博士論文

Time behavior of solutions to nonlinear Schrödinger equations

(非線形シュレディンガー方程式の解の時間挙動)

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Abstract. In this paper, we deal with nonlinear Schrödinger system (NLS) in the mass-subcritical case and nonlinear Schrödinger equation with a potential (NLS_V) (or (NLS_γ)) in the inter-critical case. We consider time behavior of solutions to these equations. For (NLS), we define a scattering threshold, by focusing structure of the nonlinearity, which corresponds to the best constant of small data scattering. We investigate a property of a solution on the threshold and an optimizing sequence of the threshold. For (NLS_V), we prove a scattering result, a blow-up or grow-up result, and a blow-up result below the ground state without a potential. Then, we show existence of a "radial" ground state and characterize the "radial" ground state by the virial functional. By using the "radial" ground state, we get a global well-posedness of (NLS_V). For (NLS_γ), we show blow-up results. Moreover, we obtain equivalence of conditions on initial data below the ground state without a potential by utilizing the global well-posedness results and the blow-up result.

Contents

1. Introduction

1.1. Nonlinear Schrödinger equation.

In this subsection, we consider the nonlinear Schrödinger equation with power type nonlinearity.

$$i\partial_t u(t,x) + \Delta u(t,x) = \mu |u(t,x)|^{p-1} u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{NLS_0}$$

where $i := \sqrt{-1}$, $\mu \in \{-1, 1\}$, $p > 1$, $d \geq 1$, $\partial_t := \frac{\partial}{\partial t}$, $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$, and an unknown function $u : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{C}$ is a solution to (NLS$_0$). (NLS$_0$) denotes a laser beam in optical fiber and vortex filament. (NLS$_0$) is also regarded as a non-relativistic limit of nonlinear Klein-Gordon equation (NLKG$_0$):

$$-\frac{1}{c^2}\partial_t^2 u(t,x) - i\partial_t u(t,x) + \Delta u(t,x) = \mu |u(t,x)|^{p-1} u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \qquad \text{(NLKG}_0\text{)}$$

where c is the speed of light. In other words, the following holds: for a solution u to (NLKG$_0$), the modulated wave function $u_c(t,x) = e^{ic^2 t} u(t,x)$ satisfies

$$-\frac{1}{c^2}\partial^2 u_c + i\partial_t u_c + \Delta u_c = \mu |u_c|^{p-1} u_c$$

and (NLS$_0$) is formally deduced as $c \to \infty$.

(NLS$_0$) preserves in time t the following mass, energy (or Hamiltonian), and momentum:

$$\text{(Mass)} \quad M(u) := \|u\|_{L_x^2}^2,$$

$$\text{(Energy)} \quad E_0(u) := \frac{1}{2}\|\nabla u\|_{L_x^2}^2 + \frac{\lambda}{p+1}\|u\|_{L_x^{p+1}}^{p+1},$$

$$\text{(Momentum)} \quad \mathcal{M}(u) := \operatorname{Im} \int_{\mathbb{R}^d} \overline{u(t,x)} \nabla u(t,x) dx.$$

$L_x^2(\mathbb{R}^d)$ is called mass space and $H_x^1(\mathbb{R}^d)$ is called energy space. Also, (NLS$_0$) has the scale invariance as follows: If u is a solution to (NLS$_0$), then $u_\lambda := \lambda^{\frac{2}{p-1}} u(\lambda^2 \cdot, \lambda \cdot)$ is also a solution to (NLS$_0$) for $\lambda > 0$. For this transformation, it follows that $\|u_\lambda(0,\cdot)\|_{\dot{H}_x^{s_c}} = \|u(0,\cdot)\|_{\dot{H}_x^{s_c}}$, where

$$s_c := \frac{d}{2} - \frac{2}{p-1}. \tag{1.1}$$

In this sense, $\dot{H}_x^{s_c}(\mathbb{R}^d)$ is scale critical space of (NLS$_0$) and "(NLS$_0$) is called mass-subcritical when $s_c < 0$ ($\Longleftrightarrow 1 < p < 1 + \frac{4}{d}$), mass-critical when $s_c = 0$ ($\Longleftrightarrow p = 1 + \frac{4}{d}$), inter-critical (or mass-supercritical and energy-subcritical) when $0 < s_c < 1$ ($\Longleftrightarrow 1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$), and energy-critical when $s_c = 1$ ($\Longleftrightarrow p = 1 + \frac{4}{d-2}$)".

From now on, we consider the Cauchy problem of (NLS$_0$). That is, we treat (NLS$_0$) with initial condition:

$$u(0,x) = u_0(x). \qquad \text{(IC}_0\text{)}$$

We state local well-posedness theory of (NLS$_0$), where local well-posedness implies existence of time local solution, uniqueness of the solution, and continuous dependence on initial data of the solution.

Theorem 1.1 (Local well-posedness in L^2, [115]). *Let $d \geq 1$, $1 < p < 1 + \frac{4}{d}$, and $\lambda \in \{-1, 1\}$. Let $u_0 \in L_x^2(\mathbb{R}^d)$. Then, there exists $T = T(\|u_0\|_{L_x^2}) > 0$ such that (NLS$_0$) with (IC$_0$) has a unique solution*

$$u \in C_t([-T,T]; L_x^2(\mathbb{R}^d)) \cap L_t^q([-T,T]; L_x^{p+1}(\mathbb{R}^d)),$$

where the exponent q satisfies $\frac{1}{q} = \frac{d}{2}(\frac{1}{2} - \frac{1}{p+1})$. Moreover, continuous dependence on initial data hods, that is,

$$\lim_{n\to\infty} \|u_n - u\|_{L_t^\infty L_x^2} = 0$$

for any $u_{0,n} \in L_x^2(\mathbb{R}^d)$ satisfying $u_{0,n} \longrightarrow u_0$ in $L_x^2(\mathbb{R}^d)$, where u_n is a solution to (NLS$_0$) with data $u_n(0,\cdot) = u_{0,n}$. Furthermore, the solution u to (NLS$_0$) preserves its mass in time t ($M(u(t)) = M(u_0)$).

Theorem 1.2 (Local well-posedness in H^1, [4, 43, 74])**.** *Let $d \geq 1$, $1 < p < \infty$ if $d = 1, 2$, $1 < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, and $\lambda \in \{-1, 1\}$. Let $u_0 \in H_x^1(\mathbb{R}^d)$. Then, there exists $T = T(\|u_0\|_{H_x^1}) > 0$ such that (NLS$_0$) with (IC$_0$) has a unique solution $u \in C_t([-T, T]; H_x^1(\mathbb{R}^d))$. Moreover, continuous dependence on initial data holds, that is,*

$$\lim_{n\to\infty} \|u_n - u\|_{L_t^\infty(I;H_x^1)} = 0$$

for any $u_{0,n} \in H_x^1(\mathbb{R}^d)$ satisfying $u_{0,n} \longrightarrow u_0$ in $H_x^1(\mathbb{R}^d)$ and any compact time interval $I \subset (T_{min}, T_{max})$, where u_n is a solution to (NLS$_0$) with data $u_n(0, \cdot) = u_{0,n}$ and (T_{min}, T_{max}) denotes the maximal existence time of the solution u.

We turn to time behavior of solutions to (NLS$_0$). We will consider the following time behaviors.

Definition 1.3 (Time behaviors of solutions to (NLS$_0$))**.** Let X be a Hilbert space and $u_0 \in X$. Let u be a solution to (NLS$_0$) on $(T_{\min}, T_{\max})$, where $(T_{\min}, T_{\max})$ denotes the maximal existence time of the solution u.

- (Scattering) We say that u scatters in positive time (resp. negative time) if $T_{\max} = \infty$ (resp. $T_{\min} = -\infty$) and there exists $\psi_+ \in X$ (resp. $\psi_- \in X$) such that

$$\lim_{t\to+\infty} \|e^{-it\Delta}u(t) - \psi_+\|_X = 0, \quad \left(\text{resp. } \lim_{t\to-\infty} \|e^{-it\Delta}u(t) - \psi_-\|_X = 0\right),$$

  which implies that the nonlinear solution u approaches a linear solution $e^{it\Delta}\psi_+$ (resp. $e^{it\Delta}\psi_-$) in X as $t \to +\infty$ (resp. $t \to -\infty$).
- (Blow-up) We say that u blows up in positive time (resp. negative time) if $T_{\max} < \infty$ (resp. $T_{\min} > -\infty$).
- (Grow-up) We say that u grows up in positive time (resp. negative time) if $T_{\max} = \infty$ (resp. $T_{\min} = -\infty$) and

$$\limsup_{t\to+\infty} \|u(t)\|_X = \infty, \quad \left(\text{resp. } \limsup_{t\to-\infty} \|u(t)\|_X = \infty\right).$$

- (Standing wave) We say that u is standing wave if $u = e^{i\omega t}Q_{\omega,0}$ for $\omega \in \mathbb{R}$, where $Q_{\omega,0}$ satisfies

$$-\omega Q_{\omega,0} + \Delta Q_{\omega,0} = -|Q_{\omega,0}|^{p-1}Q_{\omega,0}, \quad x \in \mathbb{R}^d. \qquad (\text{SP}_{\omega,0})$$

Remark 1.4. If the Schrödinger group $e^{it\Delta}$ is unitary on X (e.g. L^2, $\dot{H}^1$, H^1), then the definition of scattering can be written as

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta}\psi_+\|_X = 0, \quad \left(\text{resp. } \lim_{t\to-\infty} \|u(t) - e^{it\Delta}\psi_-\|_X = 0\right).$$

We also define the ground state solutions to (SP$_{\omega,0}$).

Definition 1.5. A set of the all ground state $\mathcal{G}_{\omega,0}$ is defined as

$$\mathcal{G}_{\omega,0} := \{\phi \in \mathcal{A}_{\omega,0} : S_{\omega,0}(\phi) \leq S_{\omega,0}(\psi) \text{ for any } \psi \in \mathcal{A}_{\omega,0}\},$$
$$S_{\omega,0}(\phi) := \frac{\omega}{2}M(\phi) + E_0(\phi),$$
$$\mathcal{A}_{\omega,0} := \{\phi \in H_x^1(\mathbb{R}^d) \setminus \{0\} : S'_{\omega,0}(\phi) = 0\},$$

where $S_{\omega,0}$ is called action.

There are two contradictory effects for time behavior of solutions. The linear term has dispersive effect, which tends to flatten the solution to (NLS$_0$) as time goes on (see Figure 1). The nonlinear term with $\lambda = 1$ has a defocusing effect, which is the same effect with dispersive effect. The nonlinear term with $\lambda = -1$ has a focusing effect, which tends to concentrate the solution to (NLS$_0$) (see Figure 2).

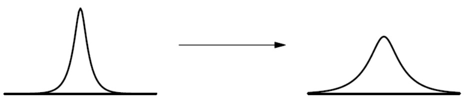

Figure 1

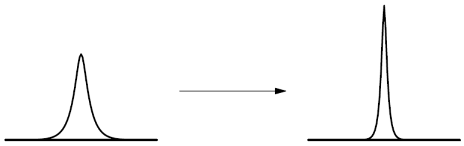

Figure 2

One may think that (NLS$_0$) with a defocusing nonlinearity has only scattering solutions. However, it is false since if the nonlinear power p is sufficiently small, then (NLS$_0$) is like another linear equation. On the other hand, if the nonlinear power p is large and the solution u is small, then the nonlinear term is more smaller, so we expect that the solution u scatters. We formally get the boundary of the nonlinear power p. We consider the final state problem:

$$\begin{cases} i\partial_t u + \Delta u = -|u|^{p-1}u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(t,x) \longrightarrow e^{it\Delta}\psi_+. \end{cases}$$

Estimating the integral equation by using a dispersive estimate (Proposition 3.3), we have

$$\begin{aligned} \|u(t) - e^{it\Delta}\psi_+\|_{L_x^2} &\lesssim \int_t^\infty \|u(s)\|_{L_x^\infty}^{p-1} \|u(s)\|_{L_x^2} ds \\ &\sim \int_t^\infty \|e^{is\Delta}\psi_+\|_{L_x^\infty}^{p-1} \|\psi_+\|_{L_x^2} ds \lesssim \int_t^\infty s^{-\frac{d(p-1)}{2}} \|\psi_+\|_{L_x^1}^{p-1} \|\psi_+\|_{L_x^2} ds. \end{aligned}$$

If $p > 1 + \frac{2}{d}$, then the last integral is finite and converges to 0 as $t \to \infty$. Barab [5] and Strauss [106] showed that if $p \le 1 + \frac{2}{d}$, then non-trivial solutions to (NLS$_0$) do not scatter in the L^2-sense, so we can not expect get a scattering result under the assumption $p \le 1 + \frac{2}{d}$. Instead, Hayashi–Naumkin [61] showed the following modified scattering result (see also Carles [12], Cazenave–Naumkin [15], Ginibre–Ozawa [47], Hayashi–Naumkin [62], and Ozawa [103]).

Theorem 1.6 (Modified scattering, [61])**.** *Let $1 \le d \le 3$ and $\frac{d}{2} < s \le 1 + \frac{2}{d}$ and $u_0 \in H^s(\mathbb{R}^d) \cap \mathcal{F}H^s(\mathbb{R}^d)$. There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \le \varepsilon_0$, if $\|u_0\|_{H^s \cap \mathcal{F}H^s} \le \varepsilon$, then there exist unique function $W_+ \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the asymptotic formula*

$$u(t,x) = \frac{1}{(2it)^{\frac{d}{2}}} e^{\frac{i|x|^2}{4t}} W_+\left(\frac{x}{2t}\right) \exp\left(\frac{\lambda}{2} i \left|W_+\left(\frac{x}{2t}\right)\right|^{\frac{2}{d}} \log t\right) + \mathcal{O}(\varepsilon t^{-\frac{3}{4}d+\delta} \log t)$$

and the estimate

$$\left\| \mathcal{F}(e^{-it\Delta}u) - W_+ \exp\left(\frac{\lambda}{2} i |W_+|^{\frac{2}{d}} \log t\right) \right\|_{L_x^2 \cap L_x^\infty} \lesssim \varepsilon t^{-\frac{d}{4}+\delta} \log t$$

hold, where u is the solution to (NLS$_0$) with (IC$_0$) and $\mathcal{F}$ is the Fourier transform.

We see scattering solutions to (NLS$_0$) with initial data near 0 (small data) under the assumption $p > 1 + \frac{2}{d}$. In this case, the small data scattering results is given by many authors in the suitable sense (e.g. see [16, 44, 45, 63, 65, 114, 116]).

We state known results for time behavior of (NLS$_0$) with a defocusing nonlinearity.

Theorem 1.7 (Scattering solutions to (NLS$_0$) in L^2-critical, [27, 29, 30])**.** *Let $d \ge 1$, $p = 1 + \frac{4}{d}$, and $\lambda = 1$. Let $u_0 \in L_x^2(\mathbb{R}^d)$. Then, a solution to (NLS$_0$) with (IC$_0$) scatters.*

Theorem 1.8 (Scattering solutions to (NLS$_0$) in inter-critical I, [42, 78, 94, 95, 96, 122])**.** *Let $d \ge 3$ and $\lambda = 1$. Let $u : (T_{min}, T_{max}) \times \mathbb{R}^d \longrightarrow \mathbb{C}$ be a maximal lifespan solution to (NLS$_0$) with (IC$_0$) satisfying $u \in L_t^\infty(T_{min}, T_{max}; \dot{H}_x^{s_c})$, where s_c is defined as (1.1). If either of the following conditions hold:*

- *$\frac{1}{2} \le s_c \le \frac{3}{4}$ if $d = 3$, $\frac{1}{2} \le s_c < 1$ if $d \ge 4$,*
- *$u_0 \in \dot{H}_{rad}^{s_c}(\mathbb{R}^d)$ and $s_c \in (0, \frac{1}{2}) \cup (\frac{3}{4}, 1)$,*

then u scatters.

Theorem 1.9 (Scattering solutions to (NLS$_0$) in inter-critical II, [7, 20, 26, 108]). *Let $d = 3$, $p = 3$, and $\lambda = 1$. If $u_0 \in H^s_x(\mathbb{R}^3)$ for $s > \frac{49}{74}$, then the solution u to (NLS$_0$) with (IC$_0$) scatters.*

Theorem 1.10 (Scattering solutions to (NLS$_0$) in $\dot{H}^1$-critical, [21, 85, 105, 117, 118]). *Let $d \geq 3$, $p = 1 + \frac{4}{d-2}$, and $\lambda = 1$. Let $u_0 \in \dot{H}^1_x(\mathbb{R}^d)$. Then, a solution to (NLS$_0$) with (IC$_0$) scatters.*

We turn to the focusing case. Before the Kenig–Merle's work [77], only time behavior of the characteristic solutions had been observed. We note that the small data scattering above is also one of the framework. The following virial functional is very useful when we consider time behavior of solutions to (NLS$_0$).

$$\text{(Virial functional)} \quad K_0(f) := \mathcal{D}^{d,2} S_{\omega,0}(f) = 2\|\nabla f\|^2_{L^2_x} - \frac{d(p-1)}{p+1}\|f\|^{p+1}_{L^{p+1}_x}.$$

where $\mathcal{D}^{\alpha,\beta}$ is defined as

$$\mathcal{D}^{\alpha,\beta} f := \frac{d}{d\lambda}\Big|_{\lambda=0} e^{\alpha\lambda} f(e^{\beta\lambda}\,\cdot\,), \quad \mathcal{D}^{\alpha,\beta}\mathscr{F}(f) := \frac{d}{d\lambda}\Big|_{\lambda=0} \mathscr{F}(e^{\alpha\lambda} f(e^{\beta\lambda}\,\cdot\,))$$

for $(\alpha,\beta) \in \mathbb{R}^2$, a function f, and a functional $\mathscr{F}$. The virial functional has the next property: If $xu_0 \in L^2_x(\mathbb{R}^d)$, then

$$\frac{d^2}{dt^2}\|xu(t)\|^2_{L^2_x} = 4K_0(u(t))$$

for each $t \in (T_{\min}, T_{\max})$. Roughly speaking, this equality implies that if $K_0(u(t)) > 0$ then the solution u to (NLS$_0$) goes far away from the origin and if $K_0(u(t)) < 0$ then the solution u to (NLS$_0$) approaches the origin. Glassey [48] and Ogawa–Tsutsumi [100] proved a existence of blow-up solutions by controlling the virial functional.

Theorem 1.11 (Blow up solutions, [48, 100]). *Let $d \geq 1$, $1 + \frac{4}{d} \leq p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} \leq p \leq 1 + \frac{4}{d-2}$ if $d \geq 3$, $\lambda = -1$, and $u_0 \in H^1_x(\mathbb{R}^d)$. Then, the followings hold:*

- *(Finite variance) If $xu_0 \in L^2_x(\mathbb{R}^d)$ and "$E_0(u_0) < 0$ or $E_0(u_0) = 0$, $Im\langle x\cdot\nabla u_0, u_0\rangle_{L^2_x} < 0$", then the solution u to (NLS$_0$) with (IC$_0$) blows up.*
- *(Radial) If $u_0 \in H^1_{rad}(\mathbb{R}^d)$, $E_0(u_0) < 0$, and we suppose an additional assumption $p < 5$ when $d = 2$, then the solution u to (NLS$_0$) with (IC$_0$) blows up.*

Berestycki–Cazenave [8] and Cazenave–Lions [14] observed solutions to (NLS$_0$) with initial data near the ground state to (SP$_\omega$).

Theorem 1.12 (Stability and instability of standing waves, [8, 14]). *Let $d \geq 1$, $1 < p < \infty$ if $d = 1, 2$, $1 < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, and $\lambda = -1$. Let $Q_{\omega,0}$ be the ground state to (SP$_{\omega,0}$).*

- *(Stability) If $1 < p < 1 + \frac{4}{d}$, then the standing wave $e^{i\omega t}Q_{\omega,0}$ is stable in the next sense: for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|u_0 - Q_{\omega,0}\|_{H^1_x} < \delta$, then the solution u to (NLS$_0$) with (IC$_0$) satisfies*

  $$\inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^d} \|u(t) - e^{i\theta}Q_{\omega,0}(\,\cdot - y)\|_{H^1_x} < \varepsilon$$

  for any $t \in \mathbb{R}$.
- *(Instability) If $1 + \frac{4}{d} \leq p < \infty$ with $d = 1, 2$ and $1 + \frac{4}{d} \leq p < 1 + \frac{4}{d-2}$ with $d \geq 3$, then the standing wave $e^{i\omega t}Q_{\omega,0}$ is unstable in the next sense: for any $\varepsilon > 0$, there exists $u_0 \in H^1(\mathbb{R}^d)$ such that $\|u_0 - Q_{\omega,0}\|_{H^1} < \varepsilon$ and the solution u to (NLS$_0$) with (IC$_0$) blows up.*

By the Kenig–Merle's work [77], we can see that the ground state $Q_{\omega,0}$ to (SP$_{\omega,0}$) is a boundary between initial data with different types of time behavior solutions. In the argument, we consider the following minimization problem and sets to control a sign of the virial functional.

$$n_{\omega,0} := \inf\{S_{\omega,0}(\phi) : \phi \in H^1_x(\mathbb{R}^d)\setminus\{0\},\ K_0(\phi) = 0\},$$

$$PW_{+,1} := \bigcup_{\omega>0} \{\phi \in H_x^1(\mathbb{R}^d) : S_{\omega,0}(\phi) < n_{\omega,0},\ K_0(\phi) > 0\},$$
$$PW_{-,1} := \bigcup_{\omega>0} \{\phi \in H_x^1(\mathbb{R}^d) : S_{\omega,0}(\phi) < n_{\omega,0},\ K_0(\phi) < 0\}.$$

Then, sets $PW_{+,1}$ and $PW_{-,1}$ are invariant under the time development of (NLS$_0$). In other words, if $u_0 \in PW_{+,1}$ then $u(t) \in PW_{+,1}$ for each $t \in (T_{\min}, T_{\max})$ and $u_0 \in PW_{-,1}$ then $u(t) \in PW_{-,1}$ for each $t \in (T_{\min}, T_{\max})$. It is well known for inter-critical case $(1 + \frac{4}{d} < p < 1 + \frac{4}{d-2})$ that a set of minimizers to $n_{\omega,0}$ corresponds to $\mathcal{G}_{\omega,0}$, that is, $\mathcal{M}_{\omega,0} = \mathcal{G}_{\omega,0}$ holds, where

$$\mathcal{M}_{\omega,0} := \{\phi \in H_x^1(\mathbb{R}^d) \setminus \{0\} : S_{\omega,0}(\phi) = n_{\omega,0},\ K_0(\phi) = 0\}.$$

Therefore, we can rewrite $PW_{+,1}$ and $PW_{-,1}$ as

$$PW_{+,1} := \bigcup_{\omega>0} \{\phi \in H_x^1(\mathbb{R}^d) : S_{\omega,0}(\phi) < S_{\omega,0}(Q_{\omega,0}),\ K_0(\phi) > 0\},$$
$$PW_{-,1} := \bigcup_{\omega>0} \{\phi \in H_x^1(\mathbb{R}^d) : S_{\omega,0}(\phi) < S_{\omega,0}(Q_{\omega,0}),\ K_0(\phi) < 0\}$$

for $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, and a ground state $Q_{\omega,0}$ to (SP$_{\omega,0}$). Using $PW_{+,1}$ and $PW_{-,1}$, the following known results for time behavior of solutions to (NLS$_0$) with a focusing nonlinearity are shown in [1, 2, 32, 33, 35, 38, 67].

Theorem 1.13 (Time behavior of solutions to (NLS$_0$) below the ground state in inter-critical)**.**

Let $d \geq 1$, $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, and $\lambda = -1$.

- *(Scattering) If $u_0 \in PW_{+,1}$, then a solution u to (NLS$_0$) with (IC$_0$) scatters.*
- *(Blow-up or grow-up) If $u_0 \in PW_{-,1}$, then a solution u to (NLS$_0$) with (IC$_0$) blows up or grows up. Moreover, if $xu_0 \in L_x^2(\mathbb{R}^d)$ or "$d \geq 2$, $p \leq 5$ if $d = 2$, and $u_0 \in H_{rad}^1(\mathbb{R}^d)$", then the solution u to (NLS$_0$) with (IC$_0$) blows up.*

The assumptions in Theorem 1.13 have the next equivalent conditions.

Proposition 1.14. *Let $d \geq 1$, $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, $\lambda = -1$, and s_c be defined as (1.1). Let $u_0 \in H_x^1(\mathbb{R}^d)$ and $Q_{\omega,0}$ be a ground state to (SP$_{\omega,0}$). Then, the following two conditions (1) and (2) are equivalence.*

(1) *There exists $\omega > 0$ such that $S_{\omega,0}(u_0) < S_{\omega,0}(Q_{\omega,0})$,* (1.2)

(2) $M(u_0)^{\frac{1-s_c}{s_c}} E_0(u_0) < M(Q_{1,0})^{\frac{1-s_c}{s_c}} E_0(Q_{1,0})$. (1.3)

Under the above condition, the following two conditions are equivalence.

- $K_0(u_0) \geq 0$
- $\|u_0\|_{L_x^2}^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_{L_x^2} < \|Q_{1,0}\|_{L_x^2}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L_x^2}$ (1.4)

Under the condition (1) (or (2)), the following two conditions are equivalence.

- $K_0(u_0) < 0$
- $\|u_0\|_{L_x^2}^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_{L_x^2} > \|Q_{1,0}\|_{L_x^2}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L_x^2}$ (1.5)

From Proposition 1.14, we can rewrite Theorem 1.13 as the next theorem.

Theorem 1.15. *Let $d \geq 1$, $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, and $\lambda = -1$. Let $Q_{1,0}$ be a ground state to (SP$_{\omega,0}$) with $\omega = 1$. Suppose that $u_0 \in H_x^1(\mathbb{R}^d)$ satisfies (1.3).*

- *(Scattering) If (1.4) holds, then a solution u to (NLS$_0$) with (IC$_0$) scatters.*

- *(Blow-up or grow-up) If (1.5) holds, then a solution u to (NLS$_0$) with (IC$_0$) blows up or grows up. Moreover, if $xu_0 \in L^2_x(\mathbb{R}^d)$ or "$d \geq 2$, $p \leq 5$ if $d = 2$, and $u_0 \in H^1_{rad}(\mathbb{R}^d)$", then the solution u to (NLS$_0$) with (IC$_0$) blows up.*

The expression in Theorem 1.15 is based on the following Gagliardo-Nirenberg inequality.

Proposition 1.16 (Gagliardo-Nirenberg inequality, [3, 40, 41, 99, 111, 121])**.** *Let $d \geq 1$, $1 < p < \infty$ if $d = 1, 2$, and $1 < p \leq 1 + \frac{4}{d-2}$ if $d \geq 3$. Then, the following inequality holds:*

$$\|f\|_{L^{p+1}}^{p+1} \leq C_{GN} \|f\|_{L^2}^{p+1-\frac{d(p-1)}{2}} \|\nabla f\|_{L^2}^{\frac{d(p-1)}{2}} \tag{1.6}$$

for any $f \in H^1(\mathbb{R}^d)$, where C_{GN} is the best constant, is attained by the ground state $Q_{1,0}$ to (SP$_{\omega,0}$) with $\omega = 1$ if $1 < p < \infty$ ($d = 1, 2$) and $1 < p < 1 + \frac{4}{d-2}$ ($d \geq 3$), and is attained by the ground state $Q_{0,0}$ to (SP$_{\omega,0}$) with $\omega = 0$ if $p = 1 + \frac{4}{d-2}$ ($d \geq 3$).

Remark 1.17. We note the following point for Proposition 1.16.

- If $d \geq 3$ and $p = 1 + \frac{4}{d-2}$, then the inequality (1.6) is called usually Sobolev inequality (Lemma 2.3).
- $Q_{0,0}$ is the Talenti function and can be written as $Q_{0,0} = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}$.

Using the expression of Theorem 1.15, the following known results in L^2-critical case is given (see [28]).

Theorem 1.18 (Scattering solutions to (NLS$_0$) below the ground state in L^2-critical)**.** *Let $d \geq 1$, $p = 1 + \frac{4}{d}$, and $\lambda = -1$. Let $Q_{1,0}$ be a ground state to (SP$_{\omega,0}$) with $\omega = 1$. If $u_0 \in L^2_x(\mathbb{R}^d)$ satisfies $\|u_0\|_{L^2_x} < \|Q_{1,0}\|_{L^2_x}$, then a solution to (NLS$_0$) with (IC$_0$) scatters.*

Using the expression of Theorem 1.15, the following known results in $\dot{H}^1$-critical case is given (see [31, 77, 84]).

