a_{i_1} a_{i_2}\Big[\frac{(12i_1^2+1)(12i_2^2+1)}{(4i_1^2-1)^2(4i_2^2-1)^2} \\
&\quad + \sum_{\substack{a,b:\text{odd}\\ a,b\neq 1}} \frac{4^2 i}{(a^2-4i_1^2)(b^2-4{i_2}^2)}\frac{a^2b^2}{(a^2-1)(b^2-1)}\sum_{\varepsilon_1,\varepsilon_2=\pm 1}\frac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+1)^2-4i^2}\Big], \\
&= -\frac{4}{\pi^3 l}\sum_{i,i_1,i_2=0}^{\infty}\Big[a_i a_{i_1} a_{i_2}\frac{(12i_1^2+1)(12i_2^2+1)}{(4i_1^2-1)^2(4i_2^2-1)^2}
\end{aligned}
$$

$$
+a_1(a_0\ a_1\ a_2\ \dots)\begin{pmatrix}
-0.0098018 & -0.0171753 & 0.00800604 & 0.00824388 & 0.00228896 & 0.00100968 & \cdots \\
-0.0171753 & -0.0306015 & 0.0169267 & 0.0144463 & 0.00298161 & 0.00129348 & \cdots \\
0.00800617 & 0.0169269 & -0.0241598 & -0.00438159 & 0.00500669 & 0.00110166 & \cdots \\
0.00824403 & 0.0144466 & -0.00438157 & -0.013504 & -0.00250348 & 0.00220992 & \cdots \\
0.00228912 & 0.00298184 & 0.00500672 & -0.00250347 & -0.00698926 & -0.00127408 & \cdots \\
0.00100984 & 0.00129372 & 0.00110169 & 0.00220993 & -0.00127408 & -0.00433055 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}
$$

$$
+a_2(a_0\ a_1\ a_2\ \dots)\begin{pmatrix}
-0.00585423 & -0.00798357 & -0.0107953 & 0.00882541 & 0.0107354 & 0.00248558 & \cdots \\
-0.00798387 & -0.0107902 & -0.0175634 & 0.0166762 & 0.0177164 & 0.00217028 & \cdots \\
-0.0107948 & -0.0175626 & 0.00990841 & -0.01239 & -0.000800288 & 0.010955 & \cdots \\
0.00882601 & 0.0166771 & -0.0123899 & -0.00630149 & -0.011251 & -0.00120814 & \cdots \\
0.010736 & 0.0177173 & -0.000800176 & -0.011251 & -0.00633698 & -0.00721143 & \cdots \\
0.00248622 & 0.00217122 & 0.0109551 & -0.00120811 & -0.00721141 & -0.00368129 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}
$$

$$
+a_3(a_0\ a_1\ a_2\ \dots)\begin{pmatrix}
-0.010169 & -0.0173908 & 0.0098944 & -0.00878926 & 0.00988501 & 0.0125713 & \cdots \\
-0.0173914 & -0.0308041 & 0.0218814 & -0.0139982 & 0.0187884 & 0.0208632 & \cdots \\
0.00989557 & 0.0218833 & -0.0334693 & 0.00254736 & -0.0174167 & -0.00322272 & \cdots \\
-0.0087879 & -0.0139962 & 0.00254755 & 0.00573298 & -0.00157069 & -0.0105505 & \cdots \\
0.00988643 & 0.0187905 & -0.0174165 & -0.00157063 & -0.00275234 & -0.00445332 & \cdots \\
0.0125728 & 0.0208654 & -0.00322244 & -0.0105504 & -0.0044533 & -0.00391895 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}
$$

$$+\cdots,$$

$$
\begin{aligned}
PK_2[\bar{W}_1,\bar{W}_2] &= -\frac{16l}{\pi^4}\sqrt{\tfrac{2}{l}}\sum_{i,i_1=0}^{\infty} a_i a_{i_1}\Big[\frac{12i_1^2+1}{4(4i_1^2-1)^2}\Big(\frac{3}{4}-\log 2\Big) \\
&\qquad + \sum_{\substack{a,b:\text{odd}\\ a,b\neq 1}}\frac{1}{a^2-4i_1^2}\frac{a^2}{b(a^2-1)(b^2-1)}\sum_{\varepsilon_1,\varepsilon_2=\pm 1}\frac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+1)^2-4i^2}\Big], \\
&\simeq -\frac{16l}{\pi^4}\sqrt{\tfrac{2}{l}}\Big[\sum_{i,i_1=0}^{\infty} a_i a_{i_1}\frac{12i_1^2+1}{4(4i_1^2-1)^2}\Big(\frac{3}{4}-\log 2\Big) \\
&\quad - a_1\begin{pmatrix} 0.00319975a_0+0.00572494a_1-0.00326835a_2 \\ -0.00273496a_3-0.000536221a_4-0.00021051a_5+\cdots \end{pmatrix} \\
&\quad - a_2\begin{pmatrix} 0.00145234a_0+0.00201291a_1+0.00301828a_2 \\ -0.00322987a_3-0.00330068a_4-0.000328207a_5+\cdots \end{pmatrix} \\
&\quad - a_3\begin{pmatrix} 0.00319406a_0+0.00574215a_1-0.00437078a_2 \\ +0.00241329a_3+0.00358896a_4-0.00385563a_5+\cdots \end{pmatrix}+\cdots\Big], \\
PK_2[\bar{W}_2,\bar{W}_2] &= -\frac{8l^2}{\pi^5}\sum_{i=0}^{\infty} a_i\Big[\Big(\frac{3}{4}-\log 2\Big)^2+\sum_{\substack{a,b:\text{odd}\\ a,b\neq 1}}\frac{1}{ab(a^2-1)(b^2-1)}\sum_{\varepsilon_1,\varepsilon_2=\pm 1}\frac{i^2}{(\varepsilon_1 a+\varepsilon_2 b+1)^2-4i^2}\Big] \\
&\simeq -\frac{8l^2}{\pi^5}\Big[\sum_{i=0}^{\infty} a_i\Big(\frac{3}{4}-\log 2\Big)^2-\begin{pmatrix} 0.00107265a_1+0.000380642a_2+0.00107825a_3 \\ +0.00380099a_4+0.00403667a_5+\cdots \end{pmatrix}\Big].
\end{aligned}
$$

We have

$$PK_2[\bar{W}_1, \bar{W}_1] = \frac{k_0}{l\pi}\left(\frac{4}{\pi}\right)^2, \qquad PK_2[\bar{W}_1, \bar{W}_2] = \frac{lk_1}{\pi^3}\left(\frac{4}{\pi}\right)\left(\frac{2}{l}\right)^{1/2},$$
$$PK_2[\bar{W}_2, \bar{W}_2] = \frac{l^3 k_2}{\pi^5}\frac{2}{l},$$

where k_0, k_1, k_2 are constants. The approximate values of k_i are given by

$$\begin{aligned}
k_0 \simeq& 0.248004a_0^3 + 0.926139a_0^2 a_1 + 0.331051a_0^2 a_2 + 0.249654a_0^2 a_3 + \cdots\\
&+ 1.09866a_0 a_1^2 + 0.977956a_0 a_1 a_2 + 0.747703a_0 a_1 a_3 + \cdots\\
&+ 0.0315291a_0 a_2^2 + 0.30764a_0 a_2 a_3 - 0.026177a_0 a_3^2 + \cdots\\
&+ 0.396315a_1^3 + 0.764946a_1^2 a_2 + 0.572289a_1^2 a_3 + \cdots\\
&- 0.0717457a_1 a_2^2 + 0.497689a_1 a_2 a_3 - 0.102975a_1 a_3^2 + \cdots\\
&+ 0.0510594a_2^3 - 0.212486a_2^2 a_3 + 0.00606446a_2 a_3^2 + 0.0247378a_3^2 + \cdots\\
k_1 \simeq& (0.0566244a_0 + 0.125717a_1 + 0.0629693a_2 + 0.0486808a_3 + \cdots)a_0\\
&+ (0.0589923a_1 + 0.0990681a_2 + 0.0746959a_3 + \cdots)a_1\\
&+ (0.0000811478a_2 + 0.0473895a_3 + \cdots)a_2 - 0.00482053a_3^2 + \cdots\\
k_2 \simeq& 0.0129285a_0 + 0.0086379a_1 + 0.011406a_2 + 0.00861553a_3 + \cdots.
\end{aligned}$$

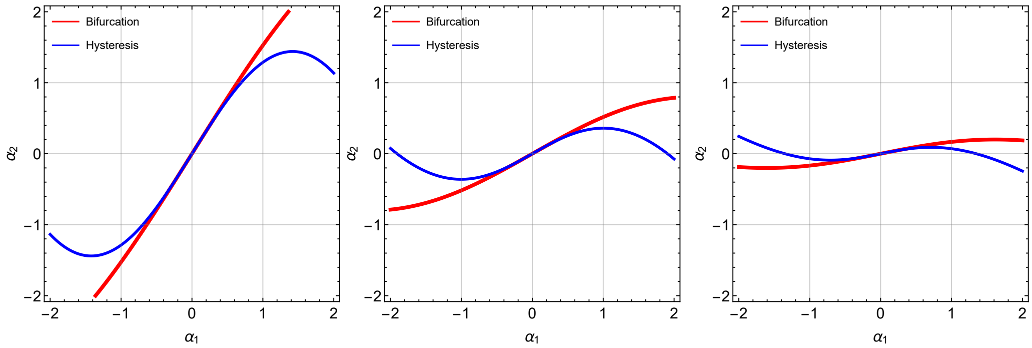
This numerical result also follows computing the summations above with $m \leq 500$.

