Non-Fredholmness for A Class of Pseudo-Differential Operators Acting on Besov Spaces

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Summary

In this paper we would prove that a class of pseudo-differential operators cannot be a Fredholm operator acting on suitable Besov spaces.

Keywords: Fredholmness, Fredholm operator, Besov space, pseudo-differential operator.

1. Introduction

This paper treats the problem of Fredholmness for a class of pseudo-differential operators. We are very interested in the issue about when the given pseudo-differential operator becomes a Fredholm operator. Actually, we are eager to investigate whether the operator in question is of Fredholm or not. The purpose of this article is to discuss the problem under what condition the operator in question is a Fredholm operator. In particular, we discuss a sufficient condition for the operator to be of non-Fredholm type. As a matter of fact, we prove in this paper that a class of pseudo-differential operators cannot be a Fredholm operator acting on suitable Besov spaces under some reasonable conditions. Theses peculiar features can be realized by key technical lemmas and some remarkable properties of Besov spaces. This article is organized as follows. The rest of this first section consists in introducing some basic notations used through this article. In Section 2, the notions of some useful mathematical tools and the definition of Besov spaces are introduced, and some valuable properties of Besov spaces are discussed in details. Section 3 is devoted to the main result of this paper, where we introduce and discuss a principal theorem for a certain class of pseudo-differential operators. We clarify the sufficient conditions for the operators in question acting on suitable Besov spaces not to be a Fredholm operator. Section 4 deals with some preliminary results whereby we can prove our main theorem. Lastly, we are going to give a proof of the main theorem in Section 5.

An element $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathbb{Z}_+^n$ is called a multi-index and the length of α is given by $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \tag{1}$$

$$D_x = \frac{1}{i}\partial_x, \quad D_{x_j} = \frac{1}{i}\partial_{x_j}, \quad \text{with} \quad i = \sqrt{-1}, \qquad D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}. \tag{2}$$

If $X \subset \mathbb{R}^n$ is open, for $k \in \mathbb{N}$, we let $C^k(X)$ denote the Fréchet space of k times continuously differentiable functions $X \to \mathbb{C}$. For $k = 0$, we get the space $C(X)$ of continuous complex-valued functions on X. We let

$$C^\infty(X) = \bigcap_{k\in\mathbb{N}} C^k(X) \tag{3}$$

be the Fréchet space of infinitely continuously differentiable functions. If I is a subset of $\mathbb{R}$, then $C^k(X; I)$ is the set of functions in $C^k(X)$ taking their values in I. For $u \in C^k(X)$, the support of u is the smallest closed subset F of X outside which u vanishes identically, and is denoted by $\operatorname{supp} u$. That is to say,

$$\operatorname{supp} u = \{\, x \in X : \quad u(x) = 0 \,\}^{-C} \tag{4}$$

where the superscript $-C$ means to take the closure of the set. For $k \in \mathbb{N} \cup \{\infty\}$,

$$C_0^k(X) = \{\, u \in C^k(X) : \quad \operatorname{supp} u \quad \text{is compact}\,\}. \tag{5}$$

Let $X \subset \mathbb{R}^n$ be an open set, and let $0 \leqslant \rho \leqslant 1$, $0 \leqslant \delta \leqslant 1$, $m \in \mathbb{R}$ and $N \in \mathbb{N} \setminus \{0\}$. Then, $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ is the space of all $a \in C^\infty(X \times \mathbb{R}^N)$ such that for all compact subsets $K \subset\subset X$ and all $\alpha \in \mathbb{Z}^n_+$, $\beta \in \mathbb{Z}^N_+$, there is a constant $C = C_{K,\alpha,\beta}(a) > 0$ such that

$$|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \leqslant C(1 + |\theta|)^{m-\rho|\beta|+\delta|\alpha|}, \qquad (x,\theta) \in K \times \mathbb{R}^N. \tag{6}$$

We say that $S^m_{\rho,\delta}$ is the space of symbols of order m and of type (ρ,δ). We observe that $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ is a Fréchet vector space with the seminorms:

$$P_{K,\alpha,\beta}(a) = \sup_{(x,\theta)\in K\times\mathbb{R}^N} \frac{|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)|}{(1+|\theta|)^{m-\rho|\beta|+\delta|\alpha|}} \tag{7}$$

for K compact in X, $\alpha \in \mathbb{Z}^n_+$ and $\beta \in \mathbb{Z}^N_+$, cf. [18] (see also [27]).