Theorem 1.19 (Time behavior of solutions to (NLS$_0$) below the ground state in $\dot{H}^1$-critical)**.**

Let $d \geq 3$, $p = 1 + \frac{4}{d-2}$, and $\lambda = -1$. Let $Q_{0,0}$ be a ground state to (SP$_{\omega,0}$) with $\omega = 0$. Suppose that $u_0 \in \dot{H}^1_x(\mathbb{R}^d)$ satisfies $E_0(u_0) < E_0(Q_{0,0})$.

- *(Scattering) If either of the followings hold:*
  - *$d \geq 4$ and $\|\nabla u_0\|_{L^2_x} < \|\nabla Q_{0,0}\|_{L^2_x}$,*
  - *$d = 3$, $u_0 \in \dot{H}^1_{rad}(\mathbb{R}^3)$, and $\|\nabla u_0\|_{L^2_x} < \|\nabla Q_{0,0}\|_{L^2_x}$,*

  then a solution u to (NLS$_0$) with (IC$_0$) scatters.
- *(Blow-up) If "$xu_0 \in L^2_x(\mathbb{R}^d)$ or $u_0 \in \dot{H}^1_{rad}(\mathbb{R}^d)$" and $\|\nabla u_0\|_{L^2_x} > \|\nabla Q_{0,0}\|_{L^2_x}$, then a solution u to (NLS$_0$) with (IC$_0$) blows up.*

As Theorem 1.13, 1.18, and 1.19, a sign of the virial functional of solutions to (NLS$_0$) with initial data below the ground state is invariant from a characterization of the ground state with the virial functional. If we try to observe on the threshold level ($M(Q_{1,0})^{\frac{1-s_c}{s_c}} E_0(Q_{1,0})$) or pull up the threshold level, then we need more detailed investigation for the ground state $Q_{\omega,0}$ to (SP$_{\omega,0}$). To consider initial data on the threshold level, that is, initial data satisfying

$$M(u_0)^{\frac{1-s_c}{s_c}} E_0(u_0) = M(Q_{1,0})^{\frac{1-s_c}{s_c}} E_0(Q_{1,0}), \tag{1.7}$$

we use a property of the ground state $Q_{\omega,0}$ to (SP$_{\omega,0}$): If

$$\|f\|_{L^{p+1}}^{p+1} = C_{\mathrm{GN}} \|f\|_{L^2}^{p+1-\frac{d(p-1)}{2}} \|\nabla f\|_{L^2}^{\frac{d(p-1)}{2}},$$

then there exist $\lambda_0 \in \mathbb{C}$ and $x_0 \in \mathbb{R}^d$ such that $f(x) = \lambda_0 Q_{1,0}(x + x_0)$ for $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$ and $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$. Fortunately, $PW_{+,2}$, PW_Q, and $PW_{-,2}$ are invariant under the time development of (NLS$_0$), where

$$PW_{+,2} := \{u_0 \in H^1_x(\mathbb{R}^d) : (1.7) \text{ and } (1.4)\},$$

$$PW_Q := \{u_0 \in H^1_x(\mathbb{R}^d) : (1.7) \text{ and } \|u_0\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_{L^2_x} = \|Q_{1,0}\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L^2_x}\},$$
$$PW_{-,2} := \{u_0 \in H^1_x(\mathbb{R}^d) : (1.7) \text{ and } (1.5)\}.$$

Theorem 1.20 (Time behavior of solutions to (NLS$_0$) on the threshold in inter-critical, [11, 37, 50])**.** *Let $d \geq 1$, $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, and $\lambda = -1$.*

- *(Existence of special solutions) There exist two radial solutions Q_+ and Q_- to (NLS$_0$) with initial data $Q_{0,+}, Q_{0,-} \in \cap_{s\in\mathbb{R}} H^s_x(\mathbb{R}^d)$ such that the followings hold:*
  - *$M(Q_+) = M(Q_-) = M(Q_{1,0})$ and $E_0(Q_+) = E_0(Q_-) = E_0(Q_{1,0})$,*
  - *Q_+ and Q_- are defined on $[0, +\infty)$ and there exist $e_0 > 0$ and $C > 0$ such that*
  $$\|Q_\pm(t) - e^{it} Q_{1,0}\|_{H^1_x} \leq Ce^{-e_0 t}$$
  for any $t \geq 0$,
  - *$\|\nabla Q_{0,+}\|_{L^2_x} < \|\nabla Q_{1,0}\|_{L^2_x}$ holds and Q_+ scatters in negative time,*
  - *$\|\nabla Q_{0,-}\|_{L^2_x} > \|\nabla Q_{1,0}\|_{L^2_x}$ holds and Q_- blows up in negative time.*
- *(Time behavior of solutions on the threshold level) Let u be a solution to (NLS$_0$) with (IC$_0$).*
  - *If $u_0 \in PW_{+,2}$, then either u scatters or $u = Q_+$ up to the symmetries.*
  - *If $u_0 \in PW_Q$, then $u = e^{it} Q_{1,0}$ up to the symmetries.*
  - *If $u_0 \in PW_{-,2}$ and "$u_0 \in H^1_{rad}(\mathbb{R}^d)$ or $xu_0 \in L^2(\mathbb{R}^d)$", then either u blows up or $u = Q_-$ up to the symmetries.*

Theorem 1.21 (Time behavior of solutions to (NLS$_0$) on the threshold in $\dot{H}^1$-critical, [11, 36, 87, 109])**.** *Let $d \geq 3$, $p = 1 + \frac{4}{d-2}$, and $\lambda = -1$.*

- *(Existence of special solutions) There exist two radial solutions Q_+ and Q_- to (NLS$_0$) with initial data $Q_{0,+}, Q_{0,-}$ such that the followings hold:*
  - *$E_0(Q_+) = E_0(Q_-) = E_0(Q_{0,0})$,*
  - *Q_+ and Q_- are defined on $[0, +\infty)$ and satisfy*
  $$\lim_{t\to+\infty} \|Q_\pm(t) - Q_{0,0}\|_{\dot{H}^1_x} = 0,$$
  - *$\|\nabla Q_{0,+}\|_{L^2_x} < \|\nabla Q_{0,0}\|_{L^2_x}$ holds and Q_+ scatters in negative time,*
  - *$\|\nabla Q_{0,-}\|_{L^2_x} > \|\nabla Q_{0,0}\|_{L^2_x}$ holds and if $d \geq 5$, then Q_- blows up in negative time.*
- *(Time behavior of solutions on the threshold level) Let $u_0 \in \dot{H}^1_x(\mathbb{R}^d)$ and u be a solution to (NLS$_0$) with (IC$_0$).*
  - *If $u_0 \in PW_{+,2}$ and "$u_0 \in \dot{H}^1_{rad}(\mathbb{R}^d)$ when $d = 3, 4$", then either u scatters or $u = Q_+$ up to the symmetries.*
  - *If $u_0 \in PW_Q$, then $u = Q_{0,0}$ up to the symmetries.*
  - *If $u_0 \in PW_{-,2}$ and $u_0 \in H^1_{rad}(\mathbb{R}^d)$, then either u blows up or $u = Q_-$ up to the symmetries.*

Next, we pull up the threshold level. We consider initial data satisfying
$$M^{\frac{1-s_c}{s_c}}(u_0)E_0(u_0) < M^{\frac{1-s_c}{s_c}}(Q_{1,0})(E_0(Q_{1,0}) + \varepsilon_0^2) \tag{1.8}$$
for $\varepsilon_0 > 0$. Unfortunately, it does not seem easy for us to find invariant sets under the time development of (NLS$_0$). In the scene, Nakanishi–Schlag [98] observe time behavior of the solutions to (NLS$_0$) by developing and using the one-pass theorem, which implies that if a solution u to (NLS$_0$) passes in and out of a small neighborhood $\{\pm Q_{\omega,0}\}$, then it can never come back again.

Theorem 1.22 (Time behavior of solutions to (NLS$_0$) above the ground state, [98])**.** *Let $d = p = 3$ and $\lambda = -1$. Then, there exists $\varepsilon_0 > 0$ such that the solution u to (NLS$_0$) with radial data $u_0 \in H^1_{rad}(\mathbb{R}^3)$ in (1.8) satisfies one of the followings:*

(1) *scattering in both time,*
(2) *blow-up in both time,*
(3) *scattering in positive time and blow-up in negative time,*

(4) *blow-up in positive time and scattering in negative time,*
(5) *trapped by $\mathcal{CSM}$ in positive time and scattering in negative time,*
(6) *scattering in positive time and trapped by $\mathcal{CSM}$ in negative time,*
(7) *trapped by $\mathcal{CSM}$ in positive time and blow-up in negative time,*
(8) *blow-up in positive time and trapped by $\mathcal{CSM}$ in negative time,*
(9) *trapped by $\mathcal{CSM}$ in both time,*

where "trapped by $\mathcal{CSM}$ in positive (resp. negative) time" means the solution u to (NLS$_0$) *stays in the ε-neighborhood of $\mathcal{CSM}$ in $H^1_x(\mathbb{R}^3)$ forever after some time (resp. before some time), where the center-stable manifold $\mathcal{CSM}$ is defined as*

$$\mathcal{CSM} := \bigcup_{\omega>0} \{e^{i\theta} Q_{\omega,0} : \theta \in \mathbb{R}\}.$$

Moreover, the all sets of initial data is not empty, whose solution satisfies (1) $\sim$ (9).

In the point of stability of the ground state $Q_{\omega,0}$ to (SP$_{\omega,0}$) (see Theorem 1.12), it seems that mass subcritical and "mass critical, inter critical, or energy critical" are different. Theorem 1.13 $\sim$ 1.22 (case mass critical, inter critical, or energy critical) is based on instability of the ground state $Q_{\omega,0}$ to (SP$_{\omega,0}$). On the other hand, we can not expect that the ground state $Q_{\omega,0}$ stands for a boundary between initial data with different time behavior solutions in mass subcritical case since it is stable. In the situation, Masaki and his co-author showed the following results.

Theorem 1.23 (Masaki, [89, 91])**.** *Let $d \geq 1$, $\max\{1+\frac{2}{d}, 1+\frac{4}{d+2}\} < p < 1+\frac{4}{d}$, $\lambda = -1$, and $u_0 \in \mathcal{F}\dot{H}^{|s_c|}(\mathbb{R}^d)$, where s_c be defined as* (1.1) *and $\|u_0\|_{\mathcal{F}\dot{H}^{|s_c|}} := \|\mathcal{F}^{-1}u_0\|_{\dot{H}^{|s_c|}} = \||x|^{|s_c|}u_0\|_{L^2_x}$. Then,* (NLS$_0$) *with* (IC$_0$) *is locally well-posed in $\mathcal{F}\dot{H}^{|s_c|}(\mathbb{R}^d)$. Moreover, there exists $u_{c,0} \in \mathcal{F}\dot{H}^{|s_c|}(\mathbb{R}^d)$ such that the solution u_c to* (NLS$_0$) *with initial data $u_{c,0}$ does not scatter in positive time and $\|u_{c,0}\|_{\mathcal{F}\dot{H}^{|s_c|}} = \ell_c$, where ℓ_c is defined as*

$$\ell_c := \inf\{\|u_0\|_{\mathcal{F}\dot{H}^{|s_c|}} : \textit{solution } u \textit{ to } (\mathrm{NLS}_0) \textit{ with } (\mathrm{IC}_0) \textit{ does not scatter in both time.}\}. \tag{1.9}$$

Theorem 1.24 (Killip–Masaki–Murphy–Visan, [81])**.** *Let $d \geq 1$, $\lambda \in \{\pm 1\}$, $\max\{1+\frac{2}{d} < p < 1+\frac{4}{d+2}\}$, $t_0 \in [-\infty, \infty)$, and $e^{-it_0\Delta}u_0 \in \mathcal{F}\dot{H}^{|s_c|}$. Let u be a solution to* (NLS$_0$) *with initial data $u(t_0) = u_0$. If u satisfies*

$$\sup_{t\in I_{max}} \|e^{-it\Delta}u(t)\|_{\mathcal{F}\dot{H}^{|s_c|}} < \infty,$$

then u scatters in positive time.

Theorem 1.25 (Killip–Masaki–Murphy–Visan, [82])**.** *Let $d \geq 3$, $\lambda \in \{\pm 1\}$, and s_c be defined as* (1.1).

- *(Existence of a soliton-like minimizer) Let $p_0(d) < p < 1+\frac{4}{d}$. If $E_c < \infty$, then there exists a radial almost-periodic solution $u_c : \mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{C}$ such that*
  (1) $\limsup_{t\to\infty} \|u_c(t)\|_{\dot{H}^{s_c}_x} = E_c$,
  (2) $u_c \in (C_t \cap L^\infty_t)(\mathbb{R}; H^{s_c}_x \cap H^{\frac{1}{2}}_x)$,
  (3) $N(t) \equiv 1$. *In particular, the orbit $\{u_c(t)\}_{t\in\mathbb{R}}$ is precompact in $\dot{H}^{s_c}_x(\mathbb{R}^d)$,*
  where $p_0(d)$ is

$$p_0(d) := \begin{cases} 1+\frac{15-2d+\sqrt{4d^2+100d+145}}{5(2d-1)}, & (3 \leq d \leq 8), \\ 1+\frac{4}{d+1}, & (d \geq 9), \end{cases}$$

  E_c is defined as

$$E_c := \inf\left\{ \limsup_{t\nearrow \sup I_{max}} \|u(t)\|_{\dot{H}^{s_c}_x} \;\middle|\; \begin{array}{l} u \textit{ is a radial solution to } (\mathrm{NLS}_0) \textit{ and} \\ u \textit{ does not scatter in positive time.} \end{array} \right\},$$

and almost periodic implies that there exist $N : I \longrightarrow (0,\infty)$ and $C : (0,\infty) \longrightarrow (0,\infty)$ such that a radial solution $u \in L_t^\infty(I; \dot{H}^{s_c}_{rad}) \setminus \{0\}$ to (NLS$_0$) satisfies

$$\sup_{t\in I}\left\{\int_{|x|>\frac{C(\eta)}{N(t)}}\left||\nabla|^{s_c}u(t,x)\right|^2dx+\int_{|\xi|>C(\eta)N(t)}\left||\xi|^{s_c}\widehat{u}(t,\xi)\right|^2d\xi\right\}<\eta$$

for any $\eta > 0$.

- *(Minimal counterexample) Let $1+\frac{4}{d+1} < p < 1+\frac{4}{d}$. If $E_c < \infty$, then there exists an almost periodic solution u to (NLS$_0$) such that u attains E_c and fits into one of the following scenarios:*
  - *(Self-similar scenario) $I = (0,\infty)$ and $N(t) = t^{-\frac{1}{2}}$.*
  - *(Cascade scenario) $I = \mathbb{R}$, $\sup_{t\in\mathbb{R}} N(t) \le 1$, and there exists a subsequence $\{t_n\}$ of times such that $N(t_n) \longrightarrow 0$ as $n \to \infty$.*
  - *(Soliton scenario) $I = \mathbb{R}$ and $N(t) \equiv 1$.*
- *(Case of defocusing) Let $\lambda = +1$ and $p_0(d) < p < 1+\frac{4}{d}$. If a radial solution u to (NLS$_0$) satisfies*

$$\sup_{t\in I_{max}} \|u\|_{\dot{H}^{s_c}_x} < \infty,$$

  then u scatters.
- *(Case of focusing) Let $\lambda = -1$ and $1+\frac{4}{d+1} < p < 1+\frac{4}{d}$. Let $Q_{1,0}$ be the ground state to (SP$_{\omega,0}$) with $\omega = 1$. Then, $E_c \le \|Q_{1,0}\|_{\dot{H}^{s_c}}$ holds and E_c is attained. Moreover, if $p_0(d) < p < 1+\frac{4}{d}$, then there exists an almost-periodic solution u_c to (NLS$_0$) such that u_c attains E_c and has the above properties (1) $\sim$ (3).*

Remark 1.26. (NLS$_0$) is ill-posed in $\dot{H}^{s_c}(\mathbb{R}^d)$ for $s_c < 0$ (see [17, 80]). However, if we restrict to radial data, (NLS$_0$) is locally well-posed in $\dot{H}^{s_c}(\mathbb{R}^d)$ for $d \ge 3$ and $1+\frac{4}{d+1} < p < 1+\frac{4}{d}$ (see [51, 66]).

In this paper, we deal with the following equations:

- (Nonlinear Schrödinger system)

$$\begin{cases} i\partial_t u(t,x)+\Delta u(t,x) = -2v(t,x)\overline{u(t,x)}, & (t,x)\in\mathbb{R}\times\mathbb{R}^d,\\ i\partial_t v(t,x)+\kappa\Delta v(t,x) = -u(t,x)^2, & (t,x)\in\mathbb{R}\times\mathbb{R}^d,\end{cases} \tag{NLS}$$

  where $\kappa > 0$.
- (Nonlinear Schrödinger equation with a potential)

$$i\partial_t u(t,x)+\Delta_V u(t,x) = -|u(t,x)|^{p-1}u(t,x), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^d, \tag{NLS$_V$}$$

  where $\Delta_V := \Delta - V$ and $V : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a given potential.

We consider the Cauchy problem of these equation with initial data

$$\begin{cases} (u(0,x),v(0,x)) = (u_0,v_0) & \text{if (NLS)},\\ u(0,x) = u_0(x) & \text{if (NLS}_V).\end{cases} \tag{IC}$$

and time behavior of solutions to the equations.

1.2. **Nonlinear Schrödinger system.** We consider

$$\begin{cases} i\partial_t u+\dfrac{1}{2n}\Delta u = \lambda v\overline{u}, & (t,x)\in\mathbb{R}\times\mathbb{R}^d,\\ i\partial_t v+\dfrac{1}{2N}\Delta v = \mu u^2, & (t,x)\in\mathbb{R}\times\mathbb{R}^d,\end{cases} \tag{1.10}$$

where $1 \le d \le 6$ and $n, N > 0$, $\lambda, \mu \in \mathbb{C}\setminus\{0\}$ are constants. From physical point of view, (1.10) is deduced from the Raman amplification in a plasma (See [19] for more detail). When a so-called *mass resonance* condition $N = 2n$ ($\kappa = \frac{1}{2}$ in the case (NLS)) holds, (1.10) is also regarded as a non-relativistic limit of the nonlinear Klein-Gordon system

$$\begin{cases} \frac{1}{2c^2n}\partial_t^2 u - \frac{1}{2n}\Delta u + \frac{nc^2}{2}u = -\lambda v\overline{u}, \\ \frac{1}{2c^2N}\partial_t^2 v - \frac{1}{2N}\Delta v + \frac{Nc^2}{2}v = -\mu u^2 \end{cases}$$

(see [64]). If $\lambda = \nu\overline{\mu}$ for some $\nu > 0$, then mass, energy, and momentum are conserved. Then, (1.10) becomes (NLS) by the scaling

$$(u, v) \mapsto \left(\sqrt{\frac{\nu}{2}}|\mu|u\left(t, \sqrt{\frac{1}{2n}}x\right), -\frac{\lambda}{2}v\left(t, \sqrt{\frac{1}{2n}}x\right)\right).$$

Thus, we consider not (1.10) but (NLS). The equation (NLS) has scale invariance: Since the nonlinear terms are quadratic, if (u, v) is a solution to (NLS), then

$$(u_{[\lambda]}(t, x), v_{[\lambda]}(t, x)) := (\lambda^2 u(\lambda^2 t, \lambda x), \lambda^2 v(\lambda^2 t, \lambda x)) \tag{1.11}$$

is also a solution to (NLS) for $\lambda > 0$. Corresponding transform of initial data is

$$((u_0)_{\{\lambda\}}(x), (v_0)_{\{\lambda\}}(x)) := (\lambda^2 u_0(\lambda x), \lambda^2 v_0(\lambda x)) \tag{1.12}$$

for $\lambda > 0$. $\dot{H}^{s_c}$-norm ($s_c = \frac{d}{2} - 2$) is invariant under the scaling (1.12), so the scale critical Sobolev space is $\dot{H}^{s_c}(\mathbb{R}^d)$. Therefore, the equation (1.10) is called L^2-subcritical if $d \leq 3$, L^2-critical if $d = 4$, $\dot{H}^{\frac{1}{2}}$-critical if $d = 5$, and $\dot{H}^1$-critical if $d = 6$. When $\kappa = \frac{1}{2}$, the identities

$$\begin{aligned} [e^{it\Delta}(e^{ix\cdot\xi_0}f)](x) &= e^{-it|\xi_0|^2 + ix\cdot\xi_0}(e^{it\Delta}f)(x - 2t\xi_0), \\ [e^{\frac{1}{2}it\Delta}(e^{2ix\cdot\xi_0}f)](x) &= e^{-2it|\xi_0|^2 + 2ix\cdot\xi_0}(e^{\frac{1}{2}it\Delta}f)(x - 2t\xi_0) \end{aligned} \tag{1.13}$$

imply that the class of solutions to the linear Schrödinger equation is invariant under Galilean transform

$$(u, v) \mapsto (e^{ix\cdot\xi_0}e^{-it|\xi_0|^2}u(t, x - 2\xi_0 t), e^{2ix\cdot\xi_0}e^{-2it|\xi_0|^2}v(t, x - 2\xi_0 t)) \tag{1.14}$$

for $\xi_0 \in \mathbb{R}^d$. The invariance is inherited in the nonlinear equation (NLS).

Hayashi–Ozawa–Tanaka [64] showed the following local well-posedness in $L^2 \times L^2$, $H^1 \times H^1$, and $H^1 \times H^1 \cap |x|^{-1}L^2 \times |x|^{-1}L^2$.

Theorem 1.27 (Local well-posedness in $L^2 \times L^2$, [64]). *If $1 \leq d \leq 3$, then for any $\rho > 0$, there exists $T(\rho) > 0$ such that for any $(u_0, v_0) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ with $\|(u_0, v_0)\|_{L^2\times L^2} \leq \rho$, (NLS) with (IC) has the unique solution $(u, v) \in X(I) \times X(I)$ with $I = [-T(\rho), T(\rho)]$, where*

$$X(I) = (C_t \cap L_t^\infty)(I; L_x^2) \cap L_t^{q_0}(I; L_x^{r_0}),$$

$(q_0, r_0) = (4, \infty)$ if $d = 1$, $0 < \frac{2}{q_0} = 1 - \frac{2}{r_0} < 1$ with r_0 sufficiently large if $d = 2$, and $(q_0, r_0) = (2, \frac{2d}{d-2})$ if $d \geq 3$. If $d = 4$, then for any $(u_0, v_0) \in L^2(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$, there exists $T(u_0, v_0) > 0$ such that (NLS) with (IC) has the unique solution $(u, v) \in X(I) \times X(I)$ with $I = [-T(u_0, v_0), T(u_0, v_0)]$. Moreover, the unique solution (u, v) to (NLS) conserves its mass:

$$\text{(Mass)} \quad M(u, v) := \|u\|_{L_x^2}^2 + 2\|v\|_{L_x^2}^2$$

with respect to time t.

Theorem 1.28 (Local well-posedness in $H^1 \times H^1$, [64]). *If $1 \leq d \leq 5$, then for any $\rho > 0$, there exists $T(\rho) > 0$ such that for any $(u_0, v_0) \in H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ with $\|(u_0, v_0)\|_{H^1\times H^1} \leq \rho$, (NLS) with (IC) has the unique solution $(u, v) \in Y(I) \times Y(I)$ with $I = [-T(\rho), T(\rho)]$, where*

$$Y(I) = (C_t \cap L_t^\infty)(I; H_x^1) \cap L_t^{q_0}(I; W_x^{1,r_0}),$$

$(q_0, r_0) = (4, \infty)$ if $d = 1$, $0 < \frac{2}{q_0} = 1 - \frac{2}{r_0} < 1$ with r_0 sufficiently large if $d = 2$, and $(q_0, r_0) = (2, \frac{2d}{d-2})$ if $d \geq 3$. If $d = 6$, then for any $(u_0, v_0) \in H^1(\mathbb{R}^6) \times H^1(\mathbb{R}^6)$, there exists

$T(u_0,v_0)>0$ such that (NLS) with (IC) has the unique solution $(u,v)\in Y(I)\times Y(I)$ with $I=[-T(u_0,v_0),T(u_0,v_0)]$. Moreover, the unique solution (u,v) to (NLS) conserves its mass, energy, and momentum:

$$\text{(Energy)}\quad E(u,v):=\|\nabla u\|_{L^2}^2+\kappa\|\nabla v\|_{L^2}^2-2Re\int_{\mathbb{R}^d}v(x)\overline{u(x)}^2dx,$$

$$\text{(Momentum)}\quad \mathcal{M}(u,v):=Im\int_{\mathbb{R}^d}\{\overline{u(x)}\nabla u(x)+\overline{v(x)}\nabla v(x)\}dx$$

with respect to time t.

Theorem 1.29 (Local well-posedness in $H^1\times H^1\cap|x|^{-1}L^2\times|x|^{-1}L^2$, [64])**.** *If $1\le d\le 5$, then for any $\rho>0$, there exists $T(\rho)>0$ such that for any $(u_0,v_0)\in H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$ with $(xu_0,xv_0)\in L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ and $\|(u_0,v_0)\|_{H^1\times H^1}+\|(xu_0,xv_0)\|_{L^2\times L^2}\le\rho$, (NLS) with (IC) has the unique solution $(u,v)\in Z(I)\times Z(I)$ with $I=[-T(\rho),T(\rho)]$, where*

$$Z(I):=\{u\in Y(I):xu\in X(I)\},\quad \|u\|_{Z(I)}:=\|u\|_{Y(I)}+\|xu\|_{X(I)}.$$

If $d=6$, then for any $(u_0,v_0)\in H^1(\mathbb{R}^6)\times H^1(\mathbb{R}^6)$ with $(xu_0,xv_0)\in L^2(\mathbb{R}^6)\times L^2(\mathbb{R}^6)$, there exists $T(u_0,v_0)>0$ such that (NLS) with (IC) has the unique solution $(u,v)\in Z(I)\times Z(I)$ with $I=[-T(u_0,v_0),T(u_0,v_0)]$.

We turn to time behavior of solution to (NLS), which is defined as a similar to (NLS_0). We define time behaviors of solutions to (NLS) clearly.

Definition 1.30 (Time behavior of solutions to (NLS))**.** Let $(u_0,v_0)\in X\times X$ for a Hilbert space X and (u,v) be a solution to (NLS) with (IC). Let $(T_{\min},T_{\max})$ be the maximal lifespan of the solution (u,v) to (NLS).

- (Scattering) We say that the solution (u,v) to (NLS) scatters in positive time (resp. negative time) if $T_{\max}=\infty$ (resp. $T_{\min}=-\infty$) and there exists $(\phi_+,\psi_+)\in X\times X$ (resp. $(\phi_-,\psi_-)\in X\times X$) such that
$$\lim_{t\to+\infty}\|(e^{-it\Delta}u(t),e^{-\kappa it\Delta}v(t))-(\phi_+,\psi_+)\|_{X\times X}=0,$$
$$\Big(\text{resp.}\quad \lim_{t\to-\infty}\|(e^{-it\Delta}u(t),e^{-\kappa it\Delta}v(t))-(\phi_-,\psi_-)\|_{X\times X}=0\Big).$$
- (Blow-up) We say that (u,v) blows up in positive time (resp. negative time) if $T_{\max}<\infty$ (resp. $T_{\min}>-\infty$).
- (Grow-up) We say that the solution (u,v) grows up in positive time (resp. negative time) if $T_{\max}=\infty$ (resp. $T_{\min}=-\infty$) and
$$\limsup_{t\to+\infty\,(\text{resp.}\,t\to-\infty)}\|(u(t),v(t))\|_{X\times X}=\infty.$$
- (Standing waves) We say that the solution (u,v) is a standing wave if (u,v) forms $(e^{i\omega t}\phi_\omega,e^{2i\omega t}\psi_\omega)$ for $\omega>0$, where $(\phi_\omega,\psi_\omega)$ is a solution to the following elliptic equation:
$$\begin{cases}-\omega\phi_\omega+\Delta\phi_\omega=-2\psi_\omega\phi_\omega, & x\in\mathbb{R}^d,\\ -2\omega\psi_\omega+\kappa\Delta\psi_\omega=-\phi_\omega^2, & x\in\mathbb{R}^d\end{cases}\tag{SP_ω}$$

Hayashi–Li–Ozawa [60] investigated solutions to (NLS) near the trivial scattering solution $(0,0)$ under $(u_0,v_0)\in H^{\frac{d}{2}-2}(\mathbb{R}^d)\times H^{\frac{d}{2}-2}(\mathbb{R}^d)$ $(d\ge4)$ and $(u_0,v_0)\in\mathcal{F}H^{\frac12}(\mathbb{R}^3)\times\mathcal{F}H^{\frac12}(\mathbb{R}^3)$ with $\|f\|_{\mathcal{F}H^{\frac12}}:=\|f\|_{L^2}+\|f\|_{\mathcal{F}\dot{H}^{\frac12}}$. Hayashi–Ozawa–Tanaka [64] proved also the existence of the ground sate to the elliptic equation (SP_ω) for $1\le d\le5$. We recall the ground state to (SP_ω). A set of the all ground state $\mathcal{G}_\omega$ is defined as

$$\mathcal{G}_\omega:=\{(\phi,\psi)\in\mathcal{A}_\omega:S_\omega(\phi,\psi)\le S_\omega(\Phi,\Psi)\text{ for any }(\Phi,\Psi)\in\mathcal{A}_\omega\},$$

$$\text{(Action)}\quad S_\omega(\phi,\psi):=\frac{\omega}{2}M(\phi,\psi)+\frac12E(\phi,\psi),$$

$$\mathcal{A}_\omega:=\{(\phi,\psi)\in H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)\setminus\{(0,0)\}:S_\omega'(\phi,\psi)=0\}.$$

Dinh [23, 24] observed solutions to (NLS) with data (u_0, v_0), which is near the ground state to (SP_ω) and showed a stability result with $d = 3$ and an instability result with $d = 4, 5$. The existence of blow-up solutions is known in [64, 101]. Hayashi–Ozawa–Tanaka [64] proved a blow-up result under the finite variance condition and Ogawa–Uriya [101] gave a blow-up result under the radial condition. In [53] (or a masters's thesis), the author showed the following result under the mass-resonance condition $\kappa = \frac{1}{2}$.