Remark 9.1.2. *If $a_0 = 1$, $a_i = 0$ $(i \geq 1)$, then*

$$PK_2[\alpha_1\bar{W}_1 + \alpha_2\bar{W}_2, \alpha_1\bar{W}_1 + \alpha_2\bar{W}_2] = -\frac{8}{l\pi}\left(\frac{1}{\pi}\alpha_1 + \frac{3 - 4\log 2}{4}\frac{l^2}{\pi^2}\sqrt{\frac{2}{l}}\alpha_2\right)^2.$$

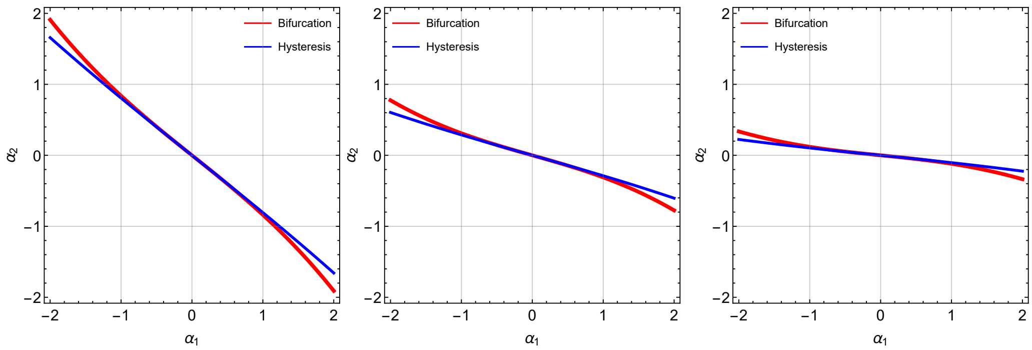
9.2 Figures of B and H

In this section, we show below the figures of the zeros of (9.1.3) and (9.1.4) in several cases. We take the length π, 2π and 4π respectively for each cases and we put the values of κ where κ depends on the values of a_0, a_i and b_i.



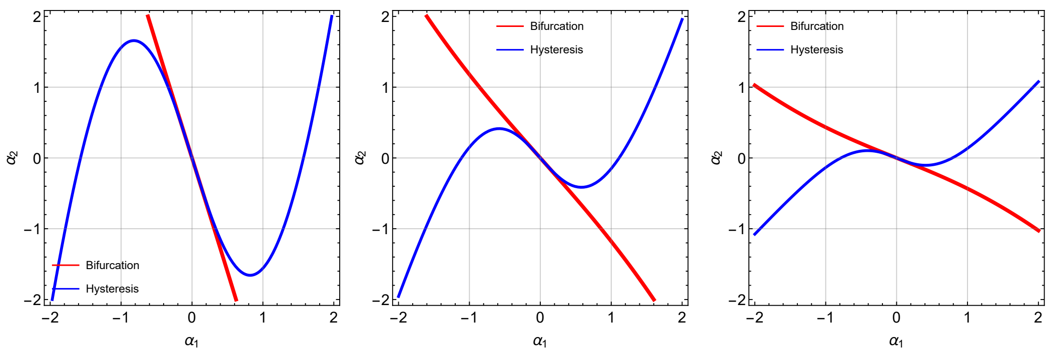
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.1: Approximations of B and H ($a_0 = -1, a_{i\geq 1} = 0, b_i = 0$)



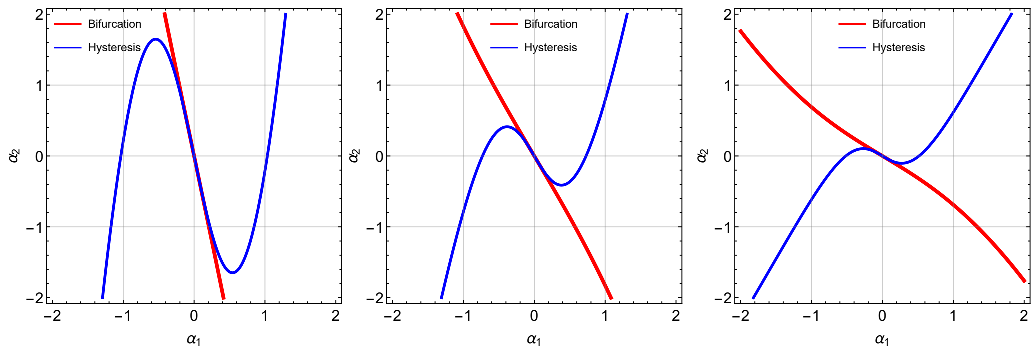
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.2: Approximations of B and H ($a_0 = 0.5, a_{i\geq 1} = 0, b_i = 0$)



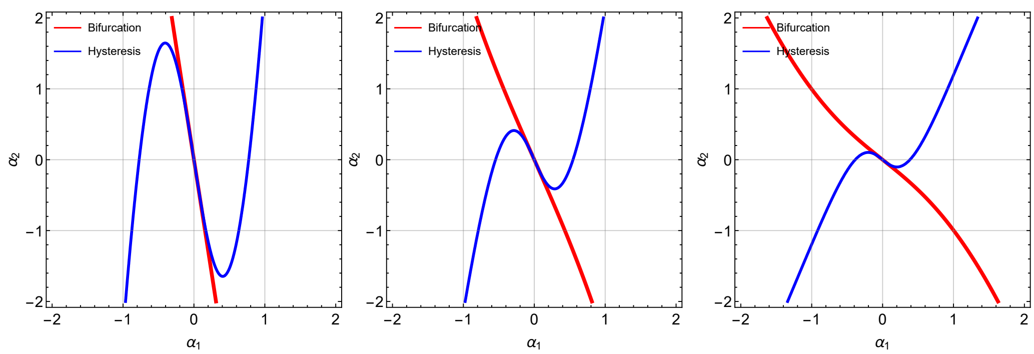
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.3: Approximations of B and H ($a_0 = 2, a_{i\geq 1} = 0$)



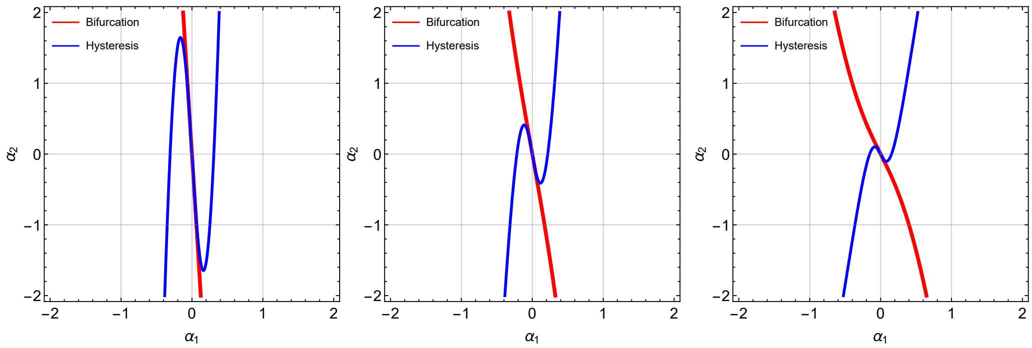
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.4: Approximations of B and H ($a_0 = 3, a_{i\geq 1} = 0$)