2. Bezov space and preliminary

In what follows, just for simplicity, we often make use of the symbol $\Xi(a; m)$ which means the power sign a^m. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$ with the Euclidean norm $|\xi|$ for $\xi \in \mathbb{R}^n$. Next we shall define the Fourier transform $\mathcal{F}$ of u as

$$\mathcal{F}u(\xi) = \hat{u}(\xi) := \int e^{-ix\cdot\xi} u(x)\mathrm{d}x \qquad \text{with} \quad i = \sqrt{-1} \tag{8}$$

for both $u \in \mathcal{S}(\mathbb{R}^n)$ and its extension to $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class on $\mathbb{R}^n$, namely, the space of smooth functions $u(x)$ being rapidly decreasing, and $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of $\mathcal{S}(\mathbb{R}^n)$, called the space of tempered distributions on $\mathbb{R}^n$. Notice that the inverse transform $\mathcal{F}^{-1}$ of $\mathcal{F}$ is given by

$$\mathcal{F}^{-1}f(x) = \check{f}(x) := (2\pi)^{-n} \int e^{i\xi\cdot x} f(\xi)\mathrm{d}\xi \tag{9}$$

for the element f in $\mathcal{S}(\mathbb{R}^n)$, cf. [20].

Let us fix a function $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\text{supp}(\varphi_0) \subset \{\xi \in \mathbb{R}^n : \ |\xi| < 2\} \qquad \text{and} \qquad \varphi_0(\xi) = 1 \quad \forall\, \xi \in U(B_0(1)) \tag{10}$$

(which is a neighborhood of the unit ball). We define functions $\varphi_j \in C_0^\infty({}^B R^n)$, $j \geq 1$, by

$$\varphi_j(\xi) := \varphi_0(\Xi(2; -j)\xi) - \varphi_0(\Xi(2; -j+1)\xi)$$

and the sets

$$K_0 := \{\xi \in \mathbb{R}^n : \ |\xi| \leqslant 2\} \qquad \text{and} \qquad K_j := \{\xi \in \mathbb{R}^n : \ \Xi(2; j-1) \leqslant |\xi| \leqslant \Xi(2; j+1)\}, \quad \forall\, j \geq 1.$$

The sequence of functions $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is called a dyadic partition of unity [17]. Associated to a dyadic partition of unity, we can always define operators

$$\varphi_j(D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

by

$$\varphi_j(D)u = Op(\varphi_j)u = \mathcal{F}^{-1}(\varphi_j \mathcal{F}(u)).$$

Let E and F be Banach spaces. We denote by the symbol $\mathcal{B}(E, F)$ the totality of bounded linear operators from E to F. For $T \in \mathcal{B}(E, F)$, T^* is an adjoint operator of T, namely, $T^* : F \to E$ when $T : E \to F$. Then the kernel of the operator T is defined by

$$\mathcal{N}(T) \equiv Ker\, T := \{\, u \in E : \quad Tu = 0 \quad \text{in} \quad F\,\}. \tag{11}$$

The set $\mathcal{N}(T)$ is a closed subspace of E. If T satisfies the following three conditions:
(C1) $\alpha(T)* = \dim \mathcal{N}(T) < \infty$.
(C2) $\mathcal{R}(T) \equiv Im(T)$ is a closed subspace of F, that is, $\mathcal{R}(T) \subset F$.
(C3) $\beta(T) := \dim \mathcal{N}(T^*) < \infty$,
then T is said to be a Fredholm operator or T is said to be of Fredholm type.