Theorem 1.31 (H. [53])**.** *Let $d = 5$ and $\kappa = \frac{1}{2}$. Let $(u_0, v_0) \in H^1(\mathbb{R}^5) \times H^1(\mathbb{R}^5)$ and $(\phi_\omega, \psi_\omega)$ be the ground state to (SP_ω). We assume that there exists $\omega > 0$ such that $S_\omega(u_0, v_0) < S_\omega(\phi_\omega, \psi_\omega)$.*

(1) *If $K(u_0, v_0) \geq 0$, then the solution (u, v) to (NLS) with (IC) scatters.*
(2) *If $K(u_0, v_0) < 0$, then the solution (u, v) to (NLS) with (IC) blows up or grows up. Moreover, if $(xu_0, xv_0) \in L^2(\mathbb{R}^5) \times L^2(\mathbb{R}^5)$ or (u_0, v_0) is radially symmetric, then the solution (u, v) to (NLS) with (IC) blows up,*

where a functional K is called the virial functional and is defined as

$$K(f, g) := \mathcal{D}^{20,8} S_\omega(f, g) = 8\|\nabla f\|_{L^2}^2 + 8\kappa\|\nabla g\|_{L^2}^2 - 20 Re\langle g, f^2\rangle_{L^2}.$$

The Galilean invariance plays an important role in the argument of scattering part in Theorem 1.31. Roughly speaking, (NLS) with the mass-resonance condition is similar to the single nonlinear Schrödinger equation (NLS_0). On the other hand, it is not clear whether (NLS) is similar to the single one in general. There are few works related to the global dynamics of the (NLS) without mass resonance condition. Inui–Kishimoto–Nishimura [72] obtained a scattering result below the ground state in the L^2-critical case ($d = 4$) under the assumption of radial symmetry. Moreover, they also showed a blow-up result below in the case $d = 5, 6$ and a blow-up or grow-up result in the case $d = 4$ under the assumption of radial symmetry in [73]. The author–Inui–Nishimura [58] showed a scattering result below the ground state for (NLS) without the mass resonance condition.

Theorem 1.32 (H.–Inui–Nishimura, [58])**.** *Let $d = 5$ and $\kappa \neq \frac{1}{2}$. Let $(u_0, v_0) \in H^1_{rad}(\mathbb{R}^5) \times H^1_{rad}(\mathbb{R}^5)$ and $(\phi_\omega, \psi_\omega)$ be the ground state to (SP_ω). We assume that there exists $\omega > 0$ such that $S_\omega(u_0, v_0) < S_\omega(\phi_\omega, \psi_\omega)$ and $K(u_0, v_0) \geq 0$. Then, the solution (u, v) to (NLS) with (IC) scatters.*

Remark 1.33.

(1) If $(u_0, v_0) \in H^1_{\text{rad}}(\mathbb{R}^5) \times H^1_{\text{rad}}(\mathbb{R}^5)$, then the solution $(u(t), v(t))$ is also radially symmetric for all time and its momentum $\mathcal{M}(u(t), v(t))$ is identically zero.
(2) In the opposite case $K(u_0, v_0) < 0$, Inui–Kishimoto–Nishimura [73] showed that the solution blows up in both time directions. And so, the behavior of the radially symmetric solution to (NLS) below the ground state completely determined by the sign of the functional K at initial time.

Remark 1.34. After Main theorem 1.32 was announced on arXiv, Wang–Yang [120] gave the same result independently by using the argument in [32]. Moreover, they also proved a scattering result for non-radial solutions under $|\kappa - \frac{1}{2}| < \varepsilon$ by using the argument in [33].

In the four (mass critical case) and five (inter critical case) dimensions, the ground state $(\phi_\omega, \psi_\omega)$ to (SP_ω) expresses a sharp threshold between scattering solutions and other solutions (see [72], Theorem 1.31, and Theorem 1.32). However, three dimensional case seems to be different from those cases as the single equation in the point of stability of the ground state $(\phi_\omega, \psi_\omega)$ (see [23, 24]). In three dimensions, we study (NLS) with mass resonance condition ($\kappa = \frac{1}{2}$) in a homogeneous weighted space $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, not in the homogeneous Sobolev space $\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$ since we want to work with the scaling critical space without radial symmetry. Let us make the notion of the solution clear. We need a slight modification of the notion compared with L^2 or H^1 solutions because the Schrödinger flow is not unitary in the homogeneous weighted space $\mathcal{F}\dot{H}^{\frac{1}{2}}$.

Definition 1.35 (Solution)**.** Let $I \subset \mathbb{R}$ be a nonempty time interval. We say that a pair of functions $(u,v) : I \times \mathbb{R}^3 \to \mathbb{C}^2$ is a solution to (NLS) on I if $(e^{-it\Delta}u(t), e^{-\frac{1}{2}it\Delta}v(t)) \in C_t(I;\mathcal{F}\dot{H}^{\frac{1}{2}}) \times C_t(I;\mathcal{F}\dot{H}^{\frac{1}{2}})$ and the Duhamel formula

$$\begin{cases} e^{-it\Delta}u(t) = e^{-i\tau\Delta}u(\tau) + 2i\int_\tau^t e^{-is\Delta}(v\overline{u})(s)ds, \\ e^{-\frac{1}{2}it\Delta}v(t) = e^{-\frac{1}{2}i\tau\Delta}v(\tau) + i\int_\tau^t e^{-\frac{1}{2}is\Delta}(u^2)(s)ds \end{cases}$$

holds in $\mathcal{F}\dot{H}^{\frac{1}{2}}$ for any $t,\tau \in I$.

This definition of solutions is not time-translation invariant. That is, if (u,v) is a solution to (NLS), then $(u(\cdot+\tau), v(\cdot+\tau))$ is not necessarily a solution for $\tau \in \mathbb{R}$. To state the local well-posedness for (NLS), we introduce the function spaces $\dot{X}^{s,r}_m(t)$, W_1, and W_2 defined by norms

$$\|f\|_{\dot{X}^{s,r}_m(t)} := \left\| \left(-\frac{t^2}{m^2}\Delta\right)^{\frac{s}{2}} e^{-\frac{im|x|^2}{2t}} f \right\|_{L^r_x}, \quad \|f\|_{W_j} := \|f\|_{L^{6,2}_t \dot{X}^{\frac{1}{2},\frac{18}{7}}_{2^{j-2}}}. \tag{1.15}$$

For a space $\dot{X}^{s,r}_m(t)$, we omit the second exponent when $r=2$, that is, $\dot{X}^s_m(t) = \dot{X}^{s,2}_m(t)$. We discuss these function spaces in more detail in subsubsection 3.1.2 and 3.1.3, below. The following is our result on the local well-posedness. A more detailed version is given later as Theorem 3.18.

Theorem 1.36 (Local well-posedness in $\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}$)**.** *Let $d=3$ and $\kappa = \frac{1}{2}$. For any initial time $t_0 \in \mathbb{R}$ and data $(u_0,v_0) \in \dot{X}^{1/2}_{1/2}(t_0) \times \dot{X}^{1/2}_1(t_0)$, there exist an open interval $I \ni t_0$ and a unique solution $(u,v) \in (C_t(I;\dot{X}^{1/2}_{1/2}) \cap W_1(I)) \times (C_t(I;\dot{X}^{1/2}_1) \cap W_2(I))$ to (NLS) with the initial condition $(u(t_0), v(t_0)) = (u_0,v_0)$. Moreover, the solution depends continuously on the initial data.*

Now, we turn to the large time behavior of solutions to (NLS). It can be said that the main purpose is to investigate threshold phenomena between scattering solutions near a prescribed "trivial scattering set" and non-scattering solutions, taking a system nature into account. We define $\mathcal{S}_+$ as the set of initial data $(u_0,v_0) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ for which the corresponding solution scatters. A straightforward generalization of the quantity (1.9) is

$$\inf\left\{ (\|u_0\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + \alpha\|v_0\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}})^{1/2} : (u_0,v_0) \notin \mathcal{S}_+ \right\} \tag{1.16}$$

with some constant $\alpha > 0$. However, there may not be a strong motivation to study the distance from the trivial solution $(0,0)$ other than the similarity to that in the single equation case, (1.9). We want to find a different way of sizing which is connected with a system nature. To this end, we look at the fact that not only the zero solution $(0,0)$ but also all solutions of the form $(0, e^{\frac{1}{2}it\Delta}v_0)$ can be also regarded as a trivial scattering solution for arbitrary $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Taking this fact into account, one natural choice of the scattering threshold would be with respect to the distance of a data from the set $\{0\} \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. This choice leads us to consider the following optimization problem:

$$\ell_{v_0} := \inf\{\|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} : (u_0,v_0) \notin \mathcal{S}_+\} \in (0,\infty]. \tag{1.17}$$

By using a stability type argument, we will show that $\ell_{v_0} > 0$ for any $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ (see, Proposition 3.20). The following criterion is obvious by the definition of ℓ_{v_0}.

Proposition 1.37 (Sharp small data scattering)**.** *Let $d=3$ and $\kappa=\frac{1}{2}$. Let $(u_0,v_0) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and (u,v) be the solution to (NLS) with (IC). If $\|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} < \ell_{v_0}$, then (u,v) scatters.*

The above criterion "$\|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} < \ell_{v_0}$" is sharp in such a sense that the number ℓ_{v_0} may not be replaced with any larger number. The following two questions arise: (a) to obtain a condition which implies ℓ_{v_0} is finite; (b) to show the existence of a minimizer to ℓ_{v_0} (when ℓ_{v_0} is finite).

It will turn out that the following quantity $\ell_{v_0}^{\dagger}$ plays an important role in the analysis of ℓ_{v_0}.

Definition 1.38. For $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $0 \le \ell < \infty$, we let

$$L_{v_0}(\ell) := \sup\left\{ \|(u,v)\|_{W_1([0,T_{\max}))\times W_2([0,T_{\max}))} \,\middle|\, \begin{array}{l} (u,v) \text{ is a solution to (NLS) on } [0,T_{\max}), \\ v(0)=v_0,\ \|u(0)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \ell,\ u(0)\in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \end{array} \right\},$$

where $W_j([0,T_{\max}))$ $(j=1,2)$ is a Strichartz-like function space defined in subsubsection 3.1.3, below. Further, define

$$\ell_{v_0}^{\dagger} := \sup\{\ell : L_{v_0}(\ell) < \infty\} \in (0,\infty].$$

We have $\ell_{v_0}^{\dagger} \le \ell_{v_0}$ by their definitions (see Lemma 3.27 for more detail). Intuitively, this can be seen by noticing that if $\|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} < \ell_{v_0}^{\dagger}$ then not only (u,v) scatters but also we have a priori bound $\|(u,v)\|_{W_1([0,\infty))\times W_2([0,\infty))} \le L_{v_0}(\|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}) < \infty$. As for the single-equation (NLS_0), it is known that these two kinds of quantities coincide each other (see [90]). Our first result is as follows.

Main theorem 1.39 (H.–Masaki, [59])**.** *Let $d=3$ and $\kappa=\frac{1}{2}$. $\ell_{v_0}^{\dagger} = \min\{\ell_0, \ell_{v_0}\}$ holds for any $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, including the case where the both sides are infinite. In particular, $\ell_0^{\dagger} = \ell_0$ holds.*

It is worth noting that $\ell_{v_0}^{\dagger} = \infty$ guarantees $\ell_{v_0} = \infty$ but the inverse is not necessarily true. Our interest in the sequel is to see what happens when $\ell_{v_0}^{\dagger} < \infty$. In the case $v_0 = 0$, we have $\ell_0^{\dagger} = \ell_0$, including the case of the both are infinite, as seen in Main theorem 1.39.

Main theorem 1.40 (H.–Masaki, [59])**.** *Let $d=3$ and $\kappa=\frac{1}{2}$. If $\ell_0^{\dagger} < \infty$, then there exists a minimizer $(u^{(0)}, v^{(0)})$ to $\ell_0 (= \ell_0^{\dagger})$ such that*

(1) *$v^{(0)}(0) = 0$ and $\|u^{(0)}(0)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_0$,*
(2) *$(u^{(0)}, v^{(0)})$ does not scatter.*

So far, we do not know whether $\ell_0^{\dagger} < \infty$ or not. It will turn out that this question is important to understand the attainability of ℓ_{v_0} for all non-zero v_0. One quick consequence of $\ell_0^{\dagger} = \infty$ is $\ell_{v_0} = \ell_{v_0}^{\dagger}$ for all v_0, which follows from Main theorem 1.39. We will resume this subject later.

Let us move on to the case $v_0 \neq 0$. Suppose $\ell_{v_0}^{\dagger} < \infty$. Then, we have either

$$\ell_{v_0}^{\dagger} = \ell_{v_0} \quad \text{or} \quad \ell_{v_0}^{\dagger} < \ell_{v_0}. \tag{1.18}$$

The following Main theorem 1.41 is about the first case and Main theorem 1.42 is about the second case.

Main theorem 1.41 (H.–Masaki, [59])**.** *Let $d=3$ and $\kappa=\frac{1}{2}$. Fix $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \setminus \{0\}$. Suppose that $\ell_{v_0}^{\dagger} = \ell_{v_0} < \ell_0$. Then, there exists a minimizer $(u^{(v_0)}, v^{(v_0)})$ to ℓ_{v_0}, that is,*

(1) *$v^{(v_0)}(0) = v_0$ and $\|u^{v_0}(0)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{v_0}$,*
(2) *$(u^{(v_0)}, v^{(v_0)})$ does not scatter.*

The case $\ell_{v_0}^{\dagger} = \ell_{v_0} = \ell_0 < \infty$ is excluded in the above theorem. We consider this exceptional case in Remark 1.43, below.

Let us consider the second case of (1.18). In this case, the following strange thing occurs: Take $u_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with $\|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{v_0}^{\dagger}$ and consider the corresponding solution (u,v) with the data (u_0, v_0). Then, on one hand, the solution (u,v) scatters for any choice of such u_0 since

$\|u_0\|_{\mathcal{F}\dot{H}^{\frac12}} < \ell_{v_0}$. However, on the other hand, for arbitrarily large number $N > 0$, one can choose $u_0 \in \mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$ so that the corresponding solution (u, v) satisfies

$$\|(u, v)\|_{W_1([0,\infty))\times W_2([0,\infty))} \geq N.$$

The next theorem tells us how this is "attained". Notice that the second case of (1.18) occurs only when $\ell_0 = \ell_{v_0}^\dagger < \infty$, thanks to Main theorem 1.39. Consequently, there is a minimizer to ℓ_0 in this case, by means of Main theorem 1.40.

Main theorem 1.42 (H.–Masaki, [59]). *Let $d = 3$ and $\kappa = \frac12$. Fix $v_0 \in \mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3) \setminus \{0\}$. Suppose that $\ell_{v_0}^\dagger < \ell_{v_0}$. Pick a sequence $\{u_{0,n}\}_n \subset \mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$ satisfying $\|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac12}} < \ell_{v_0}^\dagger$ for all $n \geq 1$,*

$$\lim_{n\to\infty} \|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac12}} = \ell_{v_0}^\dagger,$$

and

$$\lim_{n\to\infty} \|(u_n, v_n)\|_{W_1([0,\infty))\times W_2([0,\infty))} = \infty,$$

where (u_n, v_n) is a solution with the initial data $(u_n(0), v_n(0)) = (u_{0,n}, v_0)$. Then, there exist a subsequence of n, a minimizer $(u^{(0)}, v^{(0)})$ to ℓ_0, and two sequences $\{\xi_n\}_n \subset \mathbb{R}^3$ and $\{h_n\}_n \subset 2^{\mathbb{Z}}$ such that

$$|\log h_n| + |\xi_n| \longrightarrow \infty$$

and

$$e^{-ix\cdot h_n\xi_n}(u_{0,n})_{\{h_n\}} \longrightarrow u^{(0)}(0) \quad \text{in} \quad \mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$$

hold along the subsequence. In particular, along the same subsequence, it holds for any $\tau \in (0, T_{\max}(u^{(0)}, v^{(0)}))$ that

$$(u_n(t), v_n(t)) = \left(e^{-it|\xi_n|^2+ix\cdot\xi_n} u^{(0)}_{[h_n^{-1}]}(t, x - 2\xi_n t), e^{-2it|\xi_n|^2+2ix\cdot\xi_n} v^{(0)}_{[h_n^{-1}]}(t, x - 2\xi_n t)\right) + (0, e^{\frac12 it\Delta}v_0) + o_{\dot{X}^{1/2}_{1/2}(t)\times\dot{X}^{1/2}_1(t)}(1)$$

for $0 \leq t \leq \tau h_n^2$, where $T_{\max}(u^{(0)}, v^{(0)})$ is the maximal existence time of $(u^{(0)}, v^{(0)})$.

Remark 1.43. The special case $\ell_{v_0}^\dagger = \ell_0 = \ell_{v_0} < \infty$ ($v_0 \neq 0$) is not included in the above two theorems. In this exceptional case, the conclusion of Main theorem 1.41 or Main theorem 1.42 holds. Namely, if there does not exist a minimizer to ℓ_{v_0} as in Main theorem 1.41, then the conclusion of Main theorem 1.42 holds.

Let us summarize the above results. Let $v_0 \in \mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$ be a given function. If $\ell_{v_0}^\dagger = \infty$, then we have $\ell_{v_0} = \infty$ (Main theorem 1.39) and hence any solution satisfying $v(0) = v_0$ scatters (Proposition 1.37). On the other hand, if $\ell_{v_0}^\dagger < \infty$ and $v_0 \neq 0$, then we have either Main theorem 1.41 or Main theorem 1.42 according to the dichotomy (1.18). Remark that the first case in (1.18) contains an exceptional case discussed in Remark 1.43. When $v_0 = 0$, we do not have the dichotomy, we have $\ell_0 = \ell_0^\dagger$ (Main theorem 1.39). If ℓ_0 is finite then there exists a minimizer (Main theorem 1.40).

The question whether $\ell_0 = \infty$ or not would be an interesting question to the system (NLS). We do not have the answer yet. Let us formulate the problem without using our terminology:

Question. *In (NLS), does $v_0 = 0$ implies scattering of the corresponding solution for any u_0?*

If it were true, that is, if $\ell_0 = \ell_0^\dagger = \infty$, then Main theorem 1.39 tells us that $\ell_{v_0}^\dagger = \ell_{v_0}$ is true for any v_0, as mentioned above.

Although we do not know the exact value of ℓ_{v_0}, we are able to have a condition which implies the finiteness of ℓ_{v_0} and to give an upper bound for ℓ_{v_0}. A simple one is a condition in terms of the energy.

Theorem 1.44 (H.–Masaki, [59]). *Let $d = 3$ and $\kappa = \frac{1}{2}$. Fix nontrivial $u_0, v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. If $E(u_0, v_0) \le 0$, then the corresponding solution (u, v) does not scatter. In particular, $\ell_{v_0} \le \|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$.*

As a consequence, one sees that a standing wave solution, not only the ground state but also all excited states, is not a minimizer of the optimizing problem (1.17). Let $(\phi_\omega, \psi_\omega)$ be a solution to the elliptic equation (SP_ω). We have $E(\phi_\omega, \psi_\omega) < 0$ (see [64, Theorem 4.1]). As a result of Theorem 1.44, there exists an open neighborhood $\mathcal{N} \subset \mathbb{R}^2$ of $(1, 1) \in \mathbb{R}^2$ such that $(c_1\phi_\omega, c_2\psi_\omega) \notin \mathcal{S}_+$ for all $(c_1, c_2) \in \mathcal{N}$. Hence, any solution to (SP_ω) is not an optimizer to (1.17). In particular, ℓ_{ψ_ω} is strictly smaller than $\|\phi_\omega\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$. Similarly, $(\phi_\omega, \psi_\omega)$ is not a solution to (1.16) for any $\alpha > 0$. We can also find intuitively the fact from the orbital stability of a standing wave $(e^{i\omega t}\phi_\omega, e^{2i\omega t}\psi_\omega)$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ (see [23] for more detail).

In our context, we want to find a condition which is stated in terms of v_0 only. We give two criteria in this direction. The first one is for large data case:

Corollary 1.45 (H.–Masaki, [59]). *Let $d = 3$ and $\kappa = \frac{1}{2}$. For any $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ with $v_0 \neq 0$, there exists $c_0 > 0$ such that the estimate $\ell_{cv_0} \lesssim_{v_0} c^{\frac{1}{2}}$ holds for any $c \ge c_0$.*

The second one is criterion for a specific v_0:

Corollary 1.46 (H.–Masaki, [59]). *Let $d = 3$ and $\kappa = \frac{1}{2}$. Let $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. If there exists $\theta \in \mathbb{R}$ such that a Schrödinger operator $-\Delta - 2\mathrm{Re}(e^{i\theta}v_0)$ has a negative eigenvalue then $\ell_{v_0} < \infty$. Moreover, if $\varphi \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ is a real-valued eigenfunction associated with a negative eigenvalue $\tilde{e} < 0$ of $-\Delta - 2\mathrm{Re}(e^{i\theta}v_0)$ then the estimate*

$$\ell_{v_0} \le \frac{\|\varphi\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}}{\sqrt{2|\tilde{e}|}\|\varphi\|_{L^2}} \|\nabla v_0\|_{L^2}$$

holds.

Remark 1.47. The estimate given in Corollary 1.46 is scaling invariant. Indeed, if $\varphi(x)$ is an eigenfunction of $-\Delta - 2\mathrm{Re}(e^{i\theta}v_0)$ associated with a negative eigenvalue $\tilde{e}$ then $\varphi_{\{\lambda\}}(x)$ is an eigenfunction of $-\Delta - 2\mathrm{Re}(e^{i\theta}(v_0)_{\{\lambda\}})$ and the corresponding eigenvalue is $\lambda^2\tilde{e}$, where $f_{\{\lambda\}}$ denotes the scaling (1.12).

Remark 1.48. It is possible to study the optimizing problem (1.16). Let us introduce a slightly different formulation: For $\rho \ge 0$, we let

$$\mathcal{B}(\rho) := \inf \left\{ \|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} : (u_0, v_0) \notin \mathcal{S}_+,\ \|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \rho \right\}. \tag{1.19}$$

Then, for any $\rho \ge 0$ such that $\mathcal{B}(\rho)$ is finite, there exists a minimizer, say (u_ρ, v_ρ), to $\mathcal{B}(\rho)$. The minimizer satisfies $\ell^\dagger_{v_\rho} = \ell_{v_\rho}$ and u_ρ is a minimizer to ℓ_{v_ρ}, that is, $\ell_{v_\rho} = \|u_\rho\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$ (Theorem 3.44). Further, it turns out that the analysis of $\mathcal{B}(\rho)$ is applicable to that of (1.16) (Theorem 3.45). Remark that $\mathcal{B}(0) = \ell_0$ holds and that $\mathcal{B}(\rho)$ is non-increasing in ρ. Hence, $\ell_0 < \infty$, which is our question, can be also phrased as "$\mathcal{B}(\rho) < \infty$ for any $\rho \ge 0$."

Remark 1.49. One can deduce similar results in the frame work of $\mathcal{F}H^1(\mathbb{R}^3) \times \mathcal{F}H^1(\mathbb{R}^3)$ or $(\mathcal{F}H^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)) \times (\mathcal{F}H^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3))$. The main difference of the results in these settings is that one can show that the minimizers belong to the corresponding space, and hence they are global-in-time solutions due to the mass conservation. We would remark that it is not clear our minimizers given in the paper coincide those obtained in the above setting. Since we are working with minimization at fixed time $t = 0$ and there is no time translation invariance, we do not know whether the minimizers have compact orbit nor enjoy additional regularity.

1.3. **Nonlinear Schrödinger equation with a potential.** (NLS$_V$) has physical background as follows: (NLS$_V$) with $V \in L^\infty(\mathbb{R}^d)$ is a model proposed to describe the local dynamics at a nucleation site (see [104]) and (NLS$_V$) with a harmonic potential $V(x) = |x|^2$ is a model proopsed to describe the Bose-Einstein condensate with attractive inter-particle interactions under a magnetic trap (see [6, 49, 119]).

We introduce local well-posedness results of (NLS$_V$) before we consider time behavior of solutions to (NLS$_V$). Cazenave [13, Theorem 4.3.1] showed the following:

Theorem 1.50 (Local well-posedness of (NLS$_V$) I, [13])**.** *Let $d \geq 1$, $1 < p < \infty$ if $d = 1, 2$, and $1 < p < 1 + \frac{4}{d-2}$ if $d \geq 3$. Let V be a real-valued function and $V \in L^\eta(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for $\eta \geq 1$ if $d = 1$ and $\eta > \frac{d}{2}$ if $d \geq 2$. Then, for any $u_0 \in H^1_x(\mathbb{R}^d)$, (NLS$_V$) is locally well-posed, that is, the followings hold:*

- *(Existence and Uniqueness) (NLS$_V$) with (IC) has the unique solution*
$$u \in C_t((T_{min}, T_{max}); H^1_x) \cap C^1_t((T_{min}, T_{max}); H^{-1}_x).$$
- *(Blow-up alternative) If $T_{max} < \infty$ (resp. $T_{min} > -\infty$), then*
$$\lim_{t \nearrow T_{max}} \|u(t)\|_{H^1_x} = \infty, \quad \left(resp. \lim_{t \searrow T_{min}} \|u(t)\|_{H^1_x} = \infty \right).$$
- *(Continuous dependence on the initial data) If $u_{0,n} \longrightarrow u_0$ in $H^1_x(\mathbb{R}^d)$, then for any compact time interval $I \subset (T_{min}, T_{max})$, there exists $n_0 \in \mathbb{N}$ such that the solution u_n to (NLS$_V$) with initial data $u_{0,n}$ is defined on I for any $n \geq n_0$ and satisfies $u_n \longrightarrow u$ in $C_t(I; H^1_x)$ as $n \to \infty$.*

To state a local well-posedness result in Hong [68] and the author–Ikeda [54], we define a potential class $\mathcal{K}_0(\mathbb{R}^3)$ as the norm closure of bounded and compactly supported functions with respect to the global Kato norm
$$\|f\|_{\mathcal{K}} := \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|f(y)|}{|x-y|} dy,$$
that is,
$$\mathcal{K}_0(\mathbb{R}^3) := \overline{\{f \in L^\infty(\mathbb{R}^3) : \text{supp}\, f \text{ is compact.}\}}^{\|\cdot\|_{\mathcal{K}}}.$$
If V satisfies $\|V_-\|_{\mathcal{K}} < 4\pi$ for $V_-(x) := \min\{V(x), 0\}$, then $-\Delta_V$ and $1 - \Delta_V$ are non-negative. More precisely, Hong [68] proved
$$\langle (-\Delta_V) f, f \rangle_{L^2} = \|\nabla f\|_{L^2}^2 + \int_{\mathbb{R}^3} V(x) |f(x)|^2 dx \geq \left(1 - \frac{\|V_-\|_{\mathcal{K}}}{4\pi}\right) \|\nabla f\|_{L^2}^2 > 0,$$
$$\langle (1 - \Delta_V) f, f \rangle_{L^2} = \|f\|_{L^2}^2 + \langle (-\Delta_V) f, f \rangle_{L^2} > 0$$
for any $f \in H^1(\mathbb{R}^3) \setminus \{0\}$. Therefore, the fractional operators $(-\Delta_V)^{\frac{1}{2}}$ and $(1 - \Delta_V)^{\frac{1}{2}}$ are well-defined on the domain $H^1_V(\mathbb{R}^3)$, whose norm is defined as $\|f\|_{H^1_V}^2 := \|(1 - \Delta_V)^{\frac{1}{2}} f\|_{L^2}^2$. We also define Sobolev spaces with a potential $\dot{W}^{s,p}_V(\mathbb{R}^3) := (-\Delta_V)^{-\frac{s}{2}} L^p(\mathbb{R}^3)$ and $W^{s,p}_V(\mathbb{R}^3) := (1 - \Delta_V)^{-\frac{s}{2}} L^p(\mathbb{R}^3)$. If $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$ satisfies $\|V_-\|_{\mathcal{K}} < 4\pi$ for $V_-(x) := \min\{V(x), 0\}$, then $-\Delta_V$ has no eigenvalues, $-\Delta_V$ is self-adjoint into $L^2(\mathbb{R}^3)$ (see [68]), so the Schrödinger evolution group $\{e^{it\Delta_V}\}_{t \in \mathbb{R}}$ is generated on $L^2(\mathbb{R})$ by the Stone's theorem. The following local well-posedness is proved by the fixed point argument with the Strichartz estimate for $\{e^{it\Delta_V}\}_{t \in \mathbb{R}}$ (Proposition 4.4).