(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

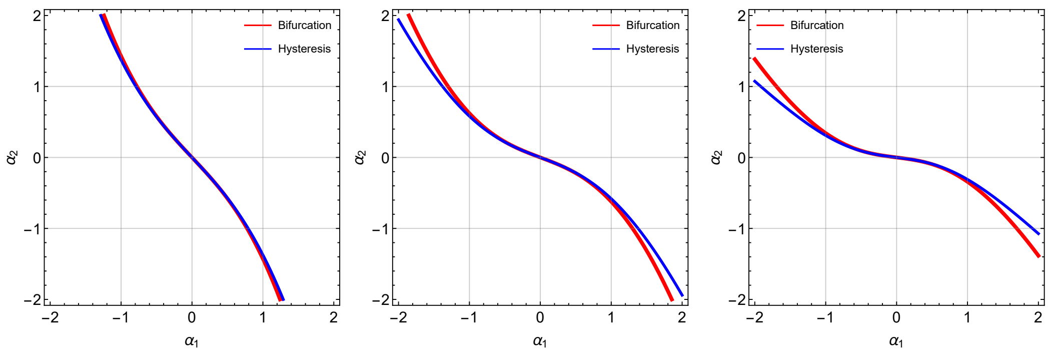
FIGURE 9.5: Approximations of B and H ($a_0 = 4, a_{i\geq 1} = 0$)



(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

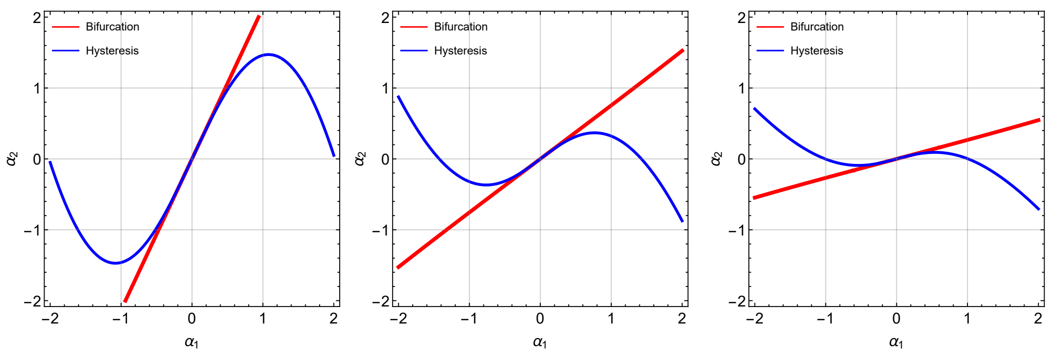
FIGURE 9.6: Approximations of B and H ($a_0 = 10, a_{i\geq 1} = 0$)

We observe from the figures above that the bifurcation and hysteresis sets change as we change κ. When $a_i = 0$ for $i \geq 1$, the aspect of the bifurcation sets changes slightly, and the aspect of the hysteresis sets sometimes changes considerably according to the change of a_0.



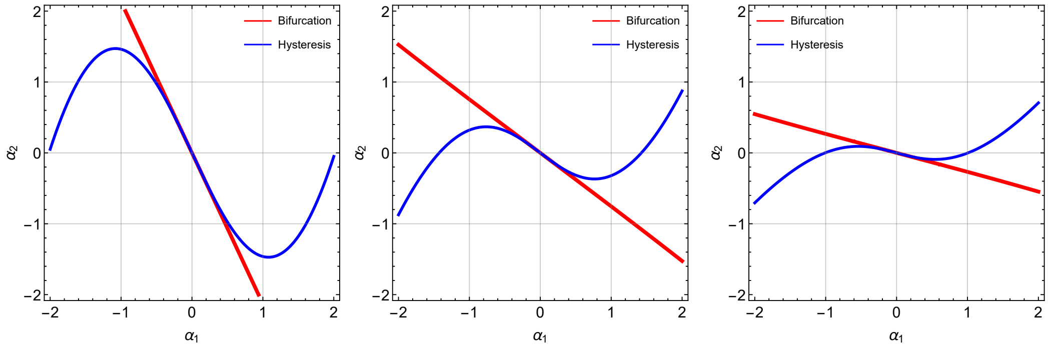
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.7: Approximations of B and H ($a_0 = 1, a_1 = 1, a_{i\geq 2} = 0$)



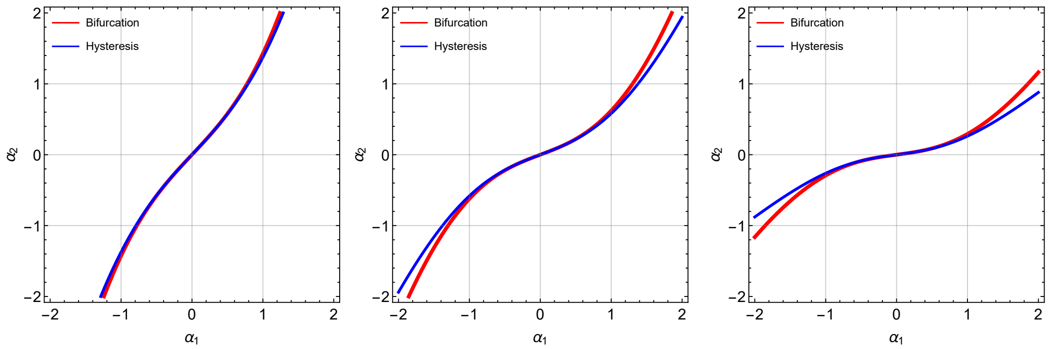
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.8: Approximations of B and H ($a_0 = -1, a_1 = 1, a_{i\geq 2} = 0$)



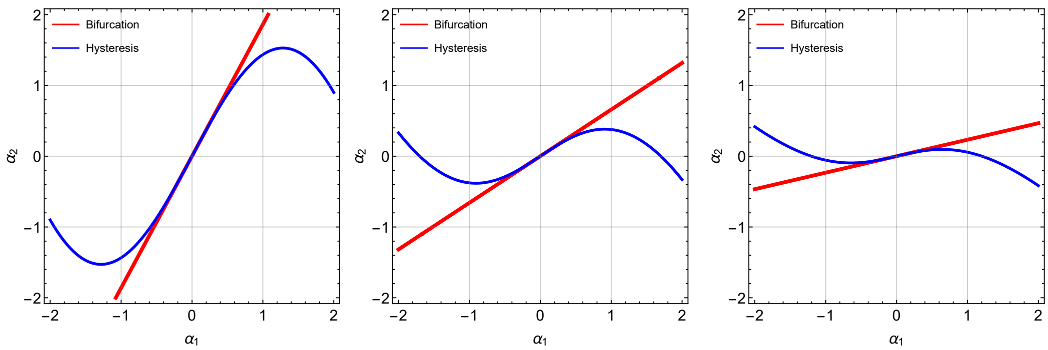
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.9: Approximations of B and H ($a_0 = 1, a_1 = -1, a_{i\geq 2} = 0$)



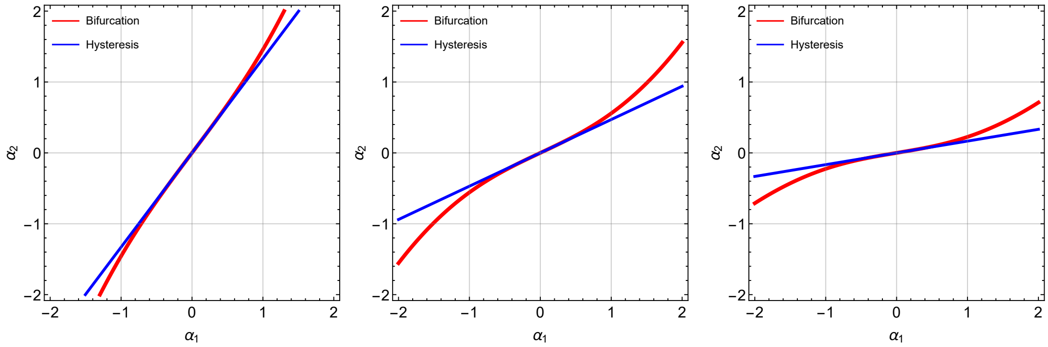
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.10: Approximations of B and H ($a_0 = -1, a_1 = -1, a_{i\geq 2} = 0$)