Now we are in a position to introduce the Besov space, which plays an important role in this paper. Let $s \in \mathbb{R}$ and $1 < p, q < \infty$. The Besov space $B^s_{pq}(\mathbb{R}^n)$ is the space of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{B^s_{pq}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} \Xi(2; jsq) \|\varphi_j(D)u\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q} < \infty. \tag{12}$$

It is a Banach space with norm $\|\cdot\|_{B^s_{pq}(\mathbb{R}^n)}$. The Besov spaces admits the following perculiar properties [25]. For an approximation method, the set $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ and $1 < p, q < \infty$. As for the duality, the dual of $B^s_{pq}(\mathbb{R}^n)$ can be identified with $B^{-s}_{p'q'}(\mathbb{R}^n)$ using the dual pair

$$(f, g) := \int_{\mathbb{R}^n} f(x)g(x)dx, \qquad \forall\, f, g \in \mathcal{S}(\mathbb{R}^n).$$

As to the interpolation,

$$(B^{s_1}_{pq_1}(\mathbb{R}^n), B^{s_2}_{pq_2}(\mathbb{R}^n))_{\theta,q} = B^s_{pq}(\mathbb{R}^n), \qquad \forall\, 1 < p, q, q_1, q_2 < \infty \quad \text{and} \quad s = (1-\theta)s_1 + \theta s_2$$

holds.

3. Main result

In this section we shall introduce the main result of this article, which asserts that the pseudo-differential operator $A = Op(a)$ with symbol $a = a(x,\xi) \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ cannot be a Fredholm operator acting on suitable Besov spaces. As a matter of fact, it is the Besov space $B^0_{pq}(\mathbb{R}^n)$ for any $1 < p, q < \infty$ of this type that we are going to treat below.

LEMMA 1. *For each $1 < p, q < \infty$, there exists a positive constant $C = C(p,q) > 0$ such that*

$$\frac{1}{C(p,q)}\|u\|_{B^0_{pq}(\mathbb{R}^n)} \leqslant \|u\|_{L^p(\mathbb{R}^n)} \leqslant C(p,q)\|u\|_{B^0_{pq}(\mathbb{R}^n)} \tag{13}$$

holds, whenever $u \in \mathcal{S}(\mathbb{R}^n)$ is a function with

$$\text{supp}(\mathcal{F}(u)) \subset \bigcup_{k=m}^{m+2} K_k, \qquad \exists m \in \mathbb{N}_0,$$

where the sets $\{K_j\}_{j\in\mathbb{N}_0}$ are series of dyadic partitions of unity. Note that the constants C do not depend upon m.

Proof. First of all, note that if $u \in \mathcal{S}(\mathbb{R}^n)$ is such that

$$\mathcal{F}(u)(\xi) = \sum_{j=m-1}^{m+3} \varphi_j(\xi)\mathcal{F}(u)(\xi),$$

using the convention that $\varphi_{-1}(\xi) = 0$. Hence it is easy to see that

$$u = \sum_{j=m-1}^{m+3} \varphi_j(\xi)\varphi_j(D)u \qquad \text{and} \qquad \|u\|_{B^0_{pq}(\mathbb{R}^n)} = \left(\sum_{j=m-1}^{m+3} \|\varphi_j(D)u\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}. \tag{14}$$

Due to the equivalence of norms in finite dimensional vector spaces, together with Young's inequality, the results yields form the following estimate

$$\|u\|_{L^p(\mathbb{R}^n)} = \left\|\sum_{j=m-1}^{m+3} \varphi_j(D)u\right\|_{L^p(\mathbb{R}-n)} \leqslant C_1\left(\sum_{j=m-1}^{m+3} |\varphi_j(D)u\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} \tag{15}$$

$$\leqslant C_2 \sum_{j=m-1}^{m+3} \|\varphi_j(D)u\|_{L^p(\mathbb{R}^n)} \leqslant \|u\|_{L^p(\mathbb{R}^n)}. \tag{16}$$