Theorem 1.51 (Local well-posedness of (NLS$_V$) II, [54, 68])**.** *Let $d = 3$ and $1 < p < 5$. Let $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$ satisfy $\|V_-\|_{\mathcal{K}} < 4\pi$ for $V_-(x) := \min\{V(x), 0\}$. Then, for any $\rho > 0$, there exists $T(\rho) > 0$ such that for any $u_0 \in H^1_x(\mathbb{R}^3)$ with $\|u_0\|_{H^1_x} \leq \rho$, (NLS$_V$) with (IC) has the unique solution*
$$u \in C_t(I; H^1_x) \cap L^2_t(I; W^{1,6}_V)$$

with $I = [-T(\rho), T(\rho)]$.

The solution u given Theorem 1.50 or 1.51 has conservation laws.

Theorem 1.52 (Conservation laws)**.** *Let u be a solution given Theorem 1.50 or 1.51. Then, the solution u conserves its mass and energy*

$$\text{(Mass)}\quad M(u) := \|u\|_{L_x^2}^2,$$
$$\text{(Energy)}\quad E_V(u) := \frac{1}{2}\|(-\Delta_V)^{\frac{1}{2}}u\|_{L_x^2}^2 - \frac{1}{p+1}\|u\|_{L_x^{p+1}}^{p+1}$$

with respect to time t.

We turn to time behavior of solutions to (NLS$_V$), which is defined as a similar manner to (NLS$_0$). We define time behaviors of solutions to (NLS$_V$) precisely.

Definition 1.53 (Time behaviors of solutions to (NLS$_V$))**.** Let $u_0 \in H_x^1(\mathbb{R}^d)$. Let u be a solution to (NLS$_V$) on $(T_{\min}, T_{\max})$, where $(T_{\min}, T_{\max})$ denotes the maximal existence time of the solution u.

- (Scattering) We say that u scatters in positive time (resp. negative time) if $T_{\max} = \infty$ (resp. $T_{\min} = -\infty$) and there exists $\psi_+ \in H_x^1(\mathbb{R}^d)$ (resp. $\psi_- \in H_x^1(\mathbb{R}^d)$) such that
$$\lim_{t\to+\infty}\|u(t) - e^{it\Delta_V}\psi_+\|_{H_x^1} = 0, \quad \left(\text{resp.} \lim_{t\to-\infty}\|u(t) - e^{it\Delta_V}\psi_-\|_{H_x^1} = 0\right).$$
- (Blow-up) We say that u blows up in positive time (resp. negative time) if $T_{\max} < \infty$ (resp. $T_{\min} > -\infty$).
- (Grow-up) We say that u grows up in positive time (resp. negative time) if $T_{\max} = \infty$ (resp. $T_{\min} = -\infty$) and
$$\limsup_{t\to+\infty}\|u(t)\|_{H_x^1} = \infty, \quad \left(\text{resp.} \limsup_{t\to-\infty}\|u(t)\|_{H_x^1} = \infty\right).$$
- (Standing wave) We say that u is standing wave if $u = e^{i\omega t}Q_{\omega,V}$ for $\omega \in \mathbb{R}$, where $Q_{\omega,V}$ satisfies
$$-\omega Q_{\omega,V} + \Delta_V Q_{\omega,V} = -|Q_{\omega,V}|^{p-1}Q_{\omega,V}. \qquad (\text{SP}_{\omega,V})$$

We introduce known results for time behavior of the solutions to (NLS$_V$). Killip–Murphy–Visan–Zheng [83] showed a scattering result and a blow-up result under $d = 3$, $p = 3$, $u_0 \in H_x^1(\mathbb{R}^3)$, and $V(x) = \frac{\gamma}{|x|^2}$ for $\gamma > -\frac{1}{4}$. Lu–Miao–Murphy [88] showed a scattering result and a blow-up result under $3 \le d \le 6$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, $u_0 \in H^1(\mathbb{R}^d)$, and $V(x) = \frac{\gamma}{|x|^2}$ for

$$\gamma > \begin{cases} -\frac{1}{4}, & (d = 3,\ \frac{7}{3} < p \le 3), \\ -\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{2} - \frac{1}{p-1}\right)^2, & (d = 3,\ 3 < p < 5 \text{ or } 4 \le d \le 6). \end{cases}$$

Zheng [124] showed a scattering result under $d \ge 3$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, $u_0 \in H_{\text{rad}}^1(\mathbb{R}^d)$, $V(x) = \frac{\gamma}{|x|^2}$, $a > -\frac{1}{4}$ if $d = 3$, $\frac{7}{3} < p \le 3$, and $\gamma > -(\frac{d-2}{2})^2 + (\frac{d-2}{2} - \frac{1}{p-1})^2$ if $d = 3$, $3 < p < 5$ or $d \ge 4$. Ikeda–Inui [71] showed a scattering result and a blow-up or grow-up result under $d = 1$, $p > 5$, $u_0 \in H^1(\mathbb{R})$, $V = m\delta_0$ (delta function) for $m > 0$. Ikeda [70] showed a scattering result under $p > 5$, V, $V' \in L_1^1(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : (1 + |\cdot|)f \in L^1(\mathbb{R})\}$, $xV' \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, $xV' \le 0$ and a blow-up or grow-up result under $p > 5$, $xV' \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, $xV' + 2V \ge 0$.

Hong [68] got Theorem 1.55 for time behavior of solutions u to (NLS$_V$) by using the following characterization (Proposition 1.54) with $d = p = 3$ of the ground state $Q_{1,0}$ to (SP$_{\omega,0}$) with $\omega = 1$.

Proposition 1.54 (Gagliardo-Nirenberg inequality with a potential)**.** *Let $d \geq 3$, $1 < p < 1 + \frac{4}{d-2}$, $V \in L^{\frac{d}{2}}(\mathbb{R}^d)$, $V \geq 0$, and $V \neq 0$. Then, the following inequality holds:*

$$\|f\|_{L^{p+1}}^{p+1} < C_{GN}\|f\|_{L^2}^{p+1-\frac{d(p-1)}{2}}\|(-\Delta_V)^{\frac{1}{2}}f\|_{L^2}^{\frac{d(p-1)}{2}}$$

for any $f \in H^1(\mathbb{R}^d) \setminus \{0\}$, where C_{GN} is the best constant and is defined in Proposition 1.16.

Theorem 1.55 (Hong, [68])**.** *Let $d = 3$, $p = 3$, and $u_0 \in H^1(\mathbb{R}^3)$. Let $Q_{1,0}$ be the ground state to (SP$_{\omega,0}$) with $\omega = 1$. Assume that $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$ satisfy $V \geq 0$, $x \cdot \nabla V \leq 0$, and $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ for any $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^3$ with $|\mathfrak{a}| = 1$. We also suppose that*

$$M(u_0)^{\frac{1-s_c}{s_c}}E_V(u_0) < M(Q_{1,0})^{\frac{1-s_c}{s_c}}E_0(Q_{1,0}) \tag{1.20}$$

and

$$\|u_0\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|(-\Delta_V)^{\frac{1}{2}}u_0\|_{L_x^2} < \|Q_{1,0}\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|\nabla Q_{1,0}\|_{L_x^2}. \tag{1.21}$$

Then, the solution u to (NLS$_V$) with (IC) scatters and satisfies

$$\|u(t)\|_{L_x^2}\|(-\Delta_V)^{\frac{1}{2}}u(t)\|_{L_x^2} < \|Q_{1,0}\|_{L_x^2}\|\nabla Q_{1,0}\|_{L_x^2}$$

for any $t \in \mathbb{R}$.

Natural questions arise from this theorem. It is whether a range of the exponent p for nonlinearity can be extend or not. In addition, it is whether we can determine behaviors of a solution to (NLS$_V$) with initial data u_0 satisfying $\|u_0\|_{L_x^2}\|(-\Delta_V)^{\frac{1}{2}}u_0\|_{L_x^2} > \|Q_{1,0}\|_{L_x^2}\|\nabla Q_{1,0}\|_{L_x^2}$ or not. Then, the author and Ikeda got the following result.

Main theorem 1.56 (H.–Ikeda, [54])**.** *Let $d = 3$, $\frac{7}{3} < p < 5$, and $u_0 \in H_x^1(\mathbb{R}^3)$. Let $Q_{1,0}$ be the ground state to (SP$_{\omega,0}$) with $\omega = 1$. Suppose that V satisfies $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ for any $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^3$ with $|\mathfrak{a}| = 1$ and $V \geq 0$. We also suppose that u_0 satisfies (1.20).*

(1) *(Scattering) If $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, $x \cdot \nabla V \leq 0$, and u_0 satisfies (1.4), then the solution u to (NLS$_V$) with (IC) exists globally in time and*

$$\|u(t)\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|\nabla u(t)\|_{L_x^2} < \|Q_{1,0}\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|\nabla Q_{1,0}\|_{L_x^2}$$

for any $t \in \mathbb{R}$. Moreover, if u_0 and V are radially symmetric, then the solution u to (NLS$_V$) with (IC) scatters.

(2) *(Blow-up or grow-up) If "$V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$ or $V \in L^{\sigma}(\mathbb{R}^3)$ for some $\frac{3}{2} < \sigma \leq \infty$", $2V + x \cdot \nabla V \geq 0$, and*

$$\|u_0\|_{L^2}^{\frac{1-s_c}{s_c}}\|(-\Delta_V)^{\frac{1}{2}}u_0\|_{L_x^2} > \|Q_{1,0}\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|\nabla Q_{1,0}\|_{L_x^2}, \tag{1.22}$$

then the solution u to (NLS$_V$) with (IC) satisfies

$$\|u(t)\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|(-\Delta_V)^{\frac{1}{2}}u(t)\|_{L_x^2} > \|Q_{1,0}\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|\nabla Q_{1,0}\|_{L_x^2}$$

for any $t \in (T_{min}, T_{max})$ and blows up or grows up. Furthermore, if either the following (i) or (ii) holds:

(i) *"u_0 and V are radially symmetric", $x \cdot \nabla V \geq 0$, and $V \in L^{\infty}(\mathbb{R}^3)$,*

(ii) *$xu_0 \in L^2(\mathbb{R}^3)$ and "$V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$ or $V \in L^{\sigma}(\mathbb{R}^3)$ for some $\frac{3}{2} < \sigma \leq \infty$",*

then u blows up.

As a corollary of Main theorem 1.56, we can get the following result.

Corollary 1.57. *The similar blow-up or grow-up result and blow-up result (ii) in the mass-critical case $p = \frac{7}{3}$ to Main theorem 1.56 holds. We assume that the potential V satisfies the same assumptions as in Main theorem 1.56 (2). The initial data $u_0 \in H^1(\mathbb{R}^3)$ satisfies $E_V(u_0) < 0$ instead of (1.22). Then, the same conclusion as Main theorem 1.56 (2) holds.*

Remark 1.58. Mizutani in [93, Theorem 2.2 and Example 3.3] proved that if $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $V \geq 0$, then there exist $\psi_\pm \in H^1_x(\mathbb{R}^3)$ such that

$$\lim_{t \to \pm\infty} \|e^{it\Delta_V}\psi - e^{it\Delta}\psi_\pm\|_{H^1_x} = 0,$$

where the double sign corresponds. This implies that the scattering solution u in Main theorem 1.56 (or Theorem 1.55) approaches not only a linear solution (with a potential) but also a free solution (linear solution without a potential) as $t \to \pm\infty$.

We compare Main theorem 1.56 with Theorem 1.55.

Remark 1.59. Main theorem 1.56 extends a range of the nonlinear power p. In Main theorem 1.56, it is assumed that u_0 and V are radially symmetric in scattering part to use the argument in [32]. The author think that we can remove the radial assumption of scattering result in Main theorem 1.56 by using the argument in [77]. We characterize sufficient condition of scattering by $\|\nabla u_0\|_{L^2_x}$ not $\|(-\Delta_V)^{\frac{1}{2}} u_0\|_{L^2_x}$. Since $\|\nabla u_0\|_{L^2_x} \leq \|(-\Delta_V)^{\frac{1}{2}} u_0\|_{L^2_x}$ holds by $V \geq 0$, our result utilized weaker expression in this point. Main theorem 1.56 also contains a blow-up or grow-up result and a blow-up result.

We notice that Main theorem 1.56 does not separate

$$\{u_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^3) : M(u_0)^{\frac{1-s_c}{s_c}} E_V(u_0) < M(Q_{1,0})^{\frac{1-s_c}{s_c}} E_0(Q_{1,0})\} \tag{1.23}$$

since there exists no $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ satisfying $x \cdot \nabla V \leq 0$ and $2V + x \cdot \nabla V \geq 0$. The conditions $x \cdot \nabla V \leq 0$ and $2V + x \cdot \nabla V \geq 0$ are used to control the virial functional K_V, which is defined as

$$K_V(f) := \frac{d}{d\lambda}\bigg|_{\lambda=0} S_{\omega,V}(e^{d\lambda} f(e^{2\lambda}\,\cdot)) = 2\|\nabla f\|^2_{L^2_x} - \int_{\mathbb{R}^d} (x \cdot \nabla V)|f(x)|^2 dx - \frac{d(p-1)}{p+1}\|f\|^{p+1}_{L^{p+1}_x},$$
$$S_{\omega,V}(f) := \frac{\omega}{2} M(f) + E_V(f).$$

To separate (1.23), we consider an expression corresponding to (1.2). We define a set of "radial" ground state solutions to (NLS$_V$) as

$$\mathcal{G}_{\omega,V,\mathrm{rad}} := \{\phi \in \mathcal{A}_{\omega,V,\mathrm{rad}} : S_{\omega,V}(\phi) \leq S_{\omega,V}(\psi) \text{ for any } \psi \in \mathcal{A}_{\omega,V,\mathrm{rad}}\}, \tag{1.24}$$

where

$$\mathcal{A}_{\omega,V,\mathrm{rad}} := \{\psi \in H^1_{\mathrm{rad}}(\mathbb{R}^d) \setminus \{0\} : S'_{\omega,V}(\psi) = 0\}.$$

We characterize a "radial" ground state to (NLS$_V$) by the minimization problem

$$r^{\alpha,\beta}_{\omega,V} := \inf\{S_{\omega,V}(f) : f \in H^1_{\mathrm{rad}}(\mathbb{R}^d) \setminus \{0\},\ K^{\alpha,\beta}_{\omega,V}(f) = 0\}, \tag{1.25}$$
$$K^{\alpha,\beta}_{\omega,V}(f) := \mathcal{D}^{\alpha,\beta} S_{\omega,V}(f) = \frac{d}{d\lambda}\bigg|_{\lambda=0} S_{\omega,V}(e^{\alpha\lambda} f(e^{\beta\lambda}\,\cdot))$$

for $(\alpha,\beta) \in \mathbb{R}^2$ satisfying

$$\alpha > 0, \qquad \beta \geq 0, \qquad \underline{\mu} := 2\alpha - d\beta \geq 0. \tag{1.26}$$

We want to deal with (1.25) with $(\alpha,\beta) = (d,2)$ as the case (NLS$_0$). However, $S_{\omega,V}$ does not include $x \cdot \nabla V$, so we treat (1.25) with $(\alpha,\beta) = (1,0)$ to prove the existence of a "radial" ground state first. We note that $K^{1,0}_{\omega,V}$ ($(\alpha,\beta) = (1,0)$) is called the Nehari functional and can be written as

$$\mathcal{N}_{\omega,V}(f) := K^{1,0}_{\omega,V}(f) = \omega\|f\|^2_{L^2_x} + \|(-\Delta_V)^{\frac{1}{2}} f\|^2_{L^2_x} - \|f\|^{p+1}_{L^{p+1}_x}.$$

To compare, we also consider a minimization problem, which removes the constraint of spherical symmetry from (1.25):

$$n^{\alpha,\beta}_{\omega,V} := \inf\{S_{\omega,V}(f) : f \in H^1(\mathbb{R}^d) \setminus \{0\},\ K^{\alpha,\beta}_{\omega,V}(f) = 0\}.$$

It is known that $n^{\alpha,\beta}_{\omega,0}$ and $r^{\alpha,\beta}_{\omega,0}$ are attained by the ground state $Q_{\omega,0}$ to (SP$_{\omega,0}$) (e.g. see [10]). Therefore, $n^{\alpha,\beta}_{\omega,0}$ and $r^{\alpha,\beta}_{\omega,0}$ are independent of (α,β) and the identity $n^{\alpha,\beta}_{\omega,0} = r^{\alpha,\beta}_{\omega,0}$ holds. For simplicity, we use notations $n_{\omega,0} := n^{\alpha,\beta}_{\omega,0}$ and $r_{\omega,0} := r^{\alpha,\beta}_{\omega,0}$.

For $n^{1,0}_{\omega,V}$ and $r^{1,0}_{\omega,V}$, the author and Ikeda proved the following theorem.

Main theorem 1.60 (H.–Ikeda, [55]). *Let $d=3$, $\omega>0$, $V\geq 0$, and $V\neq 0$.*

- *(non-radial case) Let $1<p\leq 5$ and $V\in L^{\frac{3}{2}}(\mathbb{R}^3)+L^{\sigma}(\mathbb{R}^3)$ for some $\frac{3}{2}<\sigma<\infty$. Then, $n^{1,0}_{\omega,V}=n_{\omega,0}$. Moreover, if $V\in\mathcal{K}_0(\mathbb{R}^3)$ or $V>0$ holds, then $n^{1,0}_{\omega,V}$ is not attained.*
- *(radial case) Let $1<p<5$ and $V\in L^{\frac{3}{2}}_{rad}(\mathbb{R}^3)$. Then, $r^{1,0}_{\omega,V}$ is attained. Moreover, a set of elements attaining $r^{1,0}_{\omega,V}$ coincides with a whole of "radial" ground states of (SP$_{\omega,V}$), that is, it follows that $\mathcal{M}^{1,0}_{\omega,V,\,rad}=\mathcal{G}_{\omega,V,\,rad}$, where*

$$\mathcal{M}^{\alpha,\beta}_{\omega,V,\,rad} := \{\phi\in H^1_{rad}(\mathbb{R}^d)\setminus\{0\} : S_{\omega,V}(\phi)=r^{\alpha,\beta}_{\omega,V},\ K^{\alpha,\beta}_{\omega,V}(\phi)=0\}. \tag{1.27}$$

As a corollary of Main theorem 1.60, the following result holds by the fact that $n^{1,0}_{\omega,V}$ is not attained and $r^{1,0}_{\omega,V}$ is attained.

Corollary 1.61. *Let $d=3$, $1<p<5$, and $\omega>0$. Let $V\in L^{\frac{3}{2}}_{rad}(\mathbb{R}^3)$, $V\geq 0$, $V\neq 0$, and "$V\in\mathcal{K}_0(\mathbb{R}^3)$ or $V>0$". Then, $n^{1,0}_{\omega,V}<r^{1,0}_{\omega,V}$ holds.*

Then, we investigate radial solutions to (NLS$_V$) with initial data u_0, whose action $S_{\omega,V}(u_0)$ is less than that of the radial ground state to (SP$_{\omega,V}$).

Theorem 1.62. *Let $d=3$, $1<p<5$, $V\in L^{\frac{3}{2}}_{rad}(\mathbb{R}^3)\cap\mathcal{K}_0(\mathbb{R}^3)$, $\|V_-\|_{\mathcal{K}}<4\pi$, and $u_0\in H^1_{rad}(\mathbb{R}^3)$. Let $Q_{\omega,V}$ be a "radial" ground state to (SP$_{\omega,V}$). Assume that there exists $\omega>0$ such that*

$$S_{\omega,V}(u_0)<S_{\omega,V}(Q_{\omega,V})\ (=r^{1,0}_{\omega,V}) \quad \text{and} \quad \mathcal{N}_{\omega,V}(u_0)\geq 0.$$

Then, the solution u to (NLS$_V$) with (IC) exists globally in time and satisfies $\mathcal{N}_{\omega,V}(u(t))\geq 0$ for any $t\in\mathbb{R}$.

We compare the condition in Theorem 1.62 with that in Main theorem 1.56 (1).

Proposition 1.63. *Let $d=3$, $V\in L^{\frac{3}{2}}_{rad}(\mathbb{R}^3)$, and $V\geq 0$.*

1. *Let $\frac{7}{3}<p<5$. The condition (1.20) in Main theorem 1.56 is equivalent to "there exists $\omega>0$ such that $S_{\omega,V}(u_0)<n^{1,0}_{\omega,V}$".*
2. *Let $1<p<5$, $V\neq 0$, and "$V\in\mathcal{K}_0(\mathbb{R}^3)$ or $V>0$". Let $Q_{\omega,V}$ be a "radial" ground state to (SP$_{\omega,V}$). For each $\omega>0$, there exists $u_0\in H^1_{rad}(\mathbb{R}^3)$ such that u_0 satisfies*

$$n^{1,0}_{\omega,V}\leq S_{\omega,V}(u_0)<r^{1,0}_{\omega,V} \quad \text{and} \quad \mathcal{N}_{\omega,V}(u_0)\geq 0.$$

Proposition 1.63 implies that Theorem 1.62 can deal with initial data, which does not satisfy the assumptions of Main theorem 1.56. However, we can not see that the solutions in Theorem 1.62 scatters or not.

We recall that it is proved that the existence of a "radial" ground state $Q_{\omega,V}$ to (SP$_{\omega,V}$) and $Q_{\omega,V}$ is characterized by $r^{1,0}_{\omega,V}$ in Main theorem 1.60. However, it is not expected that one can prove scattering result by using the above characterization of the ground state $Q_{\omega,V}$. In fact, it is unclear whether that we can control the virial functional K_V or not. Then, we study the minimization problem $r^{\alpha,\beta}_{\omega,V}$ with (α,β) satisfying (1.26), which contains not only $r^{1,0}_{\omega,V}$ but also $r^{d,2}_{\omega,V}$.

To state next main result, we define the following quantity:

$$\omega_0 := -\frac{1}{2}\operatorname*{ess\,inf}_{x\in\mathbb{R}^d}(2V+x\cdot\nabla V).$$

Main theorem 1.64 (H.–Ikeda, [56]). *Let $d \geq 3$ and $1+\frac{4}{d}<p<1+\frac{4}{d-2}$.*

- *(Non-radial case) Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{d}{2}}(\mathbb{R}^d)+L^{\sigma}(\mathbb{R}^d)$ for some $\frac{d}{2} \leq \sigma < \infty$ and any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}| \leq 1$, $V \geq 0$, $x\cdot\nabla V \leq 0$, and $2V + x\cdot\nabla V \geq 0$. Then, for each (α,β) satisfy (1.26) and each $\omega>0$, $n_{\omega,V}^{\alpha,\beta} = n_{\omega,0}\,(=S_{\omega,0}(Q_{\omega,0}))$ holds. Moreover, if we assume $x\cdot\nabla V<0$, then $n_{\omega,V}^{\alpha,\beta}$ is not attained.*
- *(Radial case) Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{d}{2}}(\mathbb{R}^d)+L^{\infty}(\mathbb{R}^d)$ for any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}| \leq 1$, $V \geq 0$, $x\cdot\nabla V \leq 0$, and $\omega_0 < \infty$. Let V be radially symmetric. Then, $r_{\omega,V}^{\alpha,\beta}$ is attained for each (α,β) with (1.26) and each $\omega>0$ satisfying $\omega \geq \omega_0$. Moreover, if $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{d}{2}}(\mathbb{R}^d)+L^{\infty}(\mathbb{R}^d)$ for any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}| \leq 2$ and $3x\cdot\nabla V + x\nabla^2 V x^T \leq 0$, then $\mathcal{M}_{\omega,V,rad}^{\alpha,\beta} = \mathcal{G}_{\omega,V,rad}$ holds, where $\nabla^2 V$ denotes Hessian matrix of V and $\mathcal{M}_{\omega,V,rad}^{\alpha,\beta}$ and $\mathcal{G}_{\omega,V,rad}$ are defined as (1.24) and (1.27) respectively.*

We note that the following results hold from the same argument as Main theorem 1.64.

Remark 1.65. The followings hold:

- (Non-radial case) If we replace $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{d}{2}}(\mathbb{R}^d)+L^{\sigma}(\mathbb{R}^d)$ for some $\frac{d}{2} \leq \sigma < \infty$ with $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\eta}(\mathbb{R}^d)+L^{\sigma}(\mathbb{R}^d)$ for some $\frac{d}{2} < \eta \leq \sigma < \infty$, then Main theorem 1.64 also holds in $d=2$. If we replace $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{d}{2}}(\mathbb{R}^d)+L^{\sigma}(\mathbb{R}^d)$ for some $\frac{d}{2} \leq \sigma < \infty$ with $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{1}(\mathbb{R}^d)+L^{\sigma}(\mathbb{R}^d)$ for some $1 \leq \sigma < \infty$, then Main theorem 1.64 also holds in $d=1$.
- (Radial case) If we replace $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{d}{2}}(\mathbb{R}^d)+L^{\infty}(\mathbb{R}^d)$ with $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\eta}(\mathbb{R}^d)+L^{\infty}(\mathbb{R}^d)$ for some $\frac{d}{2} < \eta < \infty$, then Main theorem 1.64 also holds in $d=2$.

As a corollary of Main theorem 1.64, the following result holds by the fact that $n_{\omega,V}^{\alpha,\beta}$ is not attained and $r_{\omega,V}^{\alpha,\beta}$ is attained.

Corollary 1.66. *Let $d \geq 2$, $1+\frac{4}{d}<p<1+\frac{4}{d-2}$ if $d \geq 3$, and $1+\frac{4}{d}<p<\infty$ if $d=2$. Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\eta}(\mathbb{R}^d)+L^{\sigma}(\mathbb{R}^d)$ for some $\eta>1$ if $d=2$, $\eta=\frac{d}{2}$ if $d\geq 3$, some $\eta \leq \sigma < \infty$, any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}| \leq 1$, $V \geq 0$, $x\cdot\nabla V < 0$, and $2V + x\cdot\nabla V \geq 0$. Then, $n_{\omega,V}^{\alpha,\beta} < r_{\omega,V}^{\alpha,\beta}$ holds for each (α,β) with (1.26) and $\omega>0$.*

Then, we investigate solutions to (NLS$_V$) with initial data u_0, whose action $S_{\omega,V}(u_0)$ is less than $n_{\omega,V}^{\alpha,\beta}$ or $r_{\omega,V}^{\alpha,\beta}$.