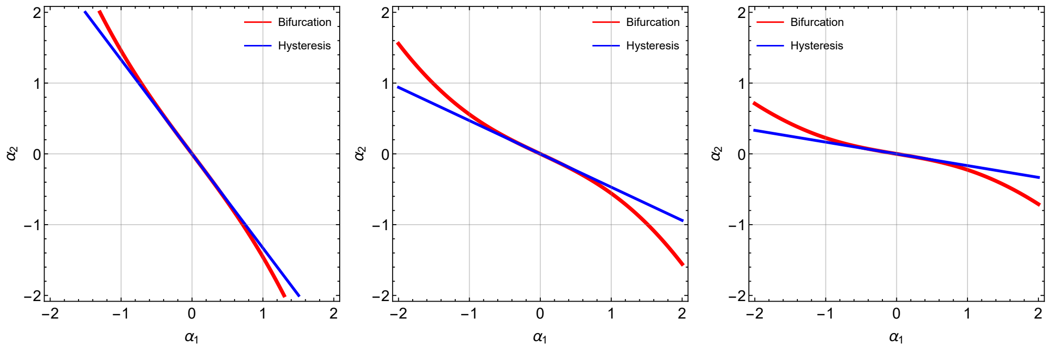
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.11: Approximations of B and H ($a_0 = -1, a_1 = 0.5, a_{i\geq 2} = 0$)



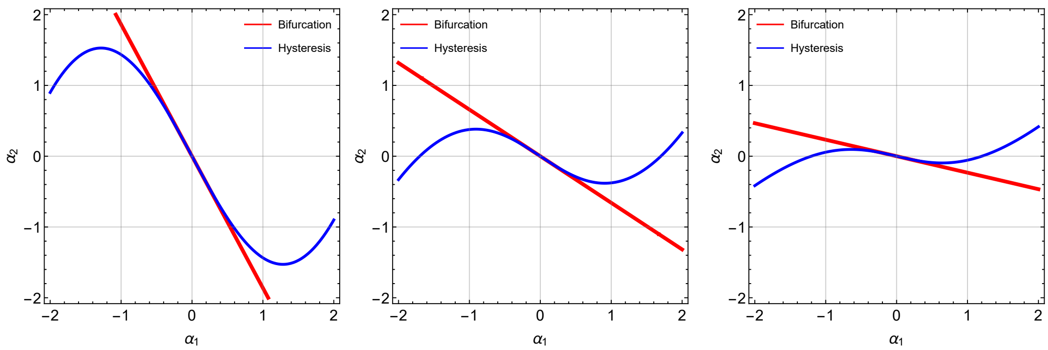
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.12: Approximations of B and H ($a_0 = -1, a_1 = -0.5, a_{i\geq 2} = 0$)



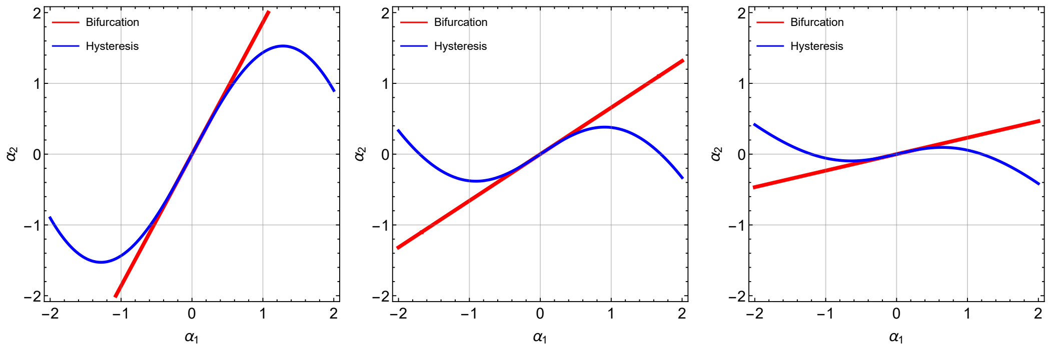
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.13: Approximations of B and H ($a_0 = 1, a_1 = 0.5, a_{i\geq 2} = 0$)



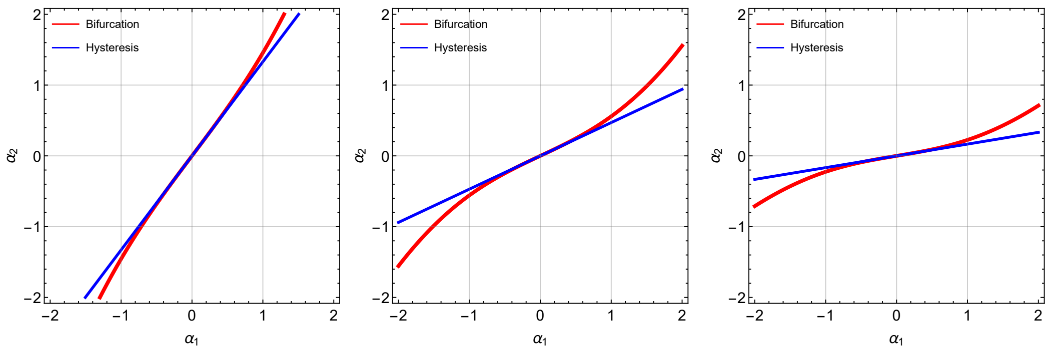
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.14: Approximations of B and H ($a_0 = 1, a_1 = -0.5, a_{i\geq 2} = 0$)



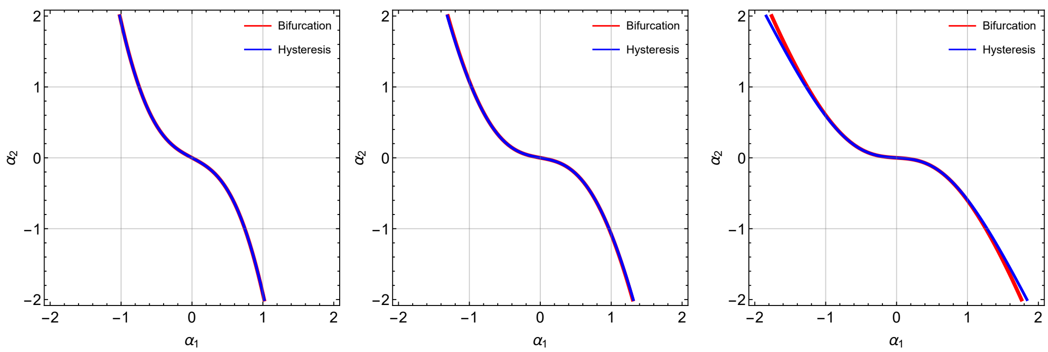
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.15: Approximations of B and H ($a_0 = -1, a_1 = 0.5, a_{i\geq 2} = 0$)



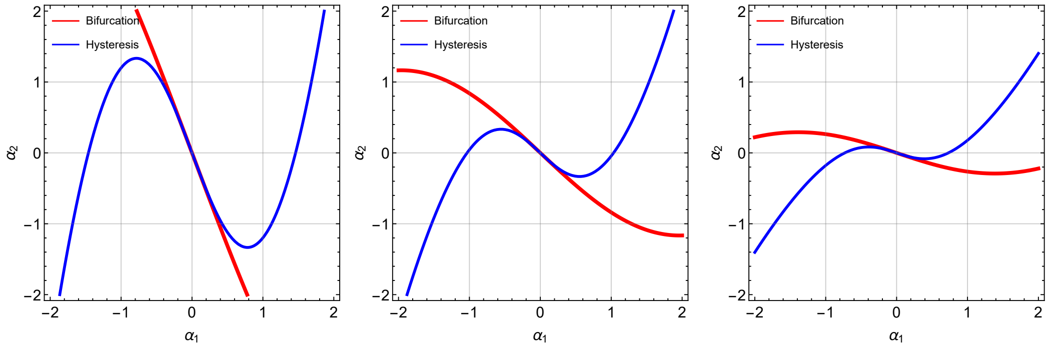
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.16: Approximations of B and H ($a_0 = -1, a_1 = -0.5, a_{i\geq 2} = 0$)