□

LEMMA 2. *If* $1 < p, q < \infty$, *then for each* $\theta \in (0,1)$, *there exists a positive constant* $C_\theta > 0$ *such that*

$$\|u\|_{B^0_{pq}(\mathbb{R}^n)} \leqslant C_\theta \, \|u\|^{1-\theta}_{L^p(\mathbb{R}^n)} \, \|u\|_{H^{1,p}(\mathbb{R}^n)}, \qquad \forall\, u \in \mathcal{S}(\mathbb{R}^n) \tag{17}$$

holds.

Proof. The following estimates with the exponents θ and $1-\theta$ follow from usual results of th interpolation theory. Indeed, the fact that

$$(L^p(\mathbb{R}^n), H^1_p(\mathbb{R}^n))_{\theta,q} = B^0_{pq}(\mathbb{R}^n)$$

implies that

$$\|u\|_{B^0_{pq}(\mathbb{R}^n)} \leqslant \|u\|_{B^\theta_{pq}(\mathbb{R}^n)} \leqslant C_\theta \|u\|^{1-\theta}_{L^p(\mathbb{R}^n)} \|u\|^\theta_{H^1_p(\mathbb{R}^n)}, \qquad \forall\, u \in \mathcal{S}(\mathbb{R}^n). \tag{18}$$

□

THEOREM 3. *Assume that* $a = a(x,\xi) \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. *If there exists a sequence*

$$\{(y_k, \eta_k)\}_{k \in \mathbb{N}_0} \subset \mathbb{R}^n \times \mathbb{R}^n$$

such that

$$\lim_{k\to\infty} |\eta_k| = \infty \qquad \text{ans} \qquad \lim_{k\to\infty} |\eta_k|^r a(y_k, \eta) = 0, \qquad \exists\, r > 0, \tag{19}$$

then the operators

$$A = Op(a) : B^0_{pq}(\mathbb{R}^n) \longrightarrow B^0_{pq}(\mathbb{R}^n)$$

are not Fredholm operators for any $1 < p, q < \infty$.

The complete proof of the above theorem shall be given in Section 5.

4. Some preliminary results

In this section we shall introduce some useful preliminary results and discuss several fruitful properties for mathematical tools which are employed in the next section. First of all, we put

$$\mathcal{K} := \left\{ u \in \mathcal{S}(\mathbb{R}^n) : \quad \text{supp}\,(\mathcal{F}(u)) \subset \left\{ \xi \in \mathbb{R}^n : \ \frac{1}{2} < |\xi| < 1 \right\} \right\}. \tag{20}$$

Let $0 < \tau < \frac{1}{3}$ fixed. For all $s \in \mathbb{R}$ and $(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$, we define the bijections

$$R_s = R_s(y,\eta) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

by $R_s u(x) := \Xi(s; \tau n/p) \cdot e^{isx\eta} u(s^\tau(x-y))$.

LEMMA 4. *For all* $1 < p, q < \infty$, *there exist some positive constants* $D(p,q) > 0$ *and* $S_0 > 0$ *such that*

$$\frac{1}{D(p,q)} \|u\|_{B^0_{pq}(\mathbb{R}^n)} \leqslant \|R_s u\|_{B^0_{pq}(\mathbb{R}^n)} \leqslant D(p,q)\, \|u\|_{B^0_{pq}(\mathbb{R}^n)} \tag{21}$$

for all $u \in \mathcal{K}$ *as far as* $s > S_0$*. Note that the constants* $D(p,q)$ *depend upon* p *and* q*, but not on* (y,η)*. If* $|\eta| \geq 1$*, then we can choose* $S_0 = \Xi(2; (1-\tau)^{-1})$*.*