Theorem 1.67. *Let $d \geq 3$ and $1+\frac{4}{d}<p<1+\frac{4}{d-2}$.*

- *(Non-radial case) Let $u_0 \in H^1(\mathbb{R}^d)$, $V \in L^{\eta}(\mathbb{R}^d)+L^{\sigma}(\mathbb{R}^d)$, $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{d}{2}}(\mathbb{R}^d)+L^{\sigma}(\mathbb{R}^d)$ for some $\frac{d}{2} < \eta \leq \sigma < \infty$ and any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}| = 1$, $V \geq 0$, $x\cdot\nabla V \leq 0$, and $2V + x\cdot\nabla V \geq 0$. We also assume that there exist (α,β) satisfying (1.26) and $\omega>0$ such that*
$$S_{\omega,V}(u_0) < n_{\omega,V}^{\alpha,\beta} = n_{\omega,0}\,(=S_{\omega,0}(Q_{\omega,0})) \quad \text{and} \quad K_{\omega,V}^{\alpha,\beta}(u_0) \geq 0.$$
Then, the solution u to (NLS$_V$) with (IC) exists globally in time and satisfies $K_{\omega,V}^{\alpha,\beta}(u(t)) \geq 0$ for any $t \in \mathbb{R}$.
- *(Radial case) Let $u_0 \in H^1_{rad}(\mathbb{R}^d)$, $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{d}{2}}(\mathbb{R}^d)+L^{\infty}(\mathbb{R}^d)$ for any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}| = 1$, $V \geq 0$, $x\cdot\nabla V \leq 0$, and $\omega_0 < \infty$. Let a radial function V satisfy the following (i) or (ii):*
  (i) $d=3$, $V \in L^{\frac{3}{2}}(\mathbb{R}^3)\cap\mathcal{K}_0(\mathbb{R}^3)$,
  (ii) $d\geq 3$, $V \in L^{\eta}(\mathbb{R}^d)+L^{\infty}(\mathbb{R}^d)$ *for some* $\frac{d}{2}<\eta<\infty$.
  We also assume that there exist (α,β) with (1.26) and $\omega>0$ satisfying $\omega \geq \omega_0$ such that
$$S_{\omega,V}(u_0) < r_{\omega,V}^{\alpha,\beta} \quad \text{and} \quad K_{\omega,V}^{\alpha,\beta}(u_0) \geq 0.$$

Then, the solution u to (NLS$_V$) with (IC) exists globally in time and satisfies $K^{\alpha,\beta}_{\omega,V}(u(t)) \geq 0$ for any $t \in \mathbb{R}$.

We compare the condition in Main theorem 1.64 with that in Main theorem 1.56 (1).

Proposition 1.68. *Let $d \geq 3$ and $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$.*

(1) *Let $u_0 \in H^1(\mathbb{R}^d)$, $V \in L^\eta(\mathbb{R}^d) + L^\sigma(\mathbb{R}^d)$, $x^{\mathfrak{a}}\partial^{\mathfrak{a}} V \in L^{\frac{d}{2}}(\mathbb{R}^d) + L^\sigma(\mathbb{R}^d)$ for some $\frac{d}{2} < \eta \leq \sigma < \infty$ and any $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^d$ with $|\mathfrak{a}| = 1$, $V \geq 0$, $x \cdot \nabla V \leq 0$, and $2V + x \cdot \nabla V \geq 0$. The condition (1.20) in Main theorem 1.56 is equivalent to "there exist (α, β) satisfying (1.26) and $\omega > 0$ such that $S_{\omega,V}(u_0) < n^{\alpha,\beta}_{\omega,V}$".*

(2) *Let V, $x \cdot \nabla V \in L^{\frac{d}{2}}(\mathbb{R}^d) + L^\sigma(\mathbb{R}^d)$ for some $\frac{d}{2} < \sigma < \infty$, $V \geq 0$, $x \cdot \nabla V < 0$, and $2V + x \cdot \nabla V \geq 0$. Let $Q_{\omega,V}$ be a "radial" ground state to (SP$_{\omega,V}$). For each (α, β) satisfying (1.26) and $\omega > 0$, there exists $u_0 \in H^1_{rad}(\mathbb{R}^d)$ such that u_0 satisfies*

$$n^{\alpha,\beta}_{\omega,V} \leq S_{\omega,V}(u_0) < r^{\alpha,\beta}_{\omega,V} \quad \text{and} \quad K^{\alpha,\beta}_{\omega,V}(u_0) \geq 0.$$

Remark 1.69. When $\beta \neq 0$, the term $x \cdot \nabla V$ appears in $K^{\alpha,\beta}_{\omega,V}$, which can be written as (4.43). Thus, we impose the assumption $x^{\mathfrak{a}}\partial^{\mathfrak{a}} V \in L^{\frac{d}{2}}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for any $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^d$ with $|\mathfrak{a}| = 1$ in Main theorem 1.64, which assures that $K^{\alpha,\beta}_{\omega,V}$ is well defined. The repulsive condition $x \cdot \nabla V \leq 0$ is used to control the fourth term in (4.43). The condition of the frequency $\omega \geq \omega_0$ implies

$$2\omega + 2V + x \cdot \nabla V \geq 0$$

and is used to control the third term in (4.44). On the other hand, when $\beta = 0$, we do not need the assumptions on $x \cdot \nabla V$ and can replace $\omega \geq \omega_0$ with $\omega > 0$.

Since we use many assumptions of a potential in Theorem 1.64, we wonder that if there is actually a potential satisfying all the assumptions. For example, a potential

$$V(x) = \frac{\gamma\{\log(1 + |x|)\}^\theta}{|x|^\mu}, \quad (\gamma > 0,\ 0 \leq \theta \leq \mu < 2,\ \mu > 0)$$

satisfies all assumptions in Main theorem 1.64. From now on, we consider the equation (NLS$_V$) with a inverse power potential $V(x) = \frac{\gamma}{|x|^\mu}$, which is the above potential with $\theta = 0$ and has a "good" property for the scaling argument:

$$i\partial_t u + \Delta_\gamma u = -|u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \tag{NLS$_\gamma$}$$

where $\Delta_\gamma = \Delta - \frac{\gamma}{|x|^\mu}$, $\gamma > 0$, and $0 < \mu < \min\{2, d\}$. The Cauchy problem of (NLS$_\gamma$) is locally well-posed in the energy space $H^1(\mathbb{R}^d)$ by Theorem 1.50.

For simplicity, we use the notations: $S_{\omega,\gamma} := S_{\omega,\frac{\gamma}{|x|^\mu}}$, $K_\gamma := K_{\frac{\gamma}{|x|^\mu}}$, $E_\gamma := E_{\frac{\gamma}{|x|^\mu}}$, $n_{\omega,\gamma} := n^{\alpha,\beta}_{\omega,\frac{\gamma}{|x|^\mu}}$, $r_{\omega,\gamma} := r^{\alpha,\beta}_{\omega,\frac{\gamma}{|x|^\mu}}$, and $Q_{\omega,\gamma} := Q_{\omega,\frac{\gamma}{|x|^\mu}}$. We note that $n_{\omega,\gamma}$ and $r_{\omega,\gamma}$ are independent of (α, β) from Main theorem 1.64 and Proposition 4.48.

Theorem 1.70 (Local well-posedness of (NLS$_\gamma$), [13])**.** *Let $d \geq 1$, $1 < p < \infty$ if $d = 1, 2$, $1 < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, $\gamma > 0$, and $0 < \mu < \min\{2, d\}$. Then, for any $u_0 \in H^1_x(\mathbb{R}^d)$, (NLS$_\gamma$) is locally well-posed, that is, the followings hold:*

- *(Existence and Uniqueness) (NLS$_\gamma$) with (IC) has the unique solution*

$$u \in C_t((T_{min}, T_{max}); H^1_x) \cap C^1_t((T_{min}, T_{max}); H^{-1}_x).$$

- *(Blow-up alternative) If $T_{max} < \infty$ (resp. $T_{min} > -\infty$), then*

$$\lim_{t \nearrow T_{max}} \|u(t)\|_{H^1_x} = \infty, \quad \left(\text{resp.} \lim_{t \searrow T_{min}} \|u(t)\|_{H^1_x} = \infty \right).$$

- *(Continuous dependence on the initial data) If $u_{0,n} \longrightarrow u_0$ in $H^1_x(\mathbb{R}^d)$, then for any compact time interval $I \subset (T_{min}, T_{max})$, there exists $n_0 \in \mathbb{N}$ such that the solution u_n to (NLS_γ) with initial data $u_{0,n}$ is defined on I for any $n \geq n_0$ and satisfies $u_n \longrightarrow u$ in $C_t(I; H^1_x)$ as $n \to \infty$.*

Moreover, the solution to (NLS_γ) preserves its mass and energy with respect to time t.

We introduce a known result for time behavior of solutions to (NLS_γ). For blow-up, Dinh [25] proved the following result.

Theorem 1.71 (Dinh, [25])**.** *Let $d \geq 1$, $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, $\gamma > 0$, and $0 < \mu < \min\{2, d\}$. Let $Q_{1,0}$ be the ground state to ($\mathrm{SP}_{\omega,0}$) with $\omega = 1$. We assume that u_0 satisfies (1.20).*

- *(Global well-posedness) If u_0 satisfies (1.4), then the solution u to (NLS_γ) with (IC) exists globally in both time directions.*
- *(Blow-up) We assume $|x|u_0 \in L^2(\mathbb{R}^d)$ or "$d \geq 2$, $1 < p \leq 5$, and $u_0 \in H^1_{rad}(\mathbb{R}^d)$". If u_0 satisfies $E_\gamma(u_0) < 0$ or "(1.5) and $E_\gamma(u_0) \geq 0$", then the solution u to (NLS_γ) with (IC) blows up in both time directions.*

To prove Theorem 1.71, Dinh [25] used the fact:

$$PW_{+,3} := \{u_0 \in H^1(\mathbb{R}^d) : (1.20) \text{ and } (1.4)\} \quad \text{and} \quad PW_{-,3} := \{u_0 \in H^1(\mathbb{R}^d) : (1.20) \text{ and } (1.5)\}$$

are invariant under the time development of (NLS_γ), which follows from the characterization of the ground state $Q_{1,0}$ to ($\mathrm{SP}_{\omega,0}$) with $\omega = 1$ by Proposition 1.16. We have invariant sets

$$PW_{+,4} := \bigcup_{\omega>0} \{u_0 \in H^1(\mathbb{R}^d) : S_{\omega,\gamma}(u_0) < S_{\omega,0}(Q_{\omega,0}) \text{ and } K_\gamma(u_0) \geq 0\}$$

and

$$PW_{-,4} := \bigcup_{\omega>0} \{u_0 \in H^1(\mathbb{R}^d) : S_{\omega,\gamma}(u_0) < S_{\omega,0}(Q_{\omega,0}) \text{ and } K_\gamma(u_0) < 0\}$$

with respect to the time development of (NLS_γ) from Proposition 1.68. Moreover, the Gagliardo–Nirenberg inequality with the inverse power potential:

Proposition 1.72 (Gagliardo-Nirenberg inequality with the inverse power potential)**.** *Let $d \geq 1$, $1 < p < \infty$ if $d = 1, 2$, $1 < p \leq 1 + \frac{4}{d-2}$ if $d \geq 3$, $\gamma > 0$, and $0 < \mu < \min\{2, d\}$. Then, the following inequality holds:*

$$\|f\|_{L^{p+1}}^{p+1} < C_{GN} \|f\|_{L^2}^{p+1-\frac{d(p-1)}{2}} \|(-\Delta_\gamma)^{\frac{1}{2}} f\|_{L^2}^{\frac{d(p-1)}{2}}$$

for any $f \in H^1(\mathbb{R}^d) \setminus \{0\}$, where C_{GN} is the best constant and is defined in Proposition 1.16.

generates the invariant sets

$$PW_{+,5} := \{u_0 \in H^1(\mathbb{R}^d) : (1.20) \text{ and } (1.21)\} \quad \text{and} \quad PW_{-,5} := \{u_0 \in H^1(\mathbb{R}^d) : (1.20) \text{ and } (1.22)\}.$$

The Proposition 1.68 unifies the sense of "below the ground state without potential". Namely, the identities

$$PW_{+,3} \cup PW_{-,3} = PW_{+,4} \cup PW_{-,4} = PW_{+,5} \cup PW_{-,5}$$

hold. We note that

$$\|u_0\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_{L^2} = \|Q_{1,0}\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L^2} \text{ and } \|u_0\|_{L^2}^{\frac{1-s_c}{s_c}} \|(-\Delta_\gamma)^{\frac{1}{2}} u_0\|_{L^2} = \|Q_{1,0}\|_{L^2}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L^2}$$

never hold under the assumption (1.20) (see Lemma 4.49). It is a natural question that the relation of the conditions on the initial data below the ground state without the potential: (1.4), (1.5), (1.21), (1.22), $K_\gamma(u_0) \geq 0$, and $K_\gamma(u_0) < 0$. First, we state the following result for the time behavior of solutions to (NLS_γ).

Theorem 1.73 (Boundedness versus unboundedness I)**.** *Let $d \geq 1$, $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, $\gamma > 0$, and $0 < \mu < \min\{2, d\}$. Let $j = 4, 5$.*

- *(Global well-posedness) If $u_0 \in PW_{+,j}$, then the solution u to (NLS$_\gamma$) with (IC) satisfies $u(t) \in PW_{+,j}$ for each $t \in (T_{min}, T_{max})$ and exists globally in time. In particular, H^1-norm of the solution u is uniformly bounded in maximal lifespan.*
- *(Brow-up or grow-up) If $u_0 \in PW_{-,j}$, then the solution u to (NLS$_\gamma$) with (IC) satisfies $u(t) \in PW_{-,j}$ for each $t \in (T_{min}, T_{max})$ and blows up or grows up. Moreover, if u_0 satisfies $u_0 \in |x|^{-1}L^2(\mathbb{R}^d)$ or "$d \geq 2$, $1 < p \leq 5$, and $u_0 \in H^1_{rad}(\mathbb{R}^d)$", then u blows up.*

Combining Theorem 1.71 and 1.73, we obtain the following equivalence of conditions on the initial data below the ground state.

Main theorem 1.74 (Equivalence of conditions on the initial data below the ground state)**.** *Let $d \geq 1$, $1+\frac{4}{d} < p < \infty$ if $d = 1, 2$, $1+\frac{4}{d} < p < 1+\frac{4}{d-2}$ if $d \geq 3$, $\gamma > 0$, and $0 < \mu < \min\{2, d\}$. We assume that u_0 satisfies (1.20). The three conditions (1.4), (1.21), $K_\gamma(u_0) \geq 0$ are equivalent. On the other hand, the three conditions (1.5), (1.22), $K_\gamma(u_0) < 0$ are equivalent. In other words, $PW_{+,3} = PW_{+,4} = PW_{+,5}$ and $PW_{-,3} = PW_{-,4} = PW_{-,5}$ hold.*

As a corollary of Main theorem 1.74, the following result holds.

Corollary 1.75. *Let $d \geq 1$, $1+\frac{4}{d} < p < \infty$ if $d = 1, 2$, $1+\frac{4}{d} < p < 1+\frac{4}{d-2}$ if $d \geq 3$, $\gamma > 0$, and $0 < \mu < \min\{2, d\}$. If $u_0 \in H^1(\mathbb{R}^d) \setminus \{0\}$ satisfies $E_\gamma(u_0) \leq 0$, then $u_0 \in PW_{-,j}$ $(j = 3, 4, 5)$.*

We also investigate time behavior of solutions to (NLS$_\gamma$) with the radial initial data below the "radial" ground state $Q_{\omega,\gamma}$ to (SP$_{\omega,V}$) with $V = \frac{\gamma}{|x|^\mu}$.

Theorem 1.76 (Boundedness versus unboundedness II)**.** *Let $d \geq 1$, $1+\frac{4}{d} < p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, $\gamma > 0$, and $0 < \mu < \min\{2, d\}$.*

- *(Global well-posedness) If $u_0 \in PW_{+,6}$, then a solution u to (NLS$_\gamma$) with (IC) satisfies $u(t) \in PW_{+,6}$ for each $t \in (T_{min}, T_{max})$ and exists globally in time, where*
$$PW_{+,6} := \bigcup_{\omega>0} \{u_0 \in H^1_{rad}(\mathbb{R}^d) : S_{\omega,\gamma}(u_0) < r_{\omega,\gamma} \text{ and } K_\gamma(u_0) \geq 0\}.$$
In particular, H^1-norm of the solution u is uniformly bounded in maximal lifespan.
- *(Blow-up or grow-up) If $u_0 \in PW_{-,6}$, then a solution u to (NLS$_\gamma$) with (IC) satisfies $u(t) \in PW_{-,6}$ for each $t \in (T_{min}, T_{max})$ and blows up or grows up, where*
$$PW_{-,6} := \bigcup_{\omega>0} \{u_0 \in H^1_{rad}(\mathbb{R}^d) : S_{\omega,\gamma}(u_0) < r_{\omega,\gamma} \text{ and } K_\gamma(u_0) < 0\}.$$
Moreover, if $d \geq 2$ and $p \leq 5$, then u blows up.

1.4. **Organization of the paper.** The organization of the rest of the paper is as follows.

In Section 2, we define some notations and collect some tool, which are used throughout this paper.

In Section 3, we prove theorems for nonlinear Schrödinger system (NLS). In Subsection 3.1, we define spaces, which are used to prove theorems for (NLS). In Subsection 3.2, we collect some tools for Section 3. In Subsection 3.3, we give local well-posedness of (NLS) and equivalence conditions to scattering. In Subsection 3.4, we prove that solutions to (NLS) with nonpositive energy does not scatter. In Subsection 3.6, we investigate properties of L_{v_0} and $\ell^\dagger_{v_0}$. In Subsection 3.7 and 3.8, we show linear profile decomposition. In Subsection 3.9, we prove Main theorem 1.39, 1.41, and 1.42. In Subsection 3.10, we deal with the other optimization problems. In Subsection 3.11, we show corollaries of Theorem 1.44.

In Section 4, we prove main theorems for nonlinear Schrödinger equation with a potential (NLS$_V$) and (NLS$_\gamma$). In Subsection 4.1, we collect some tools for Section 4. In Subsection 4.2 $\sim$ 4.6, we prove theorems for (NLS$_V$). In Subsection 4.7, we prove theorems for (NLS$_\gamma$).

2. Preliminaries

In this section, we define some notations and collect some tools, which are used throughout this paper.

2.1. **Notations.** For non-negative X and Y, $X \lesssim Y$ denotes $X \le CY$ for some $C > 0$. If $X \lesssim Y \lesssim X$ holds, we write $X \sim Y$. We use subscripts to indicate the dependence of implicit constants, e.g. $X \lesssim_u Y$ denotes $X \le CY$ for some $C = C(u)$. We write $a' \in [1,\infty]$ to denote the Hölder dual exponent to $a \in [1,\infty]$, that is, a and a' satisfy $\frac{1}{a} + \frac{1}{a'} = 1$.

$C^\infty(\mathbb{R}^d)$ is a space of smooth functions and $C_c^\infty(\mathbb{R}^d)$ is a space of smooth functions with a compact support. $L^p(\mathbb{R}^d)$ denotes a usual Lebesgue space for $1 \le p \le \infty$, that is,

$$L^p(\mathbb{R}^d) := \{f : \mathbb{R}^d \longrightarrow \mathbb{C} \text{ is a Lebesgue measurable function} : \|f\|_{L^p} < \infty\},$$

where

$$\|f\|_{L^p} := \begin{cases} \left(\displaystyle\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{\frac{1}{p}}, & (1 \le p < \infty), \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|, & (p = \infty). \end{cases}$$

For $1 \le p \le \infty$, a space ℓ^p is defined as $\ell^p := \{\{a_n\}_{n\in X} \subset \mathbb{C} : \|a_n\|_{\ell^p} < \infty\}$, where

$$\|a_n\|_{\ell^p} := \begin{cases} \left(\displaystyle\sum_{n\in X} |a_n|^p\right)^{\frac{1}{p}}, & (1 \le p < \infty), \\ \sup_{n\in X} |a_n|, & (p = \infty). \end{cases}$$

$\mathcal{S}(\mathbb{R})$ is the Schwartz space and defined as

$$\mathcal{S}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) : \|x^{\mathfrak{a}} \partial^{\mathfrak{b}} f\|_{L^\infty} < \infty \text{ for any } \mathfrak{a}, \mathfrak{b} \in (\mathbb{N} \cup \{0\})^d\}.$$

$\mathcal{S}'(\mathbb{R}^d)$ is a set of a whole of the tempered distribution, that is,

$$\mathcal{S}'(\mathbb{R}^d) := \{F : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathbb{C} : F \text{ is linear and continuous.}\}.$$

$\langle\,\cdot\,,\,\cdot\,\rangle_X$ denotes the X-inner product for a Hilbert space X. For a Banach space X, $L_t^q(I;X)$ denotes the Banach space of functions $f : I \times \mathbb{R}^d \longrightarrow \mathbb{C}$, whose norm is $\|f\|_{L_t^q(I;X)} := \big\|\|f\|_X\big\|_{L_t^q(I)} < \infty$. If a time interval is not specified, that is, if we write $\|\cdot\|_{L_t^q X}$, then the t-norm is taken over $\mathbb{R}$. The norm of $X \times Y$ and $X \cap Y$ are defined as $\|(f,g)\|_{X\times Y} = \|f\|_X + \|g\|_Y$ and $\|f\|_{X\cap Y} = \|f\|_X + \|f\|_Y$, respectively for Banach spaces X and Y.

We define the Fourier transform and the inverse Fourier transform on $\mathbb{R}^d$ respectively as

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \quad \mathcal{F}^{-1}f(x) = \check{f}(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)d\xi,$$

where $x\cdot\xi := x_1\cdot\xi_1 + \cdots + x_d\cdot\xi_d$. $W^{s,p}(\mathbb{R}^d) = (1-\Delta)^{-\frac{s}{2}} L^p(\mathbb{R}^d)$ and $\dot{W}^{s,p}(\mathbb{R}^d) = (-\Delta)^{-\frac{s}{2}} L^p(\mathbb{R}^d)$ are inhomogeneous Sobolev space and homogeneous Sobolev space, respectively for $s \in \mathbb{R}$ and $p \in [1,\infty]$, where $(1-\Delta)^{\frac{s}{2}} = \mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}$ and $(-\Delta)^{\frac{s}{2}} = |\nabla|^s = \mathcal{F}^{-1}|\xi|^s\mathcal{F}$. When $p = 2$, we express $W^{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ and $\dot{W}^{s,2}(\mathbb{R}^d) = \dot{H}^s(\mathbb{R}^d)$.

Let $(T_{\min}, T_{\max})$ be the maximal lifespan of the solution to (NLS), (NLS$_V$), or (NLS$_\gamma$). Let $[0, T_{\max})$ be the maximal positive lifespan of the solution to (NLS), (NLS$_V$), or (NLS$_\gamma$). We convert (NLS) and (NLS$_V$) respectively into the following integral system by Duhamel's principle:

$$\begin{cases} u(t) = e^{it\Delta}u_0 + 2i\displaystyle\int_0^t e^{i(t-s)\Delta}(v\overline{u})(s)ds, \\ v(t) = e^{\kappa it\Delta}v_0 + i\displaystyle\int_0^t e^{\kappa i(t-s)\Delta}(u^2)(s)ds \end{cases}$$

and

$$u(t) = e^{it\Delta_V}u_0 + i\int_0^t e^{i(t-s)\Delta_V}(|u|^{p-1}u)(s)ds,$$

where the Schrödinger group $e^{it\Delta}$ is defined as $e^{it\Delta}f(x) = (e^{-it|\xi|^2}\widehat{f})^\vee(x)$ and $\{e^{it\Delta_V}\}_{t\in\mathbb{R}}$ is the Schrödinger evolution group generated on $L^2(\mathbb{R}^d)$ by Stone's theorem.

We define the following functions for $R > 0$ and $r = |x|$. A cut-off function $\mathscr{X}_R \in C_c^\infty(\mathbb{R}^d)$ is radially symmetric and satisfies

$$\mathscr{X}_R(r) := R^2 \mathscr{X}\left(\frac{r}{R}\right), \quad \text{where} \quad \mathscr{X}(r) := \begin{cases} r^2 & (0 \leq r \leq 1), \\ smooth & (1 \leq r \leq 3), \\ 0 & (3 \leq r), \end{cases} \tag{2.1}$$

$\mathscr{X}''(r) \leq 2$ for any $r \geq 0$. A cut-off function $\mathscr{Y}_R \in C_c^\infty(\mathbb{R}^d)$ is radially symmetric and satisfies

$$\mathscr{Y}_R(r) := \mathscr{Y}\left(\frac{r}{R}\right), \quad \text{where} \quad \mathscr{Y}(r) := \begin{cases} 1 & (0 \leq r \leq 1), \\ smooth & (1 \leq r \leq 2), \\ 0 & (2 \leq r), \end{cases} \tag{2.2}$$

and $-2 \leq \mathscr{Y}'(r) \leq 0$ for any $r \geq 0$. A function $\mathscr{Z}_R \in C^\infty(\mathbb{R}^d)$ is defined as

$$\mathscr{Z}_R(r) := \mathscr{Z}\left(\frac{r}{R}\right), \quad \text{where} \quad \mathscr{Z}(r) := 1 - \mathscr{Y}(r). \tag{2.3}$$

We also define the characteristic function of A as $\mathbf{1}_A(x)$ for a set $A \subset \mathbb{R}^d$, that is, $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if $x \notin A$.

2.2. **Some tools.** In this subsection, we introduce some standard tools, which are used throughout this paper.

Lemma 2.1 (Young's inequality). *The following inequality holds:*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

for any $a, b \geq 0$ and any $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.2 (Hölder's inequality). *Let $d \geq 1$. For any $q, q' \geq 1$ with $\frac{1}{q} + \frac{1}{q'} = 1$, $f \in L^q(\mathbb{R}^d)$, and $g \in L^{q'}(\mathbb{R}^d)$, we have*

$$\|fg\|_{L^1} \leq \|f\|_{L^q}\|g\|_{L^{q'}}.$$

Lemma 2.3 (Sobolev's embedding). *Let $d \geq 1$. For any $q, r \geq 1$, $s \geq 0$ with $\frac{1}{q} = \frac{1}{r} - \frac{s}{d}$,*

$$\|f\|_{L^q} \lesssim \|f\|_{\dot{W}^{s,r}},$$

that is, $\dot{W}^{s,r}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ holds. For any $q, r \geq 1$, $s \geq 0$ with $\frac{1}{q} \geq \frac{1}{r} - \frac{s}{d}$,

$$\|f\|_{L^q} \lesssim \|f\|_{W^{s,r}},$$

that is, $W^{s,r}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ holds.

Lemma 2.4 (Radial Sobolev inequality, [100]). *Let $d \geq 2$ and $p \geq 1$. There exists $C_0 > 0$ such that for any $R > 0$ and any $f \in H^1_{rad}(\mathbb{R}^d)$, the following inequality holds:*

$$\|f\|_{L_x^{p+1}(R \leq |x|)}^{p+1} \leq C_0 R^{-\frac{(d-1)(p-1)}{2}} \|f\|_{L_x^2(R \leq |x|)}^{\frac{p+3}{2}} \|\nabla f\|_{L_x^2(R \leq |x|)}^{\frac{p-1}{2}}.$$

Lemma 2.5 (Compact embedding). *Let $d \geq 2$ and $2 < q < 1 + \frac{4}{d-2}$. Then, the embedding $H^1_{rad}(\mathbb{R}^d) \subset\subset L^q(\mathbb{R}^d)$ is compact.*

The following proposition is cited in [13, Lemma 8.1.2]

Proposition 2.6 (Pohozaev identities). *Let $d \geq 1$ and $2 < p + 1 < 2^*$. A solution $Q_{\omega,0}$ to ($\mathrm{SP}_{\omega,0}$) satisfies the following Pohozaev identities.*

$$\|Q_{\omega,0}\|_{L_x^{p+1}}^{p+1} = \frac{2(p+1)\omega}{d+2-(d-2)p}\|Q_{\omega,0}\|_{L_x^2}^2, \quad \|Q_{\omega,0}\|_{L_x^{p+1}}^{p+1} = \frac{2(p+1)}{d(p-1)}\|\nabla Q_{\omega,0}\|_{L_x^2}^2.$$

This proposition deduces the following relations:

$$E_0(Q_{\omega,0}) = \frac{d(p-1)-4}{2d(p-1)}\|\nabla Q_{\omega,0}\|_{L_x^2}^2, \tag{2.4}$$

$$C_{\rm GN} = \frac{2(p+1)}{d(p-1)} \cdot \frac{1}{\|Q_{1,0}\|_{L_x^2}^{\frac{d+2-(d-2)p}{2}} \|\nabla Q_{1,0}\|_{L_x^2}^{\frac{(p-1)d-4}{2}}}. \tag{2.5}$$

3. Proof of theorems for NLS system

3.1. **Notations for Section 3.** We define some notations and spaces, which are used in this section.