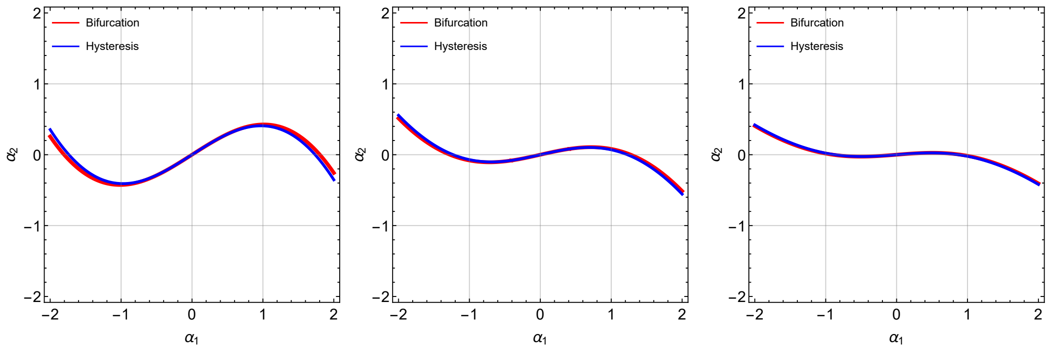
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.17: Approximations of B and H ($a_0 = 1, a_1 = 2, a_{i\geq 2} = 0$)



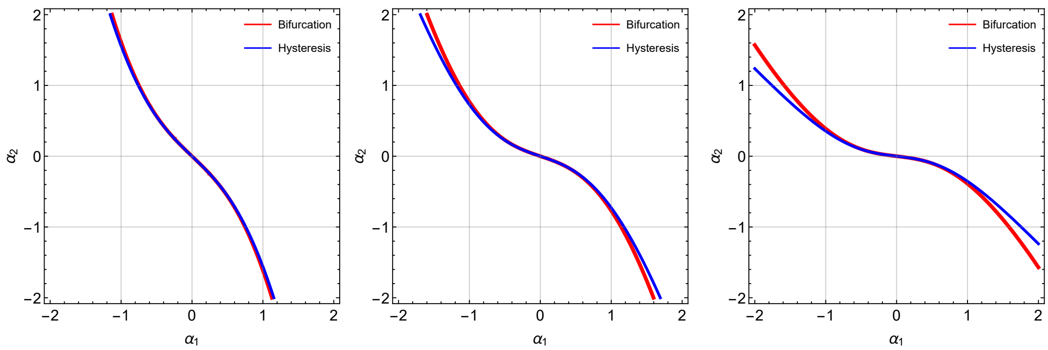
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.18: Approximations of B and H ($a_0 = 1, a_1 = -2, a_{i\geq 2} = 0$)



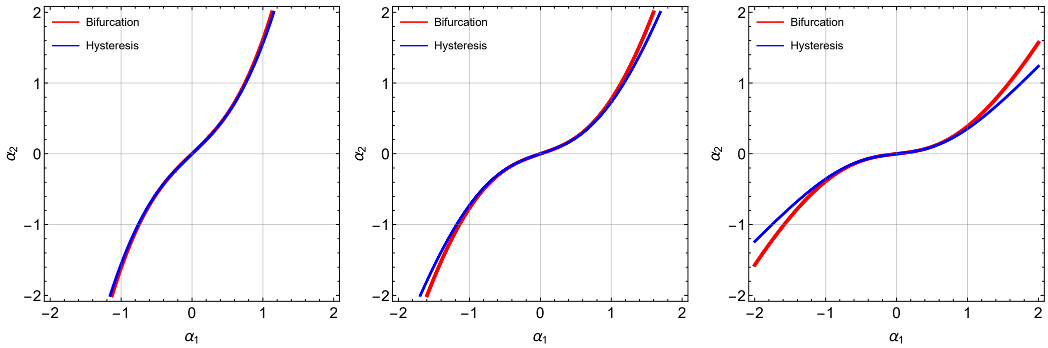
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.19: Approximations of B and H ($a_0 = 0, a_1 = 1, a_2 = 1, a_{i\geq 3} = 0$)



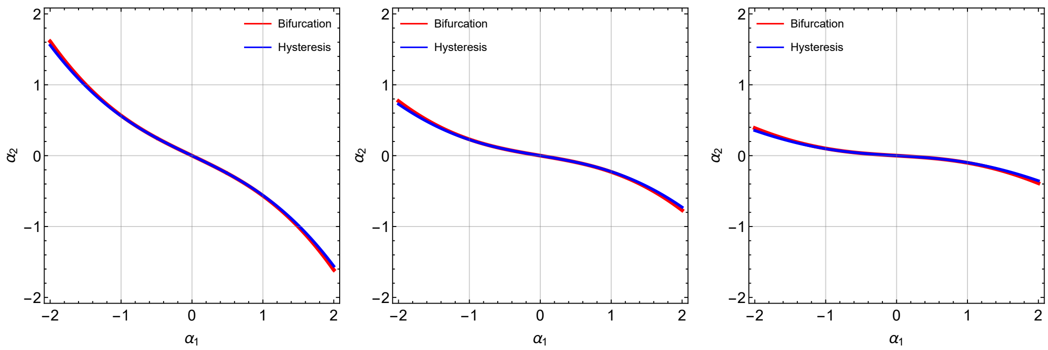
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.20: Approximations of B and H ($a_0 = 1, a_1 = 1, a_2 = 1, a_{i\geq 3} = 0$)



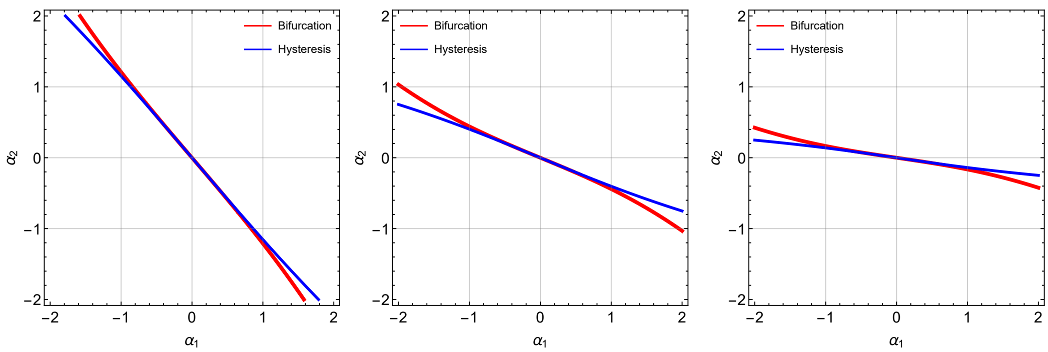
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.21: Approximations of B and H ($a_0 = -1, a_1 = -1, a_2 = -1, a_{i\geq 3} = 0$)



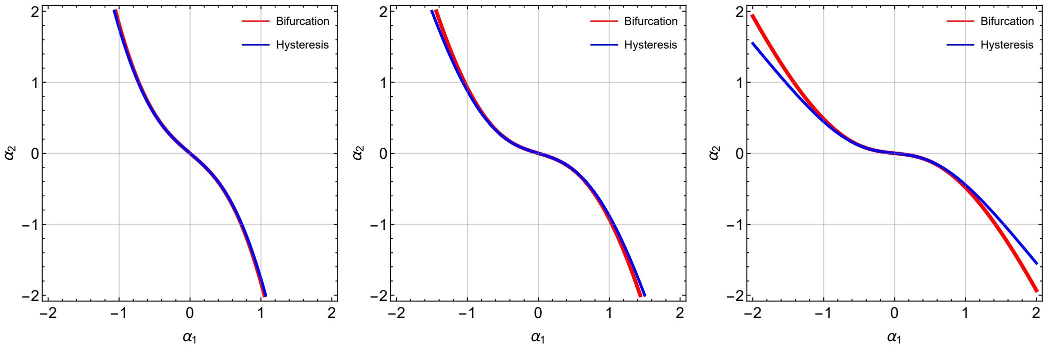
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.22: Approximations of B and H ($a_0 = 0.5, a_1 = 0.5, a_2 = 0.5, a_{i\geq 3} = 0$)



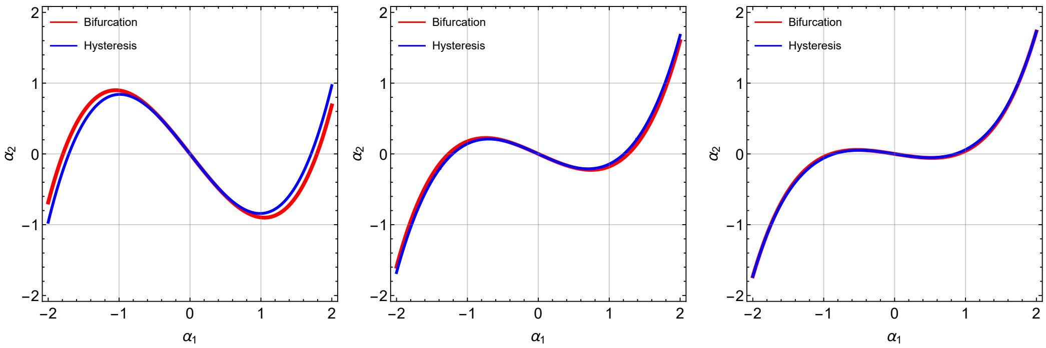
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.23: Approximations of B and H ($a_0 = 1, a_1 = 1, a_2 = -1, a_{i\geq 3} = 0$)