Proof. A simple computation leads to the Fourier transform of $R_s u$ given by

$$\mathcal{F}(R_s u)(\xi) = \Xi(s; \frac{\tau n}{p} - n\tau)\, e^{-iy(\xi - s\eta)}\, \hat{u}(s^{-\tau}(\xi - s\eta)). \tag{22}$$

Therefore if ξ satisfies $\mathcal{F}(R_s u)(\xi) \neq 0$, then we have

$$\frac{1}{2} < |\Xi(s; -\tau)(\xi - s\eta)| < 1.$$

If $\eta = 0$, then it follows immediately that

$$\mathrm{supp}(\mathcal{F}(R_s(u))) \subset \left\{ \xi \in \mathbb{R}^n : \quad \frac{1}{2} s^\tau < |\xi| < s^\tau \right\} \qquad \forall\, s > S_0, \tag{23}$$

where S_0 is chosen such that $s^\tau < (s/2)|\eta|$ for all $s > S_0$. This implies that, as far as $s \geq S_0$, there exists a constant $m \in \mathbb{N}_0$ such that

$$\mathrm{supp}(\mathcal{F}(R_s u)) \subset \bigcup_{k=m}^{m+2} K_k,$$

where the sets K_k are a series of dyadic partitions of unity described in Section 2. A consideration of Definition of the mapping R_s together with a little computation leads to

$$\|R_s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} \qquad \forall\, u \in \mathcal{S}(\mathbb{R}^n).$$

Therefore, by employing Lemma 1, we conclude that there exists a constant $C = C(p,q) > 0$ such that

$$\|u\|_{B^0_{pq}(\mathbb{R}^n)} \leqslant C(p,q)\|R_s u\|_{L^p(\mathbb{R}^n)} \leqslant C(p,q)^2 \|R_s u\|_{B^0_{pq}(\mathbb{R}^n)} \tag{24}$$

and

$$\|R_s u\|_{B^0_{pq}(\mathbb{R}^n)} \leqslant C(p,q)\|R_s u\|_{L^p(\mathbb{R}^n)} \leqslant C(p,q)^2 \|u\|_{B^0_{pq}(\mathbb{R}^n)}. \tag{25}$$

□

LEMMA 5. *Assume that* $a = a(x,\xi) \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$*. Let* $\{(y_k, \eta_k) \in \mathbb{R}^n \times \mathbb{R}^n\}_{k \in \mathbb{N}_0}$ *be a sequence such that*

$$\lim_{k \to \infty} |\eta_k| = \infty.$$

If $u \in \mathcal{S}(\mathbb{R}^n)$*, then setting*

$$s_k = |\eta_k| \qquad \text{and} \qquad R_k = R_{s_k}\left(y_k, \frac{\eta_k}{|\eta_k|}\right),$$

we have

$$\lim_{k \to \infty} \|Op(a) R_k u\|_{B^0_{pq}(\mathbb{R}^n)} = 0, \qquad \forall\, 1 < p, q < \infty, \tag{26}$$

as far as $\lim_{k\to\infty} |\eta_k|^r\, a(y_k,\eta_k) = 0$ *for some* $r > 0$.

Proof. First, observe that

$$\partial_{x_j}(R_s u(x)) = is\eta_j\, R_s u(x) + s^\tau\, R_s(\partial_{x_j} u)(x). \tag{27}$$

The above observation together with the fact that

$$\|R_s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} \qquad \forall\, u \in \mathcal{S}(\mathbb{R}^n)$$

yields to the estimate

$$\|R_s u\|_{H^1_p(\mathbb{R}^n)} \leqslant (1 + s\langle\eta\rangle)\|u\|_{H^1_P(\mathbb{R}^n)} \qquad \forall\, s \geq 1. \tag{28}$$

By virtue of Lemma 2 we can deduce that

$$\|R_s u\|_{B^0_{pq}(\mathbb{R}^n)} \leqslant C_\theta (1 + s\langle\eta\rangle)^\theta\, \|u\|_{H^1_p(\mathbb{R}^n)}. \tag{29}$$