We recall the standard Littlewood-Paley projection operators. Let ϕ be a radial cut-off function satisfies $\mathbf{1}_{\{|\xi|\le 4/3\}} \le \phi \le \mathbf{1}_{\{|\xi|\le 5/3\}}$. For $N \in 2^{\mathbb{Z}}$, the operators P_N is defined as

$$\widehat{P_N f}(\xi) := \widehat{f_N}(\xi) := \psi_N(\xi)\widehat{f}(\xi),$$

where $\phi_N(x) = \phi(x/N)$ and

$$\psi_N(x) = \phi_N(x) - \phi_{N/2}(x). \tag{3.1}$$

3.1.1. *The Galilean transform and the Galilean operator.* The Galilean operator

$$J_m(t) := x + i\frac{t}{m}\nabla,$$

which is a multiple of the infinitesimal operator for transforms appearing in (1.14), plays an important role in the scattering theory for mass-subcritical nonlinear Schrödinger equation. We define the multiplication operator

$$[\mathcal{M}_m(t)f](x) := e^{\frac{im|x|^2}{2t}} f(x) \quad (t \neq 0)$$

and the dilation operator

$$[\mathcal{D}(t)f](x) := (2it)^{-\frac{3}{2}} f\left(\frac{x}{2t}\right) \quad (t \neq 0).$$

It is known that the Schrödinger group is factorized as $e^{it\Delta} = \mathcal{M}_{\frac{1}{2}}(t)\mathcal{D}(t)\mathcal{F}\mathcal{M}_{\frac{1}{2}}(t)$ by using these operators. This factorization deduces the identity

$$e^{it\Delta}\Phi(x)e^{-it\Delta} = \mathcal{M}_{\frac{1}{2}}(t)\Phi(2it\nabla)\mathcal{M}_{\frac{1}{2}}(-t) \tag{3.2}$$

for suitable multiplier Φ, where $\Phi(i\nabla)$ denotes the Fourier multiplier operator with multiplier $\Phi(\xi)$, that is, $\Phi(i\nabla) := \mathcal{F}^{-1}\Phi(\xi)\mathcal{F}$. The Galilean operator is written as follows:

$$J_m(t) = e^{\frac{1}{2m}it\Delta} x e^{-\frac{1}{2m}it\Delta} = \mathcal{M}_m(t) i\frac{t}{m}\nabla\mathcal{M}_m(-t),$$

where the second equality holds for $t \neq 0$. We define a fractional power of J_m by

$$J_m^s(t) := e^{\frac{1}{2m}it\Delta}|x|^s e^{-\frac{1}{2m}it\Delta} = \mathcal{M}_m(t)\left(-\frac{t^2}{m^2}\Delta\right)^{\frac{s}{2}}\mathcal{M}_m(-t) \quad \text{for} \quad s \in \mathbb{R}.$$

Remark that the second formula is valid for $t \neq 0$.

3.1.2. *Function spaces.* We define a time-dependent spaces $\dot{X}_m^{s,r} = \dot{X}_m^{s,r}(t)$ by using the norm

$$\|f\|_{\dot{X}_m^{s,r}} := \|J_m^s(t)f\|_{L_x^r(\mathbb{R}^3)} \sim \||t|^s|\nabla|^s\mathcal{M}_m(-t)f\|_{L_x^r(\mathbb{R}^3)}. \tag{3.3}$$

When $r = 2$, we omit the exponent r, that is, $\dot{X}_m^s = \dot{X}_m^{s,2}$. We can see immediately by the definition of J_m^s that the equivalence of norms in (3.3) for $t \neq 0$. It is natural to write

$$f \in e^{\frac{1}{2m}it\Delta}\mathcal{F}\dot{H}^s \iff e^{-\frac{1}{2m}it\Delta}f \in \mathcal{F}\dot{H}^s.$$

Then, we have $e^{\frac{1}{2m}it\Delta}\mathcal{F}\dot{H}^s = \dot{X}^s_m(t)$. We use Lorentz-modified space-time norms. For an interval I, $1 \le q < \infty$, and $1 \le \alpha \le \infty$, the Lorentz space $L^{q,\alpha}_t(I)$ is defined by using the quasi-norm

$$\|f\|_{L^{q,\alpha}_t(I)} := \|\lambda|\{t \in I : |f(t)| > \lambda\}|^{\frac{1}{q}}\|_{L^\alpha((0,\infty),\frac{d\lambda}{\lambda})}.$$

For a Banach space X, $L^{q,\alpha}_t(I;X)$ is defined as the whole of functions $u : I \times \mathbb{R}^3 \longrightarrow \mathbb{C}$ satisfying

$$\|u\|_{L^{q,\alpha}_t(I;X)} := \|\|u(t)\|_X\|_{L^{q,\alpha}_t(I)} < \infty.$$

The following equivalence is useful:

$$\|f\|_{L^{q,\alpha}_t(I)} \sim \|\|f \cdot 1_{\{2^{k-1}\le|f|\le 2^k\}}\|_{L^q_t(I)}\|_{\ell^\alpha(k\in\mathbb{Z})}.$$

We also define Besov space as follows:

$$\dot{B}^s_{p,q}(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where $\|f\|_{\dot{B}^s_{p,q}} := \|2^{Ns}\|\psi_N * f\|_{L^p_x}\|_{\ell^q_N}$ and ψ_N is defined as (3.1).

3.1.3. *Specific function spaces.* We define an admissible pair.

Definition 3.1. If a pair (q, r) satisfies

$$2 < q < \infty, \quad 2 < r < 6, \quad \text{and} \quad \frac{2}{q} + \frac{3}{r} = \frac{3}{2},$$

then (q, r) is an admissible pair.

Remark that we do not include two end points $(\infty, 2)$ and $(2, 6)$ to admissible pairs. It is because they require exceptional treatments sometimes.

We use the following concrete choice of function spaces. The same exponents were used in [81, 89]. We define

$$\left(\frac{1}{q_1}, \frac{1}{r_1}\right) := \left(\frac{1}{6}, \frac{7}{18}\right) \quad \text{and} \quad \left(\frac{1}{\tilde{q}}, \frac{1}{\tilde{r}}\right) := \left(\frac{2}{3}, \frac{2}{9}\right).$$

The pair (q_1, r_1) is admissible. The pair $(\tilde{q}, \tilde{r})$ satisfies the critical scaling relation $\frac{2}{\tilde{q}} + \frac{3}{\tilde{r}} = 2$, and is not a admissible pair. These exponents satisfy the following relations:

$$\frac{1}{q_1'} = \frac{1}{\tilde{q}} + \frac{1}{q_1}, \quad \frac{1}{r_1'} = \frac{1}{\tilde{r}} + \frac{1}{r_1}, \quad \text{and} \quad \frac{1}{\tilde{q}} - \frac{1}{q_1} = \frac{3}{r_1} - \frac{3}{\tilde{r}} = \frac{1}{2}.$$

We define the spaces

$$S^{\text{weak}} := L^{\tilde{q},\infty}_t L^{\tilde{r}}_x = L^{\frac{3}{2},\infty}_t L^{\frac{9}{2}}_x, \quad S := L^{\tilde{q},2}_t L^{\tilde{r}}_x = L^{\frac{3}{2},2}_t L^{\frac{9}{2}}_x, \quad \text{and} \quad W_j := L^{q_1,2}_t \dot{X}^{\frac{1}{2},r_1}_{2^{j-2}} = L^{6,2}_t \dot{X}^{\frac{1}{2},\frac{18}{7}}_{2^{j-2}}$$

for the solutions and the spaces

$$N_j := L^{q_1',2}_t \dot{X}^{\frac{1}{2},r_1'}_{2^{j-2}} = L^{\frac{6}{5},2}_t \dot{X}^{\frac{1}{2},\frac{18}{11}}_{2^{j-2}}$$

for the nonlinear terms. We use a notation $S^{\text{weak}}(I)$ to indicate that the norm is taken over the space-time slab $I \times \mathbb{R}^3$, and similarly for the other spaces.

3.2. **Some tools for Section 3.** In this subsection, we introduce some standard tools in this section.

Lemma 3.2 (Riemann–Lebesgue lemma)**.** *Let $d \ge 1$ and $f \in L^1(\mathbb{R}^d)$. Then, we have*

$$\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx \longrightarrow 0 \quad \text{as} \quad |\xi| \to \infty.$$

Proposition 3.3 (Dispersive estimate)**.** *Let $d \ge 1$, $t \ne 0$, and $p \in [2, \infty]$. Then, it follows that $e^{it\Delta} : L^{p'}(\mathbb{R}^d) \longrightarrow L^p(\mathbb{R}^d)$ is continuous and*

$$\|e^{it\Delta}f\|_{L^p_x} \le (4\pi|t|)^{-\frac{d}{2}(\frac{1}{p'}-\frac{1}{p})}\|f\|_{L^{p'}_x}$$

for any $f \in L^{p'}_x(\mathbb{R}^d)$.

Proposition 3.4 (Strichartz estimate, [39, 46, 76, 86, 107]). *Let $d \geq 1$, $t_0 \in \mathbb{R}$, and $I(\ni t_0)$ be a time interval. If (q_1, r_1) and (q_2, r_2) satisfy $\frac{2}{q_1} + \frac{d}{r_1} = \frac{2}{q_2} + \frac{d}{r_2} = \frac{d}{2}$, $2 \leq q_1, q_2 \leq \infty$, $(q_1, r_1, d) \neq (2, \infty, 2)$, and $(q_2, r_2, d) \neq (2, \infty, 2)$, then*

$$\|e^{it\Delta} f\|_{L_t^{q_1} L_x^{r_1}} \lesssim \|f\|_{L_x^2}, \qquad \left\| \int_{t_0}^t e^{i(t-s)\Delta} F(\,\cdot\,, s) ds \right\|_{L_t^{q_1}(I; L_x^{r_1})} \lesssim \|F\|_{L_t^{q_2'}(I; L_x^{r_2'})}.$$

We also need Strichartz estimates for the spaces $L_t^{q,\alpha} \dot{X}_m^{s,r}$, which were proved in [90, 97].

Proposition 3.5 (Strichartz estimates, [97]). *Let $d = 3$, $s \geq 0$, and $t_0 \in I \subset \mathbb{R}$.*

(1) *For any admissible pair (q_1, r_1), we have*

$$\|e^{\frac{1}{2m} it\Delta} f\|_{L_t^\infty \dot{X}_m^s \cap L_t^{q_1,2} \dot{X}_m^{s,r_1}} \lesssim \|f\|_{\mathcal{F}\dot{H}^s}.$$

(ii) *For any admissible pairs (q_1, r_1) and (q_2, r_2), we have*

$$\left\| \int_{t_0}^t e^{\frac{1}{2m} i(t-s)\Delta} F(s) ds \right\|_{L_t^\infty(I; \dot{X}_m^s) \cap L_t^{q_1,2}(I; \dot{X}_m^{s,r_1})} \lesssim \|F\|_{L_t^{q_2',2}(I; \dot{X}_m^{s,r_2'})}.$$

Lemma 3.6 (Embeddings, [81]). *Let $d = 3$. The following inequalities hold:*

$$\|u\|_{S^{weak}} \lesssim \|u\|_S \lesssim_j \|u\|_{W_j},$$

where $j = 1, 2$.

Lemma 3.7 (Square function estimate). *For $0 \leq s \leq 2$ and $1 < p < \infty$, we have*

$$\||\nabla|^s f\|_{L_x^p} \sim \left\| \Big(\sum_{N \in 2^{\mathbb{Z}}} |P_N |\nabla|^s f|^2 \Big)^{\frac{1}{2}} \right\|_{L_x^p}.$$

Lemma 3.8 (Hölder's inequality in Lorentz spaces, [69, 102]). *Let $d \geq 1$. Let $1 \leq q, q_1, q_2 < \infty$ and $1 \leq \alpha, \alpha_1, \alpha_2 \leq \infty$ satisfy*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \quad \text{and} \quad \frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}.$$

Then, the following estimate holds:

$$\|fg\|_{L_t^{q,\alpha}} \lesssim \|f\|_{L_t^{q_1,\alpha_1}} \|g\|_{L_t^{q_2,\alpha_2}}.$$

Lemma 3.9 ([79]). *For any $1 < p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $0 < \alpha, \alpha_1, \alpha_2 < 1$ with $\alpha = \alpha_1 + \alpha_2$, we have*

$$\||\nabla|^\alpha (fg) - f|\nabla|^\alpha g - g|\nabla|^\alpha f\|_{L_x^p} \lesssim \||\nabla|^{\alpha_1} f\|_{L_x^q} \||\nabla|^{\alpha_2} g\|_{L_x^r}.$$

Lemma 3.10 ([89]). *Let $d = 3$ and $\mathcal{B}$ be a bounded subset of $\mathbb{R} \times \mathbb{R}^3$. Let $f \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $1 \leq p, q \leq \infty$. Then, it follows that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\|e^{it\Delta} |x|^{\frac{1}{2}} f\|_{L^2(\mathcal{B})} \leq \varepsilon \|f\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + C_\varepsilon \|e^{it\Delta} f\|_{L_t^{p,\infty} L_x^q}.$$

Lemma 3.11. *Let $d = 3$. Let $1 \leq r < \infty$, $0 < s < \frac{3}{r}$ and let $\chi \in \mathcal{S}(\mathbb{R}^3)$. Then, a multiplication operator $\chi \times$ is bounded on $\dot{X}_m^{s,r}$.*

Proof. We take (r_1, r_2) satisfying $\frac{1}{r_1} = \frac{1}{r} - \frac{s}{3}$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Using Lemma 2.3, we have

$$\begin{aligned}
\|\chi u\|_{\dot{X}_m^{s,r}} &\sim \||t|^s |\nabla|^s \mathcal{M}_m(-t) \chi u\|_{L_x^r} \\
&\lesssim \||t|^s |\nabla|^s \mathcal{M}_m(-t) u\|_{L_x^r} \|\chi\|_{L_x^\infty} + \||t|^s \mathcal{M}_m(-t) u\|_{L_x^{r_1}} \||\nabla|^s \chi\|_{L_x^{r_2}} \\
&\lesssim \||t|^s |\nabla|^s \mathcal{M}_m(-t) u\|_{L_x^r} \|\chi\|_{L_x^\infty} + \||t|^s |\nabla|^s \mathcal{M}_m(-t) u\|_{L_x^r} \||\nabla|^s \chi\|_{L_x^{r_2}} \\
&\lesssim \|u\|_{\dot{X}_m^{s,r}}.
\end{aligned}$$

□

Lemma 3.12 (Nonlinear estimates)**.** *Let $I \subset \mathbb{R}$. We also assume $\tau \in I$ in (5) and (6). The following inequalities hold:*

(1) $\|v\overline{u}\|_{N_1(I)} \lesssim \|u\|_{S^{weak}(I)}\|v\|_{W_2(I)} + \|u\|_{W_1(I)}\|v\|_{S^{weak}(I)} \lesssim \|u\|_{W_1(I)}\|v\|_{W_2(I)}$,
(2) $\|u_1u_2\|_{N_2(I)} \lesssim \|u_1\|_{W_1(I)}\|u_2\|_{S^{weak}(I)} + \|u_1\|_{S^{weak}(I)}\|u_2\|_{W_1(I)} \lesssim \|u_1\|_{W_1(I)}\|u_2\|_{W_1(I)}$,
(3) $\|v\overline{u}\|_{L_t^{\frac{3}{2},2}(I;\dot{X}_{1/2}^{\frac{1}{2},\frac{18}{13}})} \lesssim \|v\|_{L_t^\infty(I;\dot{X}_1^{\frac{1}{2}})}\|u\|_{S(I)} + \|v\|_{S(I)}\|u\|_{L_t^\infty(I;\dot{X}_{1/2}^{\frac{1}{2}})}$,
(4) $\|u_1u_2\|_{L_t^{\frac{3}{2},2}(I;\dot{X}_1^{\frac{1}{2},\frac{18}{13}})} \lesssim \|u_1\|_{L_t^\infty(I;\dot{X}_{1/2}^{\frac{1}{2}})}\|u_2\|_{S(I)} + \|u_1\|_{S(I)}\|u_2\|_{L_t^\infty(I;\dot{X}_{1/2}^{\frac{1}{2}})}$,
(5) $\left\|\int_\tau^t e^{i(t-s)\Delta}(v\overline{u})(s)ds\right\|_{S(I)} \lesssim \|u\|_{S(I)}\|v\|_{S(I)}$,
(6) $\left\|\int_\tau^t e^{\frac{1}{2}i(t-s)\Delta}(u_1u_2)(s)ds\right\|_{S(I)} \lesssim \|u_1\|_{S(I)}\|u_2\|_{S(I)}$.

Proof. We prove the inequality (1). Using (3.3) and Lemma 2.2, 3.8, and 3.6,

$$\begin{aligned}\|v\overline{u}\|_{N_1(I)} &= \|(-4t^2\Delta)^{\frac{1}{4}}(\mathcal{M}_1(-t)v\cdot\overline{\mathcal{M}_{\frac{1}{2}}(-t)u})\|_{L_t^{\frac{6}{5},2}(I;L_x^{\frac{18}{11}})}\\ &\lesssim \|(-4t^2\Delta)^{\frac{1}{4}}\mathcal{M}_1(-t)v\|_{L_t^{6,2}(I;L_x^{\frac{18}{7}})}\|\overline{\mathcal{M}_{\frac{1}{2}}(-t)u}\|_{L_t^{\frac{3}{2},\infty}(I;L_x^{\frac{9}{2}})}\\ &\qquad + \|\mathcal{M}_1(-t)v\|_{L_t^{\frac{3}{2},\infty}(I;L_x^{\frac{9}{2}})}\|(-4t^2\Delta)^{\frac{1}{4}}\overline{\mathcal{M}_{\frac{1}{2}}(-t)u}\|_{L_t^{6,2}(I;L_x^{\frac{18}{7}})}\\ &\sim \|J_1^{\frac{1}{2}}v\|_{L_t^{6,2}(I;L_x^{\frac{18}{7}})}\|u\|_{L_t^{\frac{3}{2},\infty}(I;L_x^{\frac{9}{2}})} + \|v\|_{L_t^{\frac{3}{2},\infty}(I;L_x^{\frac{9}{2}})}\|J_{\frac{1}{2}}^{\frac{1}{2}}u\|_{L_t^{6,2}(I;L_x^{\frac{18}{7}})}\\ &= \|v\|_{W_2(I)}\|u\|_{S^{\mathrm{weak}}(I)} + \|v\|_{S^{\mathrm{weak}}(I)}\|u\|_{W_1(I)}\\ &\lesssim \|v\|_{W_2(I)}\|u\|_{W_1(I)}.\end{aligned}$$

The inequality (2) holds by the same argument with (1). We prove the inequality (3). Applying (3.3), Lemma 2.2, and 3.8,

$$\begin{aligned}\|v\overline{u}\|_{L_t^{\frac{3}{2},2}(I;\dot{X}_{1/2}^{\frac{1}{2},\frac{18}{13}})} &= \|(-4t^2\Delta)^{\frac{1}{4}}(\mathcal{M}_1(-t)v\overline{\mathcal{M}_{\frac{1}{2}}(-t)u})\|_{L_t^{\frac{3}{2},2}(I;L_x^{\frac{18}{13}})}\\ &\lesssim \|(-4t^2\Delta)^{\frac{1}{4}}\mathcal{M}_1(-t)v\|_{L_t^{\infty,\infty}(I;L_x^2)}\|\mathcal{M}_{\frac{1}{2}}(-t)u\|_{L_t^{\frac{3}{2},2}(I;L_x^{\frac{9}{2}})}\\ &\qquad + \|\mathcal{M}_1(-t)v\|_{L_t^{\frac{3}{2},2}(I;L_x^{\frac{9}{2}})}\|(-4t^2\Delta)^{\frac{1}{4}}\mathcal{M}_{\frac{1}{2}}(-t)u\|_{L_t^{\infty,\infty}(I;L_x^2)}\\ &\lesssim \|v\|_{L_t^\infty(I;\dot{X}_1^{\frac{1}{2}})}\|u\|_{S(I)} + \|v\|_{S(I)}\|u\|_{L_t^\infty(I;\dot{X}_{1/2}^{\frac{1}{2}})}.\end{aligned}$$

The inequality (4) holds by the same argument with (3). The last two inequalities are consequences of inhomogeneous Strichartz estimate for non-admissible pairs by Kato [75]. □

Lemma 3.13 (Interpolation in $\dot{X}_m^{s,r}$)**.** *Let $I \subset \mathbb{R}$. The following inequality holds.*

$$\|u\|_{L_t^{\rho,\gamma}(I;\dot{X}_m^{s,r})} \lesssim \|u\|_{L^{\rho_1,\gamma_1}(I;\dot{X}_m^{s_1,r_1})}^{1-\theta}\|u\|_{L^{\rho_2,\gamma_2}(I;\dot{X}_m^{s_2,r_2})}^{\theta}$$

for $1 \leq \rho,\rho_1,\rho_2 < \infty$, $1 \leq \gamma,\gamma_1,\gamma_2 \leq \infty$, $1 < r,r_1,r_2 < \infty$, $0 < \theta < 1$ with $\frac{1}{\rho} = \frac{1-\theta}{\rho_1} + \frac{\theta}{\rho_2}$, $\frac{1}{\gamma} = \frac{1-\theta}{\gamma_1} + \frac{\theta}{\gamma_2}$, $s = (1-\theta)s_1 + \theta s_2$, $s_1 \neq s_2$, and $\frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}$.

Proof. Since $\|u\|_{\dot{W}_x^{s,r}} \lesssim \|u\|_{\dot{W}_x^{s_1,r_1}}^{1-\theta}\|u\|_{\dot{W}_x^{s_2,r_2}}^{\theta}$ holds (see [9, Theorem 6.4.5]), we have

$$\begin{aligned}\|u\|_{\dot{X}_m^{s,r}} &\sim |t|^s\|\mathcal{M}_m(-t)u\|_{\dot{W}_x^{s,r}}\\ &\lesssim (|t|^{s_1}\|\mathcal{M}_m(-t)u\|_{\dot{W}_x^{s_1,r_1}})^{1-\theta}(|t|^{s_2}\|\mathcal{M}_m(-t)u\|_{\dot{W}_x^{s_2,r_2}})^{\theta}\\ &\sim \|u\|_{\dot{X}_m^{s_1,r_1}}^{1-\theta}\|u\|_{\dot{X}_m^{s_2,r_2}}^{\theta}.\end{aligned}$$

Therefore, Lemma 3.8 deduces

$$\|u\|_{L_t^{\rho,\gamma}(I;\dot{X}_m^{s,r})} \lesssim \left\|\|u\|_{\dot{X}_m^{s_1,r_1}}^{1-\theta}\|u\|_{\dot{X}_m^{s_2,r_2}}^{\theta}\right\|_{L_t^{\rho,\gamma}(I)} \lesssim \|u\|_{L_t^{\rho_1,\gamma_1}(I;\dot{X}_m^{s_1,r_1})}^{1-\theta}\|u\|_{L_t^{\rho_2,\gamma_2}(I;\dot{X}_m^{s_2,r_2})}^{\theta}.$$

□

The following embedding is a general case of $W_j(I)\subset S(I)$ (see Lemma 3.6).

Lemma 3.14 (Sobolev type embedding)**.** *Let $d=3$ and $I\subset\mathbb{R}$ be a time interval. Let $1\le\rho_1,\rho_2,r_1,r_2<\infty$, $1\le\gamma_1\le\infty$, and $0<s\le 1$ satisfy $\frac{1}{\rho_1}=s+\frac{1}{\rho_2}$ and $\frac{1}{r_1}=\frac{1}{r_2}-\frac{s}{3}$. Then, the embedding $L_t^{\rho_2,\gamma_1}(I;\dot{X}_m^{s,r_2})\subset L_t^{\rho_1,\gamma_1}(I;L_x^{r_1})$ holds. In particular, the following inequality holds for any $u\in L_t^{\rho_2,\gamma_1}(I;\dot{X}_m^{s,r_2})$:*

$$\|u\|_{L_t^{\rho_1,\gamma_1}(I;L_x^{r_1})}\lesssim\|u\|_{L_t^{\rho_2,\gamma_1}(I;\dot{X}_m^{s,r_2})}.$$

Proof. By Lemma 2.3 and 3.8, we have

$$\begin{aligned}\|u\|_{L_t^{\rho_1,\gamma_1}(I;L_x^{r_1})}&=\|\mathcal{M}_m(-t)u\|_{L_t^{\rho_1,\gamma_1}(I;L_x^{r_1})}\lesssim\||\nabla|^s\mathcal{M}_m(-t)u\|_{L_t^{\rho_1,\gamma_1}(I;L_x^{r_2})}\\&\lesssim\||t|^{-s}\|_{L_t^{\frac{1}{s},\infty}(I)}\||t|^s|\nabla|^s\mathcal{M}_m(-t)u\|_{L_t^{\rho_2,\gamma_1}(I;L_x^{r_2})}\lesssim\|u\|_{L_t^{\rho_2,\gamma_1}(I;\dot{X}_m^{s,r_2})}.\end{aligned}$$

□

The following lemma is cited in [89, Proposition 2.5].

Proposition 3.15. *Let $1<\rho_j<\infty$, $s_j\in\mathbb{R}$, and $1\le q_j,r_j<\infty$ for $j=1,2$. Let $v\in L_t^{\rho_1,2}(\mathbb{R};\dot{M}_{q_1,r_1}^{s_1})\cap L_t^{\rho_2,2}(\mathbb{R};\dot{M}_{q_2,r_2}^{s_2})$. For any $\varepsilon>0$, there exist a function $\widetilde{v}(t,x)$ defined on $\mathbb{R}\times\mathbb{R}^3$ and $\delta,M,R>0$ such that supp $\widetilde{v}\subset\{(t,x)\in\mathbb{R}\times\mathbb{R}^3:\delta\le|t|\le M,|x|\le R\}$ and*

$$\sum_{j=1,2}\|v-\widetilde{v}\|_{L_t^{\rho_j,2}(\mathbb{R};\dot{M}_{q_j,r_j}^{s_j})}\le\varepsilon,$$

where the norm $\|\cdot\|_{\dot{M}_{q,r}^s(t)}$ is defined as

$$\|f\|_{\dot{M}_{q,r}^s(t)}:=\|2^{js}e^{it\Delta}\psi_N e^{-it\Delta}f\|_{\ell_N^r(2^{\mathbb{Z}};L_x^q)}\sim\||t|^s\mathcal{M}_{\frac{1}{2}}(-t)f\|_{\dot{B}_{q,r}^s}.$$

Proposition 3.16. *Let $d=3$. Let $1<\rho_j<\infty$, $s_j,m\in\mathbb{R}$, and $2<r_j<\infty$ for $j=1,2$. Let $v\in L_t^{\rho_1,2}(\mathbb{R};\dot{X}_m^{s_1,r_1})\cap L_t^{\rho_2,2}(\mathbb{R};\dot{X}_m^{s_2,r_2})$. For any $\varepsilon>0$, there exist a function $\widetilde{v}=\widetilde{v}(t,x)$ defined on $\mathbb{R}\times\mathbb{R}^3$ and $\delta,M,R>0$ such that supp $\widetilde{v}\subset\{(t,x)\in\mathbb{R}\times\mathbb{R}^3:\delta\le|t|\le M,|x|\le R\}$*

$$\sum_{j=1,2}\|v-\widetilde{v}\|_{L_t^{\rho_j,2}(\mathbb{R};\dot{X}_m^{s_j,r_j})}\le\varepsilon.$$

Proof. Since it follows from [9, Theorem 6.4.4] that $\dot{B}_{r_j,2}^{s_j}(\mathbb{R}^3)\subset\dot{W}_x^{s_j,r_j}(\mathbb{R}^3)$, that is, $\|f\|_{\dot{W}_x^{s_j,r_j}}\lesssim\|f\|_{\dot{B}_{r_j,2}^{s_j}}$ holds, we can get the desired result by using Proposition 3.15. Indeed, we have

$$\begin{aligned}\sum_{j=1,2}\|v-\widetilde{v}\|_{L_t^{\rho_j,2}(\mathbb{R};\dot{X}_m^{s_j,r_j})}&\sim\sum_{j=1,2}\||t|^{s_j}\mathcal{M}_m(-t)(v-\widetilde{v})\|_{L_t^{\rho_j,2}(\mathbb{R};\dot{W}_x^{s_j,r_j})}\\&\lesssim\sum_{j=1,2}\||t|^{s_j}\mathcal{M}_m(-t)(v-\widetilde{v})\|_{L_t^{\rho_j,2}(\mathbb{R};\dot{B}_{r_j,2}^{s_j})}\le\varepsilon.\end{aligned}$$

□

3.3. **Local well-posedness.** In this subsection, we establish a local theory in $(C_t\dot{X}_{1/2}^{1/2}\cap W_1)\times(C_t\dot{X}_1^{1/2}\cap W_2)$ for (NLS). The result is given as a consequence of Strichartz estimate (Proposition 3.5) and the estimates of the previous subsection (Lemma 3.6 and 3.12). Also, we derive a necessary and sufficient condition for scattering (Proposition 3.19) and give a scattering result for small data (Proposition 3.20).