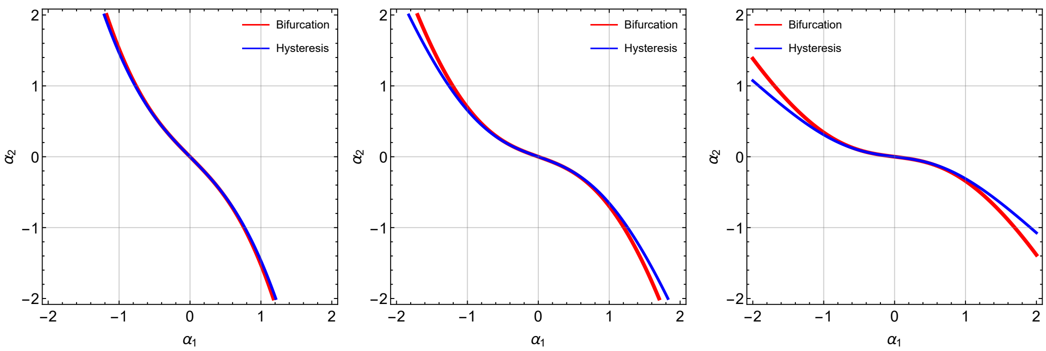
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.24: Approximations of B and H ($a_0 = 1, a_1 = 1, a_2 = 2, a_{i\geq 3} = 0$)



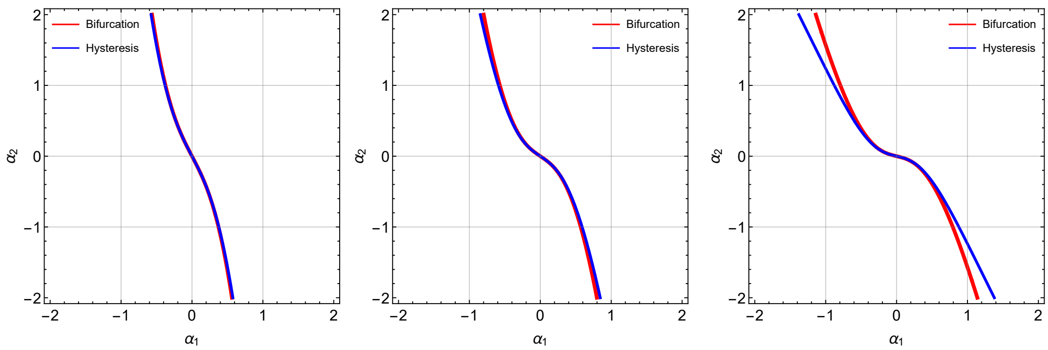
(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.25: Approximations of B and H ($a_0 = 1, a_1 = 1, a_2 = -2, a_{i\geq 3} = 0$)



(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.26: Approximations of B and H ($a_0 = 1, a_1 = 1, a_2 = 0.5, a_{i\geq 3} = 0$)



(a) Length $= \pi$ (b) Length $= 2\pi$ (c) Length $= 4\pi$

FIGURE 9.27: Approximations of B and H ($a_0 = 2, a_1 = 2, a_2 = 2, a_{i\geq 3} = 0$)

We often observe that the bifurcation set and the hysteresis set are close near the origin when the coefficients a_0, a_1 and a_2 are big.

Appendix A

Appendix

A.1 Inverse Function Theorem and Implicit Function Theorem

Inverse function theorem and implicit function theorem are very essential topic for analysing the Euler buckling problem. We recall these theorems as follows:

Theorem A.1.1. *(Inverse Function Theorem)[Ananlysis in $\mathbb{R}^n$] Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be C^1 . Let $x_0 \in U$ such that $Df(x_0)$ is nonsingular. Then there exists a neighborhood W of x_0 such that*
i. $f : W \to f(W)$ is a bijection;
ii. $f(W)$ is an open set in $\mathbb{R}^n$;
iii. $f^{-1} : f(W) \to W$ is C^1 and $Df^{-1}(f(x)) = (Df(x))^{-1}$ for $x \in W$.

Theorem A.1.2. *(Implicit Function Theorem)[Calculus2-international] Assume $F(x, y) \in C^1$ near (x_0, y_0) such that*
i. $F(x_0, y_0) = 0$,
ii. $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$.
Then there is a unique function $y = f(x) \in C^1$ in a neighborhood of x_0 such that
i. $y_0 = f(x_0)$;
ii. $F(x, f(x)) = 0 \; \forall$ near x_0;
iii. $f'(x) = -\frac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))}$.

A.2 Second variation of Energy equation of Euler buckling problem

MORSE-PALAIS theorem [A.2.1] give us the idea that the positivity of the second derivative of energy concludes the minmimality of the energy.

Theorem A.2.1. *[PALAIS, 1969, DINH and TUAN, 1973] (MORSE-PALAIS LEMMA). Let V be a real Banach space, $\mathcal{O}$ a convex neighborhood of the origin and left $f : \mathcal{O} \to \mathbb{R}$ be a C^{k+2} function $(k >= 1)$ having the origin as a nondegenerate* *critical point with $f(0) = 0$. Then there is a neighborhood U of the origin and a C^k diffeomorphism: $\Phi : U \to \mathcal{O}$ with $\Phi(0) = 0$ and $(D\Phi)_0 = id_v$ (the identity map of V) such that for x in U, $f(\Phi(x)) = \frac{1}{2}(D^2 f)_0(x, x)$.*

Here 0 is a nondgenerate critical point of f means that $(D^2 f)_0$ is an isomorphism of V onto V^* and $(Df)_0 = 0$.

Energy equation of Euler buckling problem is defined by

$$E(u, \lambda, \alpha) = \frac{1}{2}\int_0^l \Big(\frac{u''}{(1-(u')^2)^{1/2}} - \alpha_1\kappa\Big)^2 ds + \lambda \int_0^l \sqrt{1-(u')^2} ds + \alpha_2 u(l/2), \tag{A.2.1}$$

where $U = \{u \in X : \|u'\|_\infty \leq 1 - \epsilon\}$, $0 < \epsilon \ll 1$, and X is the Sobolev space

$$X = \{u \in H^2[0, l] : u(0) = u(l) = 0\}.$$

Where κ is a function defined by

$$\kappa = \frac{1}{\sqrt{l/2}}\Big[a_0 + \sum_{i=1}^{\infty}\Big(a_i \cos\frac{2i\pi s}{l} + b_i \sin\frac{2i\pi s}{l}\Big)\Big] \tag{A.2.2}$$

with $\|\kappa_0\|_\infty < \infty$. Since $u \in H^2[0, l]$, we can choose that u is C^1, and there is a constant ε_1, $0 < \epsilon_1 \ll 1$, so that $|u'(s)| < 1 - \varepsilon_1$.