Now we can choose $0 < \theta < \min\{r, \tau\}$ and finally conclude that

$$\begin{aligned}\|Op(a)R_k u\|_{B^0_{pq}(\mathbb{R}^n)} &\leqslant \left\|R_k\left(R_k^{-1}Op(a)R_k u\right)\right\|_{B^0_{pq}(\mathbb{R}^n)} \\ &\leqslant C_\theta\left(1 + |\eta_k|\left\langle\frac{\eta_k}{|\eta_k|}\right\rangle\right)^\theta \left\|R_k^{-1}Op(a)R_k u\right\|_{H^1_p(\mathbb{R}^n)} \longrightarrow 0.\end{aligned} \tag{30}$$

□

LEMMA 6. *If* $u \in \mathcal{K}$, *then*

$$\lim_{s\to\infty} R_s u = 0 \qquad \textit{weakly in} \quad B^0_{pq}(\mathbb{R}^n). \tag{31}$$

Proof. For every $u, v \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\left|\int_{\mathbb{R}^n} (R_s u)(x) v(x) dx\right| \leqslant \Xi(s; \frac{\tau n}{p} - n\tau)\|v\|_{L^\infty(\mathbb{R}^n)}\|u\|_{L^1(\mathbb{R}^n)}. \tag{32}$$

Hence it follows that

$$\lim_{s\to\infty} \int_{\mathbb{R}^n} (R_s u)(x) v(x) dx = 0. \tag{33}$$

If $u \in \mathcal{K}$, then $R_s u$ is uniformly bounded in $B^0_{pq}(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in the Besov space $B^0_{pq}(\mathbb{R}^n)$ and the dual space of $B^0_{pq}(\mathbb{R}^n)$ can be identified with

$$B^0_{p'q'}(\mathbb{R}^n) \qquad \text{with} \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1$$

according to the fundamental properties of Besov spaces described in Section 2, we finally obtain the required result. □

5. Proof of the main theorem

Let $u \in \mathcal{S}(\mathbb{R}^n)$, $u \neq 0$, satisfying

$$\text{supp}(\mathcal{F}(u)) \subset \left\{ \xi \in \mathbb{R}^n : \quad \frac{1}{2} < |\xi| < 1 \right\}.$$

Suppose that $A = Op(a) : B^0_{pq}(\mathbb{R}^n) \to B^0_{pq}(\mathbb{R}^n)$ is of Fredholm type. Then there are operators B and K in $\mathcal{B}(B^0_{pq}(\mathbb{R}^n))$ such that K is compact and

$$BA = I + K.$$

Let us now define $R_k := R_{s_k}(y_k, \eta_k/|\eta_k|)$ with $s_k = |\eta_k|$. We assume without loss of generality that $|\eta_k| \geq \Xi(2; (1-\tau)^{-1})$ for all k. Then Lemma 4 implies that

$$\|u\|_{B^0_{pq}(\mathbb{R}^n)} \leqslant D_{pq} \left(\||B|\|_{\mathcal{B}(B^0_{pq}(\mathbb{R}^n))} \|AR_k u\|_{B^0_{pq}(\mathbb{R}^n)} + \|KR_k u\|_{B^0_{pq}(\mathbb{R}^n)} \right). \tag{34}$$

On the other hand, the limit result

$$\lim_{k\to\infty} \|AR_k u\|_{B^0_{pq}(\mathbb{R}^n)} = 0$$

follows from Lemma 5, and

$$\lim_{k\to\infty} \|KR_k u\|_{B^0_{pq}(\mathbb{R}^n)} = 0$$

yields from Lemma 6, respectively. Therefore we conclude that

$$\|u\|_{B^0_{pq}(\mathbb{R}^n)} = 0.$$

However, recall that we have assumed that $u \neq 0$, which leads us to a contradiction. □

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