Let us first establish a weak version of the local well-posedness.

Proposition 3.17. *Let $d=3$, $\kappa=\frac{1}{2}$, and $\tau\in\mathbb{R}$. There exists a universal constant $\delta>0$ such that if $(u_\tau,v_\tau)\in\mathcal{S}'(\mathbb{R}^3)\times\mathcal{S}'(\mathbb{R}^3)$ satisfies*

$$\|(e^{i(t-\tau)\Delta}u_\tau,e^{\frac{1}{2}i(t-\tau)\Delta}v_\tau)\|_{S(I)\times S(I)}\le\delta$$

for some interval $I \ni \tau$, then the integral equation

$$\begin{cases} u(t) = e^{i(t-\tau)\Delta}u_\tau + 2i\displaystyle\int_\tau^t e^{i(t-s)\Delta}(v\overline{u})(s)ds, \\ v(t) = e^{\frac{1}{2}i(t-\tau)\Delta}v_\tau + i\displaystyle\int_\tau^t e^{\frac{1}{2}i(t-s)\Delta}(u^2)(s)ds \end{cases}$$

has a unique solution $(u,v) \in S(I) \times S(I)$ in $S(I) \times S(I)$ sense and (u,v) satisfies

$$\|(u,v)\|_{S(I)\times S(I)} \le 2\|(e^{i(t-\tau)\Delta}u_\tau, e^{\frac{1}{2}i(t-\tau)\Delta}v_\tau)\|_{S(I)\times S(I)}.$$

Proof. We define a map (Φ,Ψ), a set E, and a distance d on E as

$$\begin{cases} \Phi(u(t),v(t)) = e^{i(t-\tau)\Delta}u_\tau + 2i\displaystyle\int_\tau^t e^{i(t-s)\Delta}(v\overline{u})(s)ds, \\ \Psi(u(t),v(t)) = e^{\frac{1}{2}i(t-\tau)\Delta}v_\tau + i\displaystyle\int_\tau^t e^{\frac{1}{2}i(t-s)\Delta}(u^2)(s)ds, \end{cases}$$

$$E := \left\{(u,v) \in S(I)\times S(I) : \|(u,v)\|_{S(I)\times S(I)} \le 2\|(e^{i(t-\tau)\Delta}u_\tau, e^{\frac{1}{2}i(t-\tau)\Delta}v_\tau)\|_{S(I)\times S(I)}\right\},$$

$$d((u_1,v_1),(u_2,v_2)) := \|(u_1,v_1)-(u_2,v_2)\|_{S(I)\times S(I)}.$$

From the last two estimates of Lemma 3.12, we have

$$\begin{aligned} \|(\Phi(u,v),\Psi(u,v))\|_{S(I)\times S(I)} &\le \|(e^{i(t-\tau)\Delta}u_\tau, e^{\frac{1}{2}i(t-\tau)\Delta}v_\tau)\|_{S(I)\times S(I)} + c\,\|u\|_{S(I)}\|(u,v)\|_{S(I)\times S(I)} \\ &\le (1+4c\delta)\|(e^{i(t-\tau)\Delta}u_\tau, e^{\frac{1}{2}i(t-\tau)\Delta}v_\tau)\|_{S(I)\times S(I)} \end{aligned}$$

and

$$\begin{aligned} &d((\Phi(u_1,v_1),\Psi(u_1,v_1)),(\Phi(u_2,v_2),\Psi(u_2,v_2))) \\ &\quad\le c\,\|v_1\|_{S(I)}\|u_1-u_2\|_{S(I)} + c\,\|v_1-v_2\|_{S(I)}\|u_2\|_{S(I)} + c\,\|u_1+u_2\|_{S(I)}\|u_1-u_2\|_{S(I)} \\ &\quad\le c\left\{\|(u_1,v_1)\|_{S(I)\times S(I)} + \|(u_2,v_2)\|_{S(I)\times S(I)}\right\} d((u_1,v_1),(u_2,v_2)) \\ &\quad\le 4c\delta\, d((u_1,v_1),(u_2,v_2)). \end{aligned}$$

Therefore, if we take a positive constant $\delta > 0$ satisfying $4c\delta < \frac{1}{2}$, then (Φ,Ψ) is a contraction map on E. □

Theorem 3.18 (Local well-posedness)**.** *Let $d = 3$ and $\kappa = \frac{1}{2}$. For any initial time $t_0 \in \mathbb{R}$ and any data $(u_0,v_0) \in \dot{X}^{1/2}_{1/2}(t_0) \times \dot{X}^{1/2}_1(t_0)$, there exist an open interval $I \ni t_0$ and the unique solution $(u,v) \in (C_t(I;\dot{X}^{1/2}_{1/2}) \cap W_1(I)) \times (C_t(I;\dot{X}^{1/2}_1) \cap W_2(I))$ to (NLS) with the initial condition $(u(t_0),v(t_0)) = (u_0,v_0)$. Moreover, there exists a universal constant $\delta > 0$ such that if the data satisfies*

$$\|(e^{i(t-t_0)\Delta}u_0, e^{\frac{1}{2}i(t-t_0)\Delta}v_0)\|_{W_1(I)\times W_2(I)} \le \delta,$$

then the solution satisfies

$$\|(u,v)\|_{W_1(I)\times W_2(I)} \lesssim \|(e^{i(t-t_0)\Delta}u_0, e^{\frac{1}{2}i(t-t_0)\Delta}v_0)\|_{W_1(I)\times W_2(I)}.$$

Furthermore, the solution depends continuously on the initial data, that is, for any $\{(u_{0,n},v_{0,n})\}$ satisfying $(u_{0,n},v_{0,n}) \longrightarrow (u_0,v_0)$ in $\dot{X}^{1/2}_{1/2}(t_0)\times\dot{X}^{1/2}_1(t_0)$ as $n \to \infty$ and any compact time interval $\widetilde{I} \subset I$, there exists $n_0 \in \mathbb{N}$ such that (NLS) with initial data $(u_{0,n},v_{0,n})$ has a unique solution $(u_n,v_n) \in (C_t(\widetilde{I};\dot{X}^{1/2}_{1/2}) \cap W_1(\widetilde{I})) \times (C_t(\widetilde{I};\dot{X}^{1/2}_1) \cap W_2(\widetilde{I}))$ for any $n \ge n_0$ and $(u_n,v_n) \longrightarrow (u,v)$ in $(C_t(\widetilde{I};\dot{X}^{1/2}_{1/2}) \cap W_1(\widetilde{I})) \times (C_t(\widetilde{I};\dot{X}^{1/2}_1) \cap W_2(\widetilde{I}))$ as $n \to \infty$.

Proof. The strategy of the proof is as follows: We first obtain a S-solution. Then, we show it is a solution in the sense of Definition 1.35 by a persistence-of-regularity type argument.

By Lemma 3.6 and Proposition 3.5, we have

$$\|(e^{i(t-t_0)\Delta}u_0, e^{\frac{1}{2}i(t-t_0)\Delta}v_0)\|_{S(\mathbb{R})\times S(\mathbb{R})} \lesssim \|(e^{i(t-t_0)\Delta}u_0, e^{\frac{1}{2}i(t-t_0)\Delta}v_0)\|_{W_1(\mathbb{R})\times W_2(\mathbb{R})}$$

$$\lesssim \|(e^{-it_0\Delta}u_0, e^{-\frac{1}{2}it_0\Delta}v_0)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}\times\mathcal{F}\dot{H}^{\frac{1}{2}}} < \infty.$$

Hence, we can chose an open interval $I \ni t_0$ so that

$$\|(e^{i(t-t_0)\Delta}u_0, e^{\frac{1}{2}i(t-t_0)\Delta}v_0)\|_{S(I)\times S(I)} \leq \delta.$$

For this interval, we have a unique S-solution $(u,v) \in S(I)\times S(I)$ by Proposition 3.17. We shall show this is a solution in the sense of Definition 1.35. By Proposition 3.5 and Lemma 3.12, one has

$$\begin{aligned}
&\|(u,v)\|_{W_1(I)\times W_2(I)} \\
&\quad \leq \|(e^{i(t-t_0)\Delta}u_0, e^{\frac{1}{2}i(t-t_0)\Delta}v_0)\|_{W_1(I)\times W_2(I)} + c\,\|v\overline{u}\|_{N_1(I)} + c\,\|u^2\|_{N_2(I)} \\
&\quad \leq \|(e^{i(t-t_0)\Delta}u_0, e^{\frac{1}{2}i(t-t_0)\Delta}v_0)\|_{W_1(I)\times W_2(I)} + c\,\|(u,v)\|_{W_1(I)\times W_2(I)}\|(u,v)\|_{S(I)\times S(I)}
\end{aligned}$$

and

$$\begin{aligned}
&\|(u,v)\|_{L_t^\infty(I;\dot{X}^{1/2}_{1/2})\times L_t^\infty(I;\dot{X}^{1/2}_1)} \\
&\quad \leq c\,\|(u_0,v_0)\|_{\dot{X}^{1/2}_{1/2}(t_0)\times \dot{X}^{1/2}_1(t_0)} + c\,\|v\overline{u}\|_{L_t^{\frac{3}{2},2}(I;\dot{X}^{\frac{1}{2},\frac{18}{13}}_{1/2})} + c\,\|u^2\|_{L_t^{\frac{3}{2},2}(I;\dot{X}^{\frac{1}{2},\frac{18}{13}}_{1})} \\
&\quad \leq c\,\|(u_0,v_0)\|_{\dot{X}^{1/2}_{1/2}(t_0)\times \dot{X}^{1/2}_1(t_0)} + c\,\|(u,v)\|_{S(I)\times S(I)}\|(u,v)\|_{L_t^\infty(I;\dot{X}^{1/2}_{1/2})\times L_t^\infty(I;\dot{X}^{1/2}_1)}
\end{aligned}$$

We subdivide the interval I into $\cup_{j=0}^J I_j$ so that we have $c\,\|(u,v)\|_{S(I_j)\times S(I_j)} \leq \frac{1}{2}$ in each interval. Suppose $t_0 \in I_0$. We have

$$\|(u,v)\|_{W_1(I_0)\times W_2(I_0)} \leq 2\|(e^{i(t-t_0)\Delta}u_0, e^{\frac{1}{2}i(t-t_0)\Delta}v_0)\|_{W_1(I_0)\times W_2(I_0)}$$

and

$$\|(u,v)\|_{L_t^\infty(I_0;\dot{X}^{1/2}_{1/2})\times L_t^\infty(I_0;\dot{X}^{1/2}_1)} \lesssim \|(u_0,v_0)\|_{\dot{X}^{1/2}_{1/2}(t_0)\times \dot{X}^{1/2}_1(t_0)}.$$

Repeat the argument to obtain $(u,v) \in (L_t^\infty(I;\dot{X}^{1/2}_{1/2}) \cap W_1(I)) \times (L_t^\infty(I;\dot{X}^{1/2}_1) \cap W_2(I))$. The continuous dependence on initial data is a special case of Proposition 3.23. We omit the details. □

Proposition 3.19 (Scattering criterion)**.** *Let $d = 3$ and $\kappa = \frac{1}{2}$. Let (u,v) be a unique solution to (NLS) given in Theorem 3.18. Then, the following seven statements are equivalent.*

1. *(u,v) scatters in positive time;*
2. *There exists $\tau \in I_{max}$ such that $\|(u,v)\|_{W_1([\tau,T_{\max}))\times W_2([\tau,T_{\max}))} < \infty$;*
3. *There exists $\tau \in I_{max}$ such that $\|(u,v)\|_{S([\tau,T_{\max}))\times S([\tau,T_{\max}))} < \infty$;*
4. *There exists $\tau \in I_{max}$ such that $\|u\|_{W_1([\tau,T_{\max}))} < \infty$;*
5. *There exists $\tau \in I_{max}$ such that $\|v\|_{W_2([\tau,T_{\max}))} < \infty$;*
6. *There exists $\tau \in I_{max}$ such that $\|u\|_{S([\tau,T_{\max}))} < \infty$;*
7. *There exists $\tau \in I_{max}$ such that $\|v\|_{S([\tau,T_{\max}))} < \infty$.*

Proof. We prove (1) $\Longrightarrow$ (2). By the definition of scattering in positive time, we have $T_{\max} = \infty$. We set that $(u_+, v_+) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ satisfies

$$\lim_{t\to\infty} \|(e^{-it\Delta}u(t), e^{-\frac{1}{2}it\Delta}v(t)) - (u_+,v_+)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}\times\mathcal{F}\dot{H}^{\frac{1}{2}}} = 0. \tag{3.4}$$

Applying Proposition 3.5, we have

$$\|(e^{it\Delta}u_+, e^{\frac{1}{2}it\Delta}v_+)\|_{W_1([0,\infty))\times W_2([0,\infty))} \lesssim \|(u_+,v_+)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}\times\mathcal{F}\dot{H}^{\frac{1}{2}}} < \infty.$$

Thus, there exists $\tau > 0$ such that

$$\|(e^{it\Delta}u_+, e^{\frac{1}{2}it\Delta}v_+)\|_{W_1([\tau,\infty))\times W_2([\tau,\infty))} < \frac{\delta}{2},$$

where $\delta > 0$ is given in Theorem 3.18. Furthermore, we have

$$\begin{aligned}\|(e^{it\Delta}u_+, e^{\frac{1}{2}it\Delta}v_+) - (e^{i(t-t_0)\Delta}u(t_0), e^{\frac{1}{2}i(t-t_0)\Delta}v(t_0))\|_{W_1([0,\infty))\times W_2([0,\infty))} \\ \lesssim \|(u_+, v_+) - (e^{-it_0\Delta}u(t_0), e^{-\frac{1}{2}it_0\Delta}v(t_0))\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}\times\mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \frac{\delta}{2}\end{aligned}$$

for sufficiently large $t_0 \geq \tau$ by Proposition 3.5 and (3.4). Then,

$$\begin{aligned}&\|(e^{i(t-t_0\Delta)}u(t_0), e^{\frac{1}{2}i(t-t_0)\Delta}v(t_0))\|_{W_1([\tau,\infty))\times W_2([\tau,\infty))} \\ &\quad\leq \|(e^{it\Delta}u_+, e^{\frac{1}{2}it\Delta}v_+)\|_{W_1([\tau,\infty))\times W_2([\tau,\infty))} \\ &\qquad + \|(e^{it\Delta}u_+, e^{\frac{1}{2}it\Delta}v_+) - (e^{i(t-t_0)\Delta}u(t_0), e^{\frac{1}{2}i(t-t_0)\Delta}v(t_0))\|_{W_1([\tau,\infty))\times W_2([\tau,\infty))} \\ &\quad\leq \delta.\end{aligned}$$

By theorem 3.18, there exists a solution $(\widetilde{u}, \widetilde{v})$ to (NLS) such that $(\widetilde{u}(t_0), \widetilde{v}(t_0)) = (u(t_0), v(t_0))$ and $\|(\widetilde{u}, \widetilde{v})\|_{W_1([\tau,\infty))\times W_2([\tau,\infty))} \leq c\delta$. By the uniqueness of solution to (NLS), we have $(\widetilde{u}, \widetilde{v}) = (u, v)$. Therefore, we have $\|(u, v)\|_{W_1([\tau,\infty))\times W_2([\tau,\infty))} \leq c\delta$.
We prove (2) $\Longrightarrow$ (1). Let $t_1 > t_2 > 0$. Using Proposition 3.5 and Lemma 3.12, we have

$$\begin{aligned}\|e^{-it_1\Delta}u(t_1) - e^{-it_2\Delta}u(t_2)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} &= 2\left\|\int_{t_2}^{t_1} e^{-is\Delta}(v\overline{u})(s)ds\right\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \\ &= 2\left\|\int_{t_2}^{t_1} e^{i(t_1-s)\Delta}(v\overline{u})(s)ds\right\|_{\dot{X}^{\frac{1}{2}}_{1/2}(t_1)} \\ &\leq 2\left\|\int_{t_2}^{t} e^{i(t-s)\Delta}(v\overline{u})(s)ds\right\|_{L_t^\infty([t_2,t_1];\dot{X}^{\frac{1}{2}}_{1/2})} \\ &\leq c\,\|v\overline{u}\|_{N_1([t_2,t_2])} \\ &\leq c\,\|u\|_{W_1([t_2,t_1])}\|v\|_{W_2([t_2,t_1])} \longrightarrow 0 \quad \text{as} \quad t_1 > t_2 \to \infty\end{aligned}$$

and

$$\|e^{-\frac{1}{2}it_1\Delta}v(t_1) - e^{-\frac{1}{2}it_2\Delta}v(t_2)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \leq c\,\|u\|^2_{W_1([t_2,t_1])} \longrightarrow 0 \quad \text{as} \quad t_1 > t_2 \to \infty.$$

Therefore, (u, v) scatters in positive time.
From Lemma 3.6, (2) $\Longrightarrow$ (3) holds.
We prove (3) $\Longrightarrow$ (2). For any $\varepsilon > 0$, there exists $\tau < \tau_0 < T_{\max}$ such that

$$\|(u, v)\|_{S([\tau_0,T_{\max}))\times S([\tau_0,T_{\max}))} < \varepsilon.$$

Using Proposition 3.5, we have

$$\begin{aligned}\|(u, v)\|_{W_1([\tau_0,T])\times W_2([\tau_0,T])} \leq c\,\|(u(\tau_0), v(\tau_0))\|_{\dot{X}^{\frac{1}{2}}_{1/2}(\tau_0)\times\dot{X}^{\frac{1}{2}}_1(\tau_0)} \\ + c\,\|(u, v)\|_{S([\tau_0,T])\times S([\tau_0,T])}\|(u, v)\|_{W_1([\tau_0,T])\times W_2([\tau_0,T])} \\ \leq c\,\|(u(\tau_0), v(\tau_0))\|_{\dot{X}^{\frac{1}{2}}_{1/2}(\tau_0)\times\dot{X}^{\frac{1}{2}}_1(\tau_0)} + c\varepsilon\|(u, v)\|_{W_1([\tau_0,T])\times W_2([\tau_0,T])}\end{aligned}$$

for any $T \geq \tau_0$. Taking $c\varepsilon \leq \frac{1}{2}$, we obtain

$$\|(u, v)\|_{W_1([\tau_0,T])\times W_2([\tau_0,T])} \leq 2c\,\|(u(\tau_0), v(\tau_0))\|_{\dot{X}^{\frac{1}{2}}_{1/2}(\tau_0)\times\dot{X}^{\frac{1}{2}}_1(\tau_0)}.$$

Letting $T \longrightarrow T_{\max}$, we get (2).
We prove (4) $\Longleftrightarrow$ (5), which implies (2) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5). Suppose (4). Take $\tau_0 \in (\tau, T_{\max})$ to be chosen later. For any $T \in (\tau_0, T_{\max})$, we have

$$\|v\|_{W_2([\tau_0,T))} \leq c\,\|v(\tau_0)\|_{\dot{X}^{\frac{1}{2}}_1(\tau_0)} + c\,\|u\|_{W_1([\tau_0,T))}\|v\|_{W_2([\tau_0,T))}.$$

Here, we choose τ_0 satisfying $c\|u\|_{W_1([\tau_0,T))} \le \frac{1}{2}$. Then, we see

$$\|v\|_{W_2([\tau_0,T))} \le 2c\|v(\tau_0)\|_{\dot{X}_1^{\frac{1}{2}}(\tau_0)}.$$

Since $T \in (\tau_0, T_{\max})$ is arbitrary, we obtain (5). By the same argument, we have (5) $\Longrightarrow$ (4). We prove (6) $\Longleftrightarrow$ (7), which implies (3) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7). Suppose (6). One deduces from Proposition 3.5 and Lemma 3.12 that

$$\|v\|_{S([\tau,\tau_0))} \le c\|v(\tau)\|_{\dot{X}_1^{\frac{1}{2}}(\tau)} + c\|u\|_{S([\tau,\tau_0))}^2$$

for any $\tau_0 \in (\tau, T_{\max})$. Here, we note that the implicit constant is independent of T. Hence, we obtain (7) by letting $\tau \uparrow T_{\max}$. Suppose (7). Take $\tau_0 \in (\tau, T_{\max})$ to be chosen later. For any $T \in (\tau_0, T_{\max})$, we see

$$\|u\|_{S((\tau_0,T))} \le c\|u(\tau_0)\|_{\dot{X}_{1/2}^{\frac{1}{2}}(\tau_0)} + c\|u\|_{S((\tau_0,T))}\|v\|_{S((\tau_0,T))},$$

where the constant c is independent of τ_0 and T. We now choose τ_0 so that $c\|v\|_{S((\tau_0,T_{\max}))} \le \frac{1}{2}$. This is possible by the property (7). Then, the above inequality implies that

$$\|u\|_{S((\tau_0,T))} \le 2c\|u(\tau_0)\|_{\dot{X}_{1/2}^{\frac{1}{2}}(\tau_0)}.$$

Since $T \in (\tau_0, T_{\max})$ is arbitrary, we obtain the result. □

We turn to a sufficient condition for scattering. One of the simplest conditions is due to smallness of the data.

Proposition 3.20 (Small data scattering)**.** *Let $d = 3$ and $\kappa = \frac{1}{2}$. Let $(u_0, v_0) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and let (u, v) be a corresponding unique solution given in Theorem 3.18. Then, we have the followings.*

1. *There exists $\eta_1 > 0$ such that if $\|(e^{it\Delta}u_0, e^{\frac{1}{2}it\Delta}v_0)\|_{S\times S} \le \eta_1$, then (u, v) scatters.*
2. *There exists $\eta_2 > 0$ such that if $\|(e^{it\Delta}u_0, e^{\frac{1}{2}it\Delta}v_0)\|_{W_1\times W_2} \le \eta_2$, then (u, v) scatters.*
3. *There exists $\eta_3 > 0$ such that if $\|(u_0, v_0)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}\times\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \eta_3$, then (u, v) scatters.*

These follow from Proposition 3.17, 3.19, and 3.5.

3.4. **Nonpositive energy implies failure of scattering.** In this subsection, we give a proof of Theorem 1.44. To begin with, we will prove that if a data belongs $H^1 \times H^1$, in addition, then the corresponding solution given in Theorem 3.18 stays in $H^1 \times H^1$ and the mass and the energy make sense and are conserved. Furthermore, as is well-known, since our equation is mass-subcritical, the conservation of mass implies the solution is global.

Proposition 3.21. *Let $d = 3$ and $\kappa = \frac{1}{2}$. For any $t_0 \in \mathbb{R}$ and any $(u_0, v_0) \in (\dot{X}_{1/2}^{1/2}(t_0) \cap H^1) \times (\dot{X}_1^{1/2}(t_0) \cap H^1)$, there exists a unique time global solution $(u, v) \in (C_t(\mathbb{R}; \dot{X}_{1/2}^{1/2} \cap H^1) \cap W_{1,loc}(\mathbb{R})) \times (C_t(\mathbb{R}; \dot{X}_1^{1/2} \cap H^1) \cap W_{2,loc}(\mathbb{R}))$ to (NLS) with the initial condition $(u(t_0), v(t_0)) = (u_0, v_0)$. The solution has conserved its mass and energy. Furthermore, if the solution scatters in $\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}$ sense then, the solution also scatters in $H^1 \times H^1$ sense.*

This is done by a persistence-of-regularity argument. We omit the details of the proof. Now, we prove Theorem 1.44.

Proof of Theorem 1.44. Suppose that a solution (u, v) given in Proposition 3.21 scatters in $\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}$. Then, the solution scatters also in $H^1 \times H^1$, that is, there exists $(u_\pm, v_\pm)$ such that

$$\begin{aligned}\|(u,v) - (e^{it\Delta}u_\pm, e^{\frac{1}{2}it\Delta}v_\pm)\|_{L_x^3\times L_x^3} &\lesssim \|(u,v) - (e^{it\Delta}u_\pm, e^{\frac{1}{2}it\Delta}v_\pm)\|_{H_x^1\times H_x^1} \\ &\longrightarrow 0 \quad \text{as} \quad t \to \pm\infty\end{aligned} \tag{3.5}$$

By the density argument, we assume $(u_\pm, v_\pm) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$. Combining this limit and Lemma 3.3, we have

$$\|(u(t), v(t))\|_{L_x^3 \times L_x^3} \longrightarrow 0$$

as $t \to \pm\infty$. Hence,

$$\left| 2\mathrm{Re} \int_{\mathbb{R}^3} v(t,x)\overline{u(t,x)}^2 dx \right| \le 2\|v(t)\|_{L_x^3}\|u(t)\|_{L_x^3}^2 \longrightarrow 0$$

as $t \to \infty$. We deduce that

$$E(u_0, v_0) = \lim_{t\to\pm\infty} E(u(t), v(t)) = \|\nabla u_\pm\|_{L_x^2}^2 + \frac{1}{2}\|\nabla v_\pm\|_{L_x^2}^2 \ge 0.$$

Further, $E(u_0, v_0) = 0$ implies $(u_\pm, v_\pm) = (0,0)$. By (3.5) and the mass conservation implies $(u_0, v_0) = (0,0)$. □

3.5. **Stability.** In this subsection, we establish a stability result. Roughly speaking, the proposition implies that two solutions are also close each other if their initial data are close and the equations for them are close.