We see as in [3.4.1] first variation of Energy of Euler buckling problem is as follows

$$\begin{aligned}(dE)_{(u,\lambda,\alpha)}[v] &= \lim_{t\to 0}\frac{1}{t}\Big[E(u + tv, \lambda, \alpha) - E(u, \lambda, \alpha)\Big] \\ &= (d\Psi)[v] - \lambda(d\Lambda)[v] - \alpha_1(dK)[v] + \alpha_2[v](\tfrac{l}{2}),\end{aligned}$$

where

$$\begin{aligned} d\Psi[v_1] &= \int_0^l \frac{u'v_1'u''^2}{(1-u'^2)^2} ds + \int_0^l \frac{u''v_1''}{(1-u'^2)} ds, \\ d\Lambda[v_1] &= \int_0^l \frac{u'v_1'}{(1-u'^2)^{1/2}} ds, \\ dK[v_1] &= \int_0^l \kappa\frac{u''u'v_1'}{(1-u'^2)} ds - \int_0^l \kappa\frac{v_1''}{(1-u'^2)^{1/2}} ds. \end{aligned}$$

Second variation of Energy of Euler buckling problem is as follows

Lemma A.2.2.

$$(d^2E)_{(u,\lambda,\alpha)}[v,w] = \int_0^l \frac{w''v''}{1-u'^2}ds + \int_0^l \frac{2u'v'u''w'' + v'w'u''^2 + 2u''v''u'w'}{(1-u'^2)^2}ds +$$
$$\int_0^l \frac{4u'^2v'u''^2w' - 4u'^4v'u''^2w'}{(1-u'^2)^4}ds + \lambda\int_0^l \frac{w'v'}{(1-u'^2)^{3/2}}ds -$$
$$\int_0^l \kappa\alpha_1 \frac{u''v'w' + w''v'u'}{(1-u'^2)}ds - \int_0^l \alpha_1\kappa\frac{2u''u'^2v'w'}{(1-u'^2)^2}ds - \int_0^l \alpha_1\kappa\frac{v''u'w'}{(1-u'^2)^{3/2}}ds.$$

Proof.

$$(d^2E)_{(u,\lambda,\alpha)}[v,w] = \lim_{t\to 0}\frac{1}{t}\Big[dE(u+tw,\lambda,\alpha)[v] - dE(u,\lambda,\alpha)[v]\Big]$$
$$= \lim_{t\to 0}\frac{1}{t}\Big[\int_0^l \frac{(u+tw)'v_1'(u+tw)''^2}{(1-(u+tw)'^2)^2}ds + \int_0^l \frac{(u+tw)''v''}{(1-(u+tw)'^2)}ds$$
$$-\int_0^l \alpha_1\kappa\frac{(u+tw)''(u+tw)'v'}{(1-(u+tw)'^2)}ds - \int_0^l \alpha_1\kappa\frac{v''}{(1-(u+tw)'^2)^{1/2}}ds +$$
$$\lambda\int_0^l \frac{(u+tw)'v'}{(1-(u+tw)'^2)^{1/2}}ds - \int_0^l \frac{u'v'u''^2}{(1-u'^2)^2}ds - \int_0^l \frac{u''v''}{(1-u'^2)}ds + \int_0^l \alpha_1\kappa\frac{u''u'v'}{(1-u'^2)}ds$$
$$+\int_0^l \alpha_1\kappa\frac{v''}{(1-u'^2)^{1/2}}ds - \lambda\int_0^l \frac{u'v'}{(1-u'^2)^{1/2}}ds\Big]$$
$$= d^2\Psi[v,w] + \lambda d^2\Lambda[v,w] + \alpha_1 d^2K[v,w],$$

where

$$d^2\Psi[v,w] = \lim_{t\to 0}\frac{1}{t}\Big[\int_0^l \frac{(u+tw)'v'(u+tw)''^2}{(1-(u+tw)'^2)^2}ds + \int_0^l \frac{(u+tw)''v''}{(1-(u+tw)'^2)}ds$$
$$-\int_0^l \frac{u'v'u''^2}{(1-u'^2)^2}ds - \int_0^l \frac{u''v''}{(1-u'^2)}ds\Big]$$
$$= \lim_{t\to 0}\frac{1}{t}\Big[\int_0^l \frac{u'v'u''^2 + 2tu'v'u''w'' + t^2u'v'w''^2 + tv'w'u''^2 + 2t^2u''w''v'w' + t^3v'w'w''^2}{(1-(u+tw)'^2)^2}ds +$$
$$\int_0^l \frac{u''v'' + tw''v''}{(1-(u+tw)'^2)}ds - \int_0^l \frac{u'v'u''^2}{(1-u'^2)^2}ds - \int_0^l \frac{u''v''}{(1-u'^2)}ds\Big]$$
$$= \int_0^l \frac{2u'v'u''w'' + v'w'u''^2}{(1-u'^2)^2}ds + \int_0^l \frac{w''v''}{1-u'^2}ds +$$
$$\lim_{t\to 0}\frac{1}{t}\Big[\int_0^l \Big[(u'v'u''^2(1-2u'^2+u'^4-1+2u'^2+4tu'w'+2t^2v'^2-u'^4-4tu'^3w'-$$
$$6t^2u'^2w'^2 - 4t^3u'w'^3 - t^4w'^4))/((1-(u'+tw')^2)^2(1-u'^2)^2)ds$$
$$+\int_0^l \frac{u''v''(1-u'^2-1+u'^2+2tu'w'+t^2w'^2)}{(1-u'^2)(1-(u'+tw')^2)}ds\Big]$$
$$= \int_0^l \frac{2u'v'u''w'' + v'w'u''^2}{(1-u'^2)^2}ds + \int_0^l \frac{w''v''}{1-u'^2}ds + \int_0^l \frac{4u'^2v'u''^2w' - 4u'^4v'u''^2w'}{(1-u'^2)^4}ds$$

$$+\int_0^l \frac{2u''v''u'w'}{(1-u'^2)^2}ds$$
$$=\int_0^l \frac{w''v''}{1-u'^2}ds+\int_0^l \frac{2u'v'u''w''+v'w'u''^2+2u''v''u'w'}{(1-u'^2)^2}ds+\int_0^l \frac{4u'^2v'u''^2w'-4u'^4v'u''^2w'}{(1-u'^2)^4}ds,$$
$$d^2\Lambda[v_1,v_2]=\lim_{t\to 0}\frac{1}{t}\int_0^l \frac{(u+tw)'v'}{(1-(u+tw)'^2)^{1/2}}ds-\int_0^l \frac{u'v'}{(1-u'^2)^{1/2}}ds$$
$$=\int_0^l \frac{w'v'}{(1-u'^2)^{1/2}}ds+\lim_{t\to 0}\frac{1}{t}\int_0^l \frac{u'v'[(1-u'^2)^{1/2}-(1-(u'+tw')^2)^{1/2}]}{(1-u'^2)^{1/2}(1-(u+tw)'^2)^{1/2}}ds$$
$$=\int_0^l \frac{w'v'}{(1-u'^2)^{1/2}}ds+\int_0^l \frac{u'^2w'v'}{(1-u'^2)^{3/2}}ds$$
$$=\int_0^l \frac{w'v'}{(1-u'^2)^{3/2}}ds,$$
$$d^2K[v,w]=\lim_{t\to 0}\frac{1}{t}\kappa\Big[\int_0^l \frac{(u+tw)''(u+tw)'v'}{(1-(u+tw)'^2)}ds+\int_0^l \frac{v''}{(1-(u+tw)'^2)^{1/2}}ds$$
$$-\int_0^l \frac{u''u'v'}{(1-u'^2)}ds-\int_0^l \frac{v''}{(1-u'^2)^{1/2}}ds\Big]$$
$$=\lim_{t\to 0}\frac{1}{t}\Big[\int_0^l \kappa\frac{v''((1-u'^2)^{1/2}-(1-(u'+tw')^2)^{1/2})}{(1-u'^2)^{1/2}(1-(u'+tw')^2)^{1/2}}ds+$$
$$\int_0^l \kappa\frac{u'v'u''+tu'v'w''+tw'v'u''+t^2w'v'w''}{(1-(u'+tw')^2)}ds-\int_0^l \kappa\frac{u'u''v'}{(1-u'^2)}ds\Big]$$
$$=\lim_{t\to 0}\frac{1}{t}\Big[\int_0^l \kappa v''\Big[\frac{1-u'^2-1+u'^2+2tu'w'+t^2w'^2}{(1-u'^2)^{1/2}(1-(u'+tw')^2)^{1/2}[(1-u'^2)^{1/2}+(1-(u'+tw')^2)^{1/2}]}\Big]ds\Big]$$
$$+\int_0^l \kappa\frac{u''v'w'+w''v'u'}{(1-u'^2)}ds+\int_0^l \kappa\frac{2u''u'^2v'w'}{(1-u'^2)^2}ds$$
$$=\int_0^l \kappa\frac{u''v'w'+w''v'u'}{(1-u'^2)}ds+\int_0^l \kappa\frac{2u''u'^2v'w'}{(1-u'^2)^2}ds+\int_0^l \kappa\frac{v''u'w'}{(1-u'^2)^{3/2}}ds.$$

□

From the first derivative of E with respect to u are zero, we will get the equilibrium points which are the critical points. If E has a local minimum then we say the equilibrium is stable. Otherwise we say the solution is unstable

It is important to know whether the solution of $dE=0$ is stable or unstable. But it seems to be difficult to conclude stability or unstability using the second variation formula above. Instead of the second variation formula we present very elementary consideration on minimality of this critical point. We consider the graph of the energy E in (x,λ,E) space. The pitchfork bifurcation for stable solution looks like the following figure [A.1]:

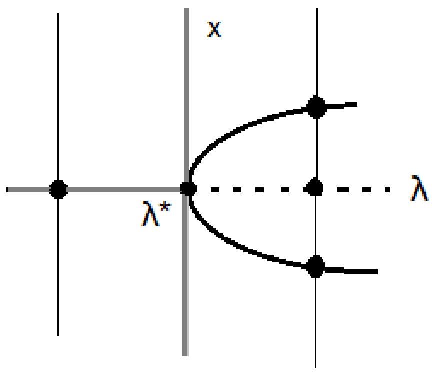


FIGURE A.1: Pitchfork bifurcation diagram.

From the figure [A.1], we see the stable solutions which are as the bold solid line and as the dotted line represents the unstable solutions. For $\lambda < \lambda^*$, the graph of E looks like the following figure [A.2]:

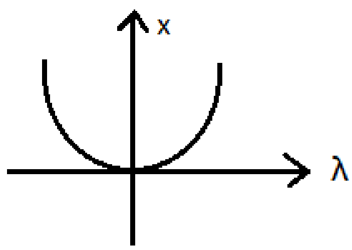


FIGURE A.2: Minimum stable for $\lambda < \lambda^*$.

For $\lambda > \lambda^*$, the graph of E looks like the following figure [A.3]:

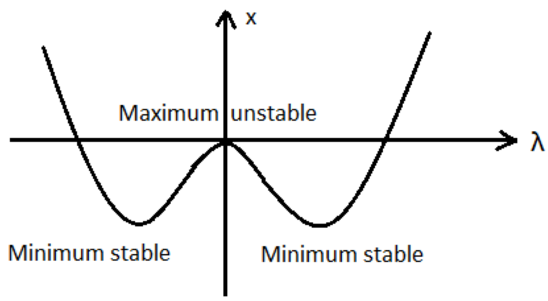


FIGURE A.3: Minimum stable and maximum unstable for $\lambda > \lambda^*$.

In figure [A.3], we get two local minimum points. Taking as imperfection, we obtain the following bifurcation diagram for example.

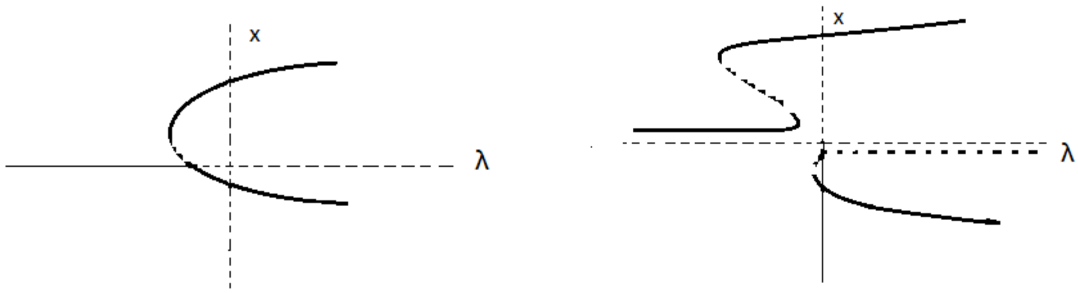


FIGURE A.4: Diagram of bifurcation and imperfect bifurcation.

From Figure [A.4], the first figure is the transcritical bifurcation and the second figure is imperfect transcritical bifurcation which is a small perturbation with a disconnected diagram. It is important problem to decide a solution which attains the minimum in the above diagram.

A.3 Reduction Method of Lyapunov-Schmidt

At the begening of the chapter 6, we use Lyapunov-Schmidt reduction for discussing the bifurcation equation which discussed in the next chapter. So, here we recall the Lyapunov-Schmidt reduction theorem as follows:

Theorem A.3.1 (Kielhöfer, 2004, Theorem I.2.3)**.** *There is a neighborhood $U_2 \times V_2$ of (x_0, y_0) in $U \times V \subset X \times Y$ such that the problem*

$$F(x,y) = 0 \ \textit{for} \ \ (x,y) \in U_2 \times V_2 \tag{A.3.1}$$

is equivalent to a finite-dimensional problem.

$$\begin{aligned}&\Phi(v,y) = 0 \ \textit{for} \ \ (v,y) \in \tilde{U}_2 \times V_2 \subset N \times Y, \ \ \textit{where} \ \ \Phi : \tilde{U}_2 \times V_2 \to Z_0 \ \ \textit{continuous}\\ &\textit{and} \ \ \Phi(v_0,y_0) = 0, \ \ (v_0,y_0) \in \tilde{U}_2 \times V_2.\end{aligned} \tag{A.3.2}$$

The function Φ, called a bifurcation function is given in (A.3.6) below. (If the parameter space Y is finite-dimensional, then (A.3.2) is indeed a purely finite-dimensional problem)

Proof. Problem (A.3.1) is obviously equivalent to the system

$$\begin{aligned}&QF(Px + (I-P)x, y) = 0,\\ &(I-Q)F(Px + (I-P)x, y) = 0,\end{aligned} \tag{A.3.3}$$

where we set $Px = v \in N$ and $(I-P)x = w \in X_0$. Next we define

$$\begin{aligned}&G : \tilde{U}_2 \times W_2 \times V_2 \to R \ \text{ via}\\ &G(v,w,y) \equiv (I-Q)F(v+w,y), \ \text{ where}\\ &v_0 = Px_0 \in \tilde{U}_2 \subset N,\\ &w_0 = (I-P)x_0 \in W_2 \subset X_0,\\ &\text{and} \ \ \tilde{U}_2, W_2 \ \text{ are neighborhood such that} \ \ \tilde{U}_2 + W_2 \subset U \subset X.\end{aligned} \tag{A.3.4}$$

We have $G(v_0, w_0, y_0) = 0$, and by the choice of the spaces, $D_w G(v_0, w_0, y_0) = (I - Q) D_x F(x_0, y_0) : X_0 \to R$ is bijective. Application of the Implicit Function Theorem then yields

$$\begin{aligned} &G(v, w, y) = 0 \ \text{ for } \ (v, w, y) \in \tilde{U}_2 \times W_2 \times V_2 \ \text{ is equivalent to} \\ &w = \psi(v, y) \ \text{ for some} \psi : \tilde{U}_2 \times V_2 \to W_2 \subset X_0 \ \text{ such that} \\ &\psi(v_0, y_0) = w_0. \end{aligned} \tag{A.3.5}$$

Insertion of the function ψ into $(A.3.3)_1$ yields

$$\Phi(x, y) \equiv QF(v + \psi(v, y), y) = 0 \tag{A.3.6}$$

□

Bibliography

Boller, John. *Ananlysis in* $\mathbb{R}^n$. Available at http://www.math.uchicago.edu/$\sim$boller/M2031 (2008).

Brezis, Haim (2010). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer.

Censor, Dr. Aviv. *Calculus2-international*. Available at Technion-International School of engineeringl (20015/12).

DINH, DANG and VU TRONG TUAN (1973). "AN ELEMENTARY PROOF OF THE MORSE-PALAIS LEMMA FOR BANACH SPACES". In: *AMERICAN MATHEMATICAL SOCIETY* 39.3, pp. 642–644.

Golubitsky, M. and D. Schaeffer (1979). "A theory for imperfect bifurcation via singularity theory". In: *Comm. Pure Appl. Math.* 32.12, pp. 21–98.

Golubitsky, M. and D. G. Schaeffer (1985). *Singularities and Groups in Bifurcation Theory, I*. Springer-Verlag.

— (1988). *Singularities and Groups in Bifurcation Theory, II*. Springer-Verlag.

Kielhöfer, Hansjörj (2004). *Bifurcation Theory An Introduction with Applications to PDEs*. Applied Mathematical Sciences 156, Springer.

Marsden, Jerrold E. and Thomas J.R. Hughes (1994). *Mathematical Foundations of Elasticity*. Courier Corporation ISBN-13:978-0-486-67865-8,1983,

PALAIS, RICHARD S. (Apr. 1969). "THE MORSE LEMMA FOR BANACH SPACES". In: *Bull. Amer. Math.Soc.* 75, pp. 968–971.

Ralph Abraham Jerrold E., Marsden and Ratiu Tudor S. (1988). *Manifolds, tensor analysis, and applications, second edition*. Applied Mathematical Sciences 75, Springer.

Thompson, J. M. T. and G. W. Hunt (1973). *A General Theory of Elastic Stability*. London: Wiley.