Proposition 3.22 (Short time perturbation)**.** *Let $d = 3$ and $\kappa = \frac{1}{2}$. Let I be a time interval and $t_0 \in I$. Let $(\widetilde{u}, \widetilde{v}) : I \times \mathbb{R}^3 \longrightarrow \mathbb{C}^2$ satisfy*

$$\begin{cases} i\partial_t \widetilde{u} + \Delta \widetilde{u} = -2\widetilde{v}\overline{\widetilde{u}} + e_1, \\ i\partial_t \widetilde{v} + \dfrac{1}{2}\Delta \widetilde{v} = -\widetilde{u}^2 + e_2 \end{cases} \tag{3.6}$$

for some function $(e_1, e_2) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. There exists a constant $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \le \varepsilon_0$, if $(u_0, v_0) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ satisfies

$$\|(\widetilde{u}, \widetilde{v})\|_{W_1(I)\times W_2(I)} \le \varepsilon_0, \quad \|(e_1, e_2)\|_{N_1(I)\times N_2(I)} \le \varepsilon,$$

and

$$\big\|\big(e^{i(t-t_0)\Delta}(u_0 - \widetilde{u}(t_0)), e^{\frac{1}{2}i(t-t_0)\Delta}(v_0 - \widetilde{v}(t_0))\big)\big\|_{W_1(I)\times W_2(I)} \le \varepsilon,$$

then a solution (u, v) to (NLS) with initial data $(u(t_0), v(t_0)) = (u_0, v_0)$ obeys

$$\|(u,v) - (\widetilde{u}, \widetilde{v})\|_{W_1(I)\times W_2(I)} \lesssim \varepsilon \quad \text{and} \quad \|(v\overline{u} - \widetilde{v}\overline{\widetilde{u}}, u^2 - \widetilde{u}^2)\|_{N_1(I)\times N_2(I)} \lesssim \varepsilon.$$

Proof. We define $(w, z) = (u, v) - (\widetilde{u}, \widetilde{v})$. Then, (w, z) satisfies

$$\begin{cases} w(t) = e^{i(t-t_0)\Delta} w(t_0) + i\displaystyle\int_{t_0}^t e^{i(t-s)\Delta}(2v\overline{u} - 2\widetilde{v}\overline{\widetilde{u}} + e_1)(s)ds, \\ z(t) = e^{\frac{1}{2}i(t-t_0)\Delta} z(t_0) + i\displaystyle\int_{t_0}^t e^{\frac{1}{2}i(t-s)\Delta}(u^2 - \widetilde{u}^2 + e_2)(s)ds. \end{cases}$$

Using the following identities

$$\begin{aligned} v\overline{u} - \widetilde{v}\overline{\widetilde{u}} &= (\overline{u} - \overline{\widetilde{u}})(v - \widetilde{v}) + (\overline{u} - \overline{\widetilde{u}})\widetilde{v} + (v - \widetilde{v})\overline{\widetilde{u}} = \overline{w}z + \overline{w}\widetilde{v} + z\overline{\widetilde{u}}, \\ u^2 - \widetilde{u}^2 &= (u - \widetilde{u})^2 + 2(u - \widetilde{u})\widetilde{u} = w^2 + 2w\widetilde{u}, \end{aligned} \tag{3.7}$$

we can rewrite the integral equation as follows:

$$\begin{cases} w(t) = e^{i(t-t_0)\Delta} w(t_0) + 2i\displaystyle\int_{t_0}^t e^{i(t-s)\Delta}(2\overline{w}z + 2\overline{w}\widetilde{v} + 2z\overline{\widetilde{u}} + e_1)(s)ds, \\ z(t) = e^{\frac{1}{2}i(t-t_0)\Delta} z(t_0) + i\displaystyle\int_{t_0}^t e^{\frac{1}{2}i(t-s)\Delta}(w^2 + 2w\widetilde{u} + e_2)(s)ds. \end{cases}$$

Proposition 3.5 and Lemma 3.12 deduce

$$\begin{aligned} \|w\|_{W_1(I)} &\le \varepsilon + c\,\|\overline{w}z + \overline{w}\widetilde{v} + z\overline{\widetilde{u}} + e_1\|_{N_1(I)} \\ &\le \varepsilon + c\,\|w\|_{W_1(I)}\|z\|_{W_2(I)} + c\,\|w\|_{W_1(I)}\|\widetilde{v}\|_{W_2(I)} + c\,\|z\|_{W_2(I)}\|\widetilde{u}\|_{W_1(I)} + c\,\|e_1\|_{N_1(I)} \\ &\le c\varepsilon + c\,\|w\|_{W_1(I)}\|z\|_{W_2(I)} + c\varepsilon_0\|w\|_{W_1(I)} + c\varepsilon_0\|z\|_{W_2(I)} \end{aligned}$$

$$\leq c\varepsilon + c\,\|(w,z)\|^2_{W_1(I)\times W_2(I)} + c\varepsilon_0\|(w,z)\|_{W_1(I)\times W_2(I)} \tag{3.8}$$

and

$$\|z\|_{W_2(I)} \leq c\varepsilon + c\,\|(w,z)\|^2_{W_1(I)\times W_2(I)} + c\varepsilon_0\|(w,z)\|_{W_1(I)\times W_2(I)}. \tag{3.9}$$

Combining these inequalities, we have

$$\|(w,z)\|_{W_1(I)\times W_2(I)} \leq c\varepsilon + c\,\|(w,z)\|^2_{W_1(I)\times W_2(I)} + c\varepsilon_0\|(w,z)\|_{W_1(I)\times W_2(I)}.$$

If ε_0 is small, we obtain

$$\|(w,z)\|_{W_1(I)\times W_2(I)} \leq c\varepsilon + c\,\|(w,z)\|^2_{W_1(I)\times W_2(I)},$$

which implies

$$\|(w,z)\|_{W_1(I)\times W_2(I)} \leq c\varepsilon \tag{3.10}$$

for $\varepsilon \leq \varepsilon_0$ if ε_0 is small. Combining (3.7), (3.8), (3.9), and (3.10), we have

$$\|(v\overline{u} - \widetilde{v}\overline{\widetilde{u}}, u^2 - \widetilde{u}^2)\|_{W_1(I)\times W_2(I)} \leq c\varepsilon.$$

□

Proposition 3.23 (Long time perturbation)**.** *Let $d = 3$ and $\kappa = \frac{1}{2}$. Let I be a time interval with $t_0 \in I$ and $M > 0$. Let $(\widetilde{u},\widetilde{v}) : I \times \mathbb{R}^3 \longrightarrow \mathbb{C}^2$ satisfy (3.6) for some functions (e_1, e_2) and $\|(\widetilde{u},\widetilde{v})\|_{W_1(I)\times W_2(I)} \leq M$. Let $(u_0, v_0) \in \dot{X}^{1/2}_{1/2}(t_0) \times \dot{X}^{1/2}_1(t_0)$ and let (u, v) be a corresponding solution to (NLS) with $(u(t_0), v(t_0)) = (u_0, v_0)$ given in Theorem 3.18. There exist $\varepsilon_1 = \varepsilon_1(M) > 0$ and $c = c(M) > 0$ such that for any $0 \leq \varepsilon < \varepsilon_1$, if*

$$\|(\widetilde{u}(t_0),\widetilde{v}(t_0)) - (u_0, v_0)\|_{\dot{X}^{1/2}_{1/2}(t_0)\times \dot{X}^{1/2}_1(t_0)} + \|(e_1, e_2)\|_{N_1(I)\times N_2(I)} \leq \varepsilon$$

then the maximal existence interval of (u, v) contains I and the solution satisfies

$$\|(u, v) - (\widetilde{u},\widetilde{v})\|_{(L^\infty_t(I;\dot{X}^{1/2}_{1/2})\cap W_1(I))\times(L^\infty_t(I;\dot{X}^{1/2}_1)\cap W_2(I))} \leq c\varepsilon.$$

Proof. By the time symmetry, we may assume $t_0 = \inf I$ without loss of generality. Take the constant ε_0 given in Proposition 3.22. Since $\|(\widetilde{u},\widetilde{v})\|_{W_1(I)\times W_2(I)} \leq M$, there exists $J \in \mathbb{N}$ such that $I = \bigcup_{j=1}^J I_j = \bigcup_{j=1}^J [t_{j-1}, t_j)$ with $\|(\widetilde{u},\widetilde{v})\|_{W_1(I_j)\times W_2(I_j)} \leq \varepsilon_0$. We set $(w, z) = (u, v) - (\widetilde{u},\widetilde{v})$. Put

$$\kappa_j = \|(v\overline{u} - \widetilde{v}\overline{\widetilde{u}}, u^2 - \widetilde{u}^2)\|_{N_1(I_j)\times N_2(I_j)}.$$

From Proposition 3.22, we see that there exists a constant $C_0 > 0$ such that if a constant $\eta_j > 0$ satisfies $\eta_j \leq \varepsilon_0$ and

$$\|(e^{i(t-t_{j-1})\Delta}w(t_{j-1}), e^{\frac{1}{2}i(t-t_{j-1})\Delta}z(t_{j-1}))\|_{W_1(I_j)\times W_2(I_j)} \leq \eta_j \tag{3.11}$$

holds, then

$$\|(w,z)\|_{W_1(I_j)\times W_2(I_j)} + \kappa_j \leq C_0\eta_j. \tag{3.12}$$

By the integral equation, we have

$$\begin{cases} e^{i(t-t_{j-1})\Delta}w(t_{j-1}) = e^{i(t-t_0)\Delta}w(t_0) + i\displaystyle\int_{t_0}^{t_{j-1}} e^{i(t-s)\Delta}(2v\overline{u} - 2\widetilde{v}\overline{\widetilde{u}} + e_1)(s)ds, \\ e^{\frac{1}{2}i(t-t_{j-1})\Delta}z(t_{j-1}) = e^{\frac{1}{2}i(t-t_0)\Delta}z(t_0) + i\displaystyle\int_{t_0}^{t_{j-1}} e^{\frac{1}{2}i(t-s)\Delta}(u^2 - \widetilde{u}^2 + e_2)(s)ds. \end{cases}$$

Using Proposition 3.5,

$$\begin{aligned} &\|e^{i(t-t_{j-1})\Delta}w(t_{j-1})\|_{W_1(I_j)} \\ &\qquad \leq c\,\|e^{-it_0\Delta}w(t_0)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + c\left\|\int_{t_0}^{t_{j-1}} e^{-is\Delta}(2v\overline{u} - 2\widetilde{v}\overline{\widetilde{v}} + e_1)(s)ds\right\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \end{aligned}$$

$$
\begin{aligned}
&= c\,\|w(t_0)\|_{\dot X^{\frac12}_{1/2}(t_0)} + c\left\|\int_{t_0}^{t_{j-1}} e^{i(t_{j-1}-s)\Delta}(2v\overline{u} - 2\widetilde{v}\overline{\widetilde{v}} + e_1)(s)ds\right\|_{\dot X^{\frac12}_{1/2}(t_{j-1})}\\
&\le c\,\|w(t_0)\|_{\dot X^{\frac12}_{1/2}(t_0)} + c\left\|\int_{t_0}^{t} e^{i(t-s)\Delta}(2v\overline{u} - 2\widetilde{v}\overline{\widetilde{v}} + e_1)(s)ds\right\|_{L^\infty(t_0,t_{j-1};\dot X^{\frac12}_{1/2})}\\
&\le c\,\|w(t_0)\|_{\dot X^{\frac12}_{1/2}(t_0)} + c\,\|v\overline{u} - \widetilde{v}\overline{\widetilde{u}}\|_{N_1(t_0,t_{j-1})} + c\,\|e_1\|_{N_1(t_0,t_{j-1})}
\end{aligned}
$$

and

$$
\|e^{\frac12 i(t-t_{j-1})\Delta}z(t_{j-1})\|_{W_2(I_j)} \le c\,\|z(t_0)\|_{\dot X^{\frac12}_1(t_0)} + c\,\|u^2 - \widetilde{u}^2\|_{N_2(t_0,t_{j-1})} + c\,\|e_2\|_{N_2(t_0,t_{j-1})},
$$

which deduce

$$
\begin{aligned}
&\left\|\left(e^{i(t-t_{j-1})\Delta}w(t_{j-1}), e^{\frac12 i(t-t_{j-1})\Delta}z(t_{j-1})\right)\right\|_{W_1(I_j)\times W_2(I_j)}\\
&\qquad\le c\,\|(w(t_0),z(t_0))\|_{\dot X^{\frac12}_{1/2}(t_0)\times \dot X^{\frac12}_1(t_0)} + c\,\|(e_1,e_2)\|_{N_1(t_0,t_{j-1})\times N_2(t_0,t_{j-1})}\\
&\qquad\qquad + c\,\|(v\overline{u} - \widetilde{v}\overline{\widetilde{u}}, u^2 - \widetilde{u}^2)\|_{N_1(t_0,t_{j-1})\times N_2(t_0,t_{j-1})}\\
&\qquad\le c\,\|(w(t_0),z(t_0))\|_{\dot X^{\frac12}_{1/2}(t_0)\times \dot X^{\frac12}_1(t_0)} + c\,\|(e_1,e_2)\|_{N_1(I)\times N_2(I)}\\
&\qquad\qquad + c\sum_{\ell=1}^{j-1}\|(v\overline{u} - \widetilde{v}\overline{\widetilde{u}}, u^2 - \widetilde{u}^2)\|_{N_1(I_\ell)\times N_2(I_\ell)}\\
&\qquad< C_1\varepsilon + C_1\sum_{\ell=1}^{j-1}\kappa_\ell
\end{aligned}
\tag{3.13}
$$

for some constant $C_1 > 1$. Let us take a constant $\alpha \ge \max\{2, 2C_0C_1\}$. For $\varepsilon' \le \varepsilon_0$, we set

$$
\eta_j = \eta_j(\varepsilon') := \alpha^{j-J-1}\varepsilon' \tag{3.14}
$$

for each $j \in [1, J+1]$. Then, we have

$$
\eta_1 < \eta_2 < \cdots < \eta_J < \eta_{J+1} = \varepsilon' \le \varepsilon_0.
$$

We also remark that η_j is increasing in ε'. We now show by induction that (3.12) holds for $j \in [1, J]$ as long as $\varepsilon' \le \varepsilon_0$ and $\varepsilon \le \frac{1}{C_1}\eta_1(\varepsilon')$. To do so, it suffices to show that (3.11) is satisfied for $j \in [1, J]$ under this condition. When $j = 1$, (3.11) is fulfilled by the assumption. Assume for induction that (3.11) is true for $1 \le j \le k$, where $k \in [1, J-1]$. Since (3.12) is also true for $j \in [1, k]$, we deduce from (3.13) that

$$
\left\|\left(e^{i(t-t_k)\Delta}w(t_k), e^{\frac12 i(t-t_k)\Delta}z(t_k)\right)\right\|_{W_1(I_{k+1})\times W_2(I_{k+1})} \le C_1\varepsilon + C_1\sum_{\ell=1}^{k}\kappa_\ell \tag{3.15}
$$

By the assumptions of ε, α, and (3.14), we have

$$
C_1\varepsilon \le \eta_1 = \alpha^{-k}\eta_{k+1}
$$

and

$$
C_1\kappa_\ell \le C_1C_0\eta_\ell \le \frac12\alpha\eta_\ell = \frac12\alpha^{\ell-k}\eta_{k+1}
$$

for $\ell \in [1, k]$. Combining these estimates, we have

$$
C_1\varepsilon + C_1\sum_{\ell=1}^{k}\kappa_\ell \le \alpha^{-k}\left(1 + \frac12\sum_{\ell=1}^{k}\alpha^\ell\right)\eta_{k+1} \le \eta_{k+1}, \tag{3.16}
$$

where we have used the assumption $\alpha \ge 2$ in the last inequality. Hence, (3.11) for $j = k+1$ follows from (3.15) and (3.16), and so we see that (3.12) holds for $1 \le j \le J$. Then, (3.16) is also true for $k \in [1, J]$. Set $\varepsilon_1 := \frac{1}{C_1}\eta_1(\varepsilon_0)$ and assume $\varepsilon \le \varepsilon_1$. We define ε' by the relation $\varepsilon = \frac{1}{C_1}\eta_1(\varepsilon')$. Notice that $\varepsilon' \le \varepsilon_0$ and $\varepsilon' = C_1\alpha^J\varepsilon$. By (3.12), $\alpha \ge 2C_1C_0$, and (3.14),

$$\begin{aligned}\|(w,z)\|_{W_1(I)\times W_2(I)} &\le \sum_{j=1}^{J}\|(w,z)\|_{W_1(I_j)\times W_2(I_j)} \le \sum_{j=1}^{J} C_0\eta_j = \frac{1}{C_1}\sum_{j=1}^{J} C_1C_0\eta_j \\ &\le \frac{1}{C_1}\sum_{j=1}^{J}\frac{1}{2}\alpha\cdot\alpha^{j-J-1}\varepsilon' = \frac{\varepsilon'}{2C_1}\cdot\frac{\alpha^{1-J}(\alpha^J-1)}{\alpha-1} \le \frac{1}{C_1}\varepsilon' = \alpha^J\varepsilon,\end{aligned}$$

where we have used the assumption $\alpha \ge 2$ in the last inequality. Further, we apply (3.16) for $k = J$. Then,

$$\begin{aligned}&\|(w,z)\|_{L_t^\infty(I;\dot X^{\frac12}_{\frac12})\times L_t^\infty(I;\dot X^{\frac12}_1)} \\ &\le c\,\|(w(t_0),z(t_0))\|_{\dot X^{\frac12}_{1/2}(t_0)\times\dot X^{\frac12}_1(t_0)} + c\,\|(v\overline{u}-\widetilde{v}\overline{\widetilde{u}}, u^2-\widetilde{u}^2)\|_{N_1(I)\times N_2(I)} + c\,\|(e_1,e_2)\|_{N_1(I)\times N_2(I)} \\ &\le C_1\varepsilon + C_1\sum_{j=1}^{J}\kappa_j \le \eta_{J+1} = \varepsilon' = C_1\alpha^J\varepsilon.\end{aligned}$$

□

3.6. **Properties of L_{v_0} and $\ell^\dagger_{v_0}$.** In this subsection, we investigate properties of L_{v_0} and $\ell^\dagger_{v_0}$.

Proposition 3.24. *Let $d = 3$ and $\kappa = \frac12$. For any $v_0 \in \mathcal{F}\dot H^{\frac12}(\mathbb{R}^3)$, there exist $\varepsilon_1 > 0$ and $\delta > 0$ such that*

$$L_{v_0}(\varepsilon) \le \|e^{\frac12 it\Delta}v_0\|_{W_2([0,\infty))} + \delta\varepsilon$$

holds for $0 \le \varepsilon < \varepsilon_1$. Here, the constants $\varepsilon_1 > 0$ and $\delta > 0$ depend only on $\|e^{\frac12 it\Delta}v_0\|_{W_2([0,\infty))}$. In particular, $\ell^\dagger_{v_0} > 0$ for any $v_0 \in \mathcal{F}\dot H^{\frac12}(\mathbb{R}^3)$.

Proof. Apply Proposition 3.23 with $(\widetilde{u},\widetilde{v}) = (0, e^{\frac12 it\Delta}v_0)$ and $(e_1,e_2) = (0,0)$, the desired result holds. □

Proposition 3.25 (Properties of L_{v_0}). *Let $d = 3$ and $\kappa = \frac12$. For each fixed $v_0 \in \mathcal{F}\dot H^{\frac12}(\mathbb{R}^3)$, the function L_{v_0} is a non-decreasing continuous function defined on $[0,\infty)$.*

Proof. It is clear that L_{v_0} is a non-decreasing function defined on $[0,\infty)$. We prove the continuity. It is obvious that

$$L_{v_0}(0) = \|e^{\frac12 it\Delta}v_0\|_{W_2([0,\infty))} < \infty.$$

The continuity of L_{v_0} at $\ell = 0$ holds by Proposition 3.24. Fix $\ell_0 \in (0,\infty)$ such that $L_{v_0}(\ell_0) < \infty$. Let us prove right continuity of $L_{v_0}(\ell)$ at $\ell = \ell_0$. Pick $\varepsilon > 0$. Take $\delta > 0$ so that $\delta < \varepsilon_1$ and $c\delta < \varepsilon$, where $\varepsilon_1 = \varepsilon_1(L_{v_0}(\ell_0))$ and $c = c(L_{v_0}(\ell_0))$ are the constants given in Proposition 3.23 with the choice $M = L_{v_0}(\ell_0)$. Fix $\ell \in (\ell_0, \ell_0+\delta)$. Then, for any $u_{0,1} \in \mathcal{F}\dot H^{\frac12}(\mathbb{R}^3)$ satisfying $\|u_{0,1}\|_{\mathcal{F}\dot H^{\frac12}}(\mathbb{R}^3) \le \ell$, the function

$$u_{0,2} = \frac{\ell_0}{\ell_0+\delta}u_{0,1}$$

satisfies $\|u_{0,2}\|_{\mathcal{F}\dot H^{\frac12}} \le \ell_0$ and $\|u_{0,1}-u_{0,2}\|_{\mathcal{F}\dot H^{\frac12}} \le \delta$. Let (u_1,v_1) and (u_2,v_2) be two solutions to (NLS) with initial data $(u_{0,1},v_0)$ and $(u_{0,2},v_0)$, respectively. Note that

$$\|(u_2,v_2)\|_{W_1([0,\infty))\times W_2([0,\infty))} \le L_{v_0}(\ell_0)$$

since $\|u_{0,2}\|_{\mathcal{F}\dot H^{\frac12}} \le \ell_0$. Hence, we have

$$\|(u_1,v_1)-(u_2,v_2)\|_{W_1([0,\infty))\times W_2([0,\infty))} \le c\delta < \varepsilon$$

by Proposition 3.23. Thus, it follows that

$$\|(u_1, v_1)\|_{W_1([0,\infty))\times W_2([0,\infty))} < \|(u_2, v_2)\|_{W_1([0,\infty))\times W_2([0,\infty))} + \varepsilon \le L_{v_0}(\ell_0) + \varepsilon.$$

Taking the supremum over such $u_{0,1} \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, we obtain

$$L_{v_0}(\ell) \le L_{v_0}(\ell_0) + \varepsilon$$

for $\ell \in (\ell_0, \ell_0 + \delta)$. This shows the right continuity of L_{v_0} at $\ell = \ell_0$ together with non-decreasing property. The left continuity is a consequence of the continuous dependence on initial data in Theorem 3.18. We omit the details.

Let us move on to the case $L_{v_0}(\ell_0) = \infty$. We may suppose that $\ell_0 := \inf\{\ell : L_{v_0}(\ell) = \infty\}$ otherwise continuity is trivial by definition. Under this assumption, we prove that $L_{v_0}(\ell)$ goes to infinity as $\ell \uparrow \ell_0$. Assume that

$$C_0 := \sup_{\ell < \ell_0} L_{v_0}(\ell) < \infty$$

for contradiction. Let $\varepsilon_1 = \varepsilon_1(C_0)$ be the constant given in Proposition 3.23. Fix $0 < \varepsilon < 1$ so that $\varepsilon\ell_0 < \varepsilon_1$. Then, for any fixed $u_{0,1} \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ with $\|u_{0,1}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \ell_0$, the function

$$u_{0,2} := (1 - \varepsilon)u_{0,1}$$

satisfies $\|u_{0,2}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le (1 - \varepsilon)\ell_0$. Let (u_1, v_1) and (u_2, v_2) be two solutions to (NLS) with initial data $(u_{0,1}, v_0)$ and $(u_{0,2}, v_0)$, respectively. One sees that

$$\|(u_2, v_2)\|_{W_1([0,\infty))\times W_2([0,\infty))} \le L_{v_0}\left((1 - \varepsilon)\ell_0\right) \le C_0 < \infty.$$

In addition, we have

$$\|u_{0,1} - u_{0,2}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \varepsilon\|u_{0,1}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \varepsilon\ell_0.$$

Applying Proposition 3.23, we obtain

$$\begin{aligned}\|(u_1, v_1)\|_{W_1([0,\infty))\times W_2([0,\infty))} &\le \|(u_2, v_2)\|_{W_1([0,\infty))\times W_2([0,\infty))} + c\varepsilon\ell_0 \\ &\le L_{v_0}\left((1 - \varepsilon)\ell_0\right) + c\varepsilon\ell_0 < \infty,\end{aligned}$$

where $c = c(C_0)$ is a constant. Taking supremum over $u_{0,1}$, it follows that

$$L_{v_0}(\ell_0) \le L_{v_0}\left((1 - \varepsilon)\ell_0\right) + c\varepsilon\ell_0 < \infty.$$

This is a contradiction. □

By using the non-decreasing property of L_{v_0}, we have the following:

Proposition 3.26 (Another characterization of $\ell^{\dagger}_{v_0}$)**.** *Let $d = 3$ and $\kappa = \frac{1}{2}$. The following identity holds:*

$$\ell^{\dagger}_{v_0} = \inf\{\ell : L_{v_0}(\ell) = \infty\}$$

for any $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.

Proof. When $L_{v_0}(\ell)$ is finite for any $\ell > 0$, we see that the both sides are infinite. Otherwise, the two sets $\{\ell : L_{v_0}(\ell) < \infty\}$ and $\{\ell : L_{v_0}(\ell) = \infty\}$ give us a Dedekind cut of a totally ordered set $[0, \infty)$, by means of Propositions 3.24 and Proposition 3.25. □

A consequence of the alternative characterization is that

$$L_{v_0}(\ell^{\dagger}_{v_0}) = \infty$$

holds for any $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. This follows from the continuity of L_{v_0}. We also have the following:

Lemma 3.27. *Let $d = 3$ and $\kappa = \frac{1}{2}$. $\ell_{v_0} \ge \ell^{\dagger}_{v_0}$ for any $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.*

Proof. If $\ell_{v_0} = \infty$, then Lemma 3.27 holds. Let $\ell_{v_0} < \infty$. By the definition of ℓ_{v_0}, for any $\varepsilon > 0$, there exists $u_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ such that $\|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} < \ell_{v_0} + \varepsilon$ holds and the corresponding solution (u, v) to (NLS) with (IC) does not scatter. Since Proposition 3.19 deduces $\|(u, v)\|_{W_1([0,T_{\max}))\times W_2([0,T_{\max}))} = \infty$ from the failure of scattering, we obtain $L_{v_0}(\ell_{v_0} + \varepsilon) = \infty$. This implies the relation $\ell^\dagger_{v_0} \le \ell_{v_0} + \varepsilon$, thanks to Proposition 3.26. Since $\varepsilon > 0$ is arbitrary, we have the desired conclusion. □

The following is one of the key property to prove Theorem 1.39.

Proposition 3.28. *Let $d = 3$ and $\kappa = \frac{1}{2}$. $\ell^\dagger_0 \ge \ell^\dagger_{v_0}$ holds for any $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$.*

Proof. Fix $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. We assume that $\ell^\dagger_0 < \ell^\dagger_{v_0}$ for contradiction. Then, we have

$$L_0\Big(\frac{\ell^\dagger_0 + \ell^\dagger_{v_0}}{2}\Big) = \infty \quad \text{and} \quad L_{v_0}\Big(\frac{\ell^\dagger_0 + \ell^\dagger_{v_0}}{2}\Big) < \infty.$$

Using the fact that $L_0(\frac{\ell^\dagger_0+\ell^\dagger_{v_0}}{2}) = \infty$ and the scaling argument, one can take data $\{(U_{0,n}, 0)\}$ so that the corresponding solution (U_n, V_n) to (NLS) satisfies

$$\|U_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \frac{\ell^\dagger_0 + \ell^\dagger_{v_0}}{2} \tag{3.17}$$

and

$$\|(U_n, V_n)\|_{W_1([0,n^{-1}])\times W_2([0,n^{-1}])} \ge n \tag{3.18}$$

for all $n \ge 1$. Let (u_n, v_n) be another solution to (NLS) with the initial data $(U_{0,n}, v_0)$. Since $L_{v_0}(\frac{\ell^\dagger_0+\ell^\dagger_{v_0}}{2}) < \infty$, one sees from (3.17) that (u_n, v_n) is global in time and

$$\|(u_n, v_n)\|_{W_1([0,\infty))\times W_2([0,\infty))} \le L_{v_0}\Big(\frac{\ell^\dagger_0 + \ell^\dagger_{v_0}}{2}\Big) < \infty.$$

We now set $(\widetilde{u}_n, \widetilde{v}_n) = (u_n, v_n) - (0, e^{\frac{1}{2}it\Delta}v_0)$. Then, $(\widetilde{u}_n, \widetilde{v}_n)$ solves

$$\begin{cases} i\partial_t \widetilde{u}_n + \Delta \widetilde{u}_n + 2\widetilde{v}_n \overline{\widetilde{u}_n} = -2(e^{\frac{1}{2}it\Delta}v_0)\overline{u_n}, \\ i\partial_t \widetilde{v}_n + \frac{1}{2}\Delta \widetilde{v}_n + \widetilde{u}_n^2 = 0, \\ (\widetilde{u}_n(0), \widetilde{v}_n(0)) = (U_{0,n}, 0) \end{cases}$$

and so it is an approximate solution to (NLS) with an error

$$e_1 = -2(e^{\frac{1}{2}it\Delta}v_0)\overline{u_n}, \quad e_2 = 0.$$

Take $\tau > 0$ and set $I = [0, \tau]$. We have

$$\|(e_1, e_2)\|_{N_1(I)\times N_2(I)} \lesssim \|e^{\frac{1}{2}it\Delta}v_0\|_{W_2(I)}\|u_n\|_{W_1(I)} \le \|e^{\frac{1}{2}it\Delta}v_0\|_{W_2(I)} L_{v_0}\Big(\frac{\ell^\dagger_0 + \ell^\dagger_{v_0}}{2}\Big).$$

The right hand side is independent of n, and tends to zero as $\tau \downarrow 0$. Now, we apply the Proposition 3.23 with $M = L_{v_0}(\frac{\ell^\dagger_0+\ell^\dagger_{v_0}}{2}) + \|e^{\frac{1}{2}it\Delta}v_0\|_{W_2([0,\infty))}$. Choose τ sufficiently small so that the above upper bound of the error becomes smaller than the corresponding ε_1. Since (U_n, V_n) is a solution with the same initial data as $(\widetilde{u}_n, \widetilde{v}_n)$, we see from Proposition 3.23 that (U_n, V_n) extends up to time τ and obeys the bound

$$\begin{aligned} \|(U_n, V_n)\|_{W_1(I)\times W_2(I)} &\le \|(\widetilde{u}_n, \widetilde{v}_n)\|_{W_1(I)\times W_2(I)} + C\varepsilon_1 \\ &\le L_{v_0}\Big(\frac{\ell^\dagger_0 + \ell^\dagger_{v_0}}{2}\Big) + \|e^{\frac{1}{2}it\Delta}v_0\|_{W_2([0,\infty))} + C\varepsilon_1. \end{aligned}$$

However, this contradicts (3.18) for large n. □

3.7. **Linear profile decomposition.** In this subsection, we obtain a linear profile decomposition (Theorem 3.38). Let us first introduce several operators and give a notion of deformation, which is a specific class of bounded operator.

Definition 3.29 (Operators)**.** We define the following operators.

(1) (Dilation)
$$(D(h)(f,g))(x) = (f_{\{h\}}, g_{\{h\}}) = (h^2 f(hx), h^2 g(hx)) \quad \text{for} \quad h \in 2^{\mathbb{Z}},$$
(2) (Translation in Fourier space)
$$(T(\xi)(f,g))(x) = (e^{ix\cdot\xi} f(x), e^{2ix\cdot\xi} g(x)) \quad \text{for} \quad \xi \in \mathbb{R}^3.$$

Definition 3.30. We say that a bounded operator
$$\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) = T(\xi)D(h) \quad \text{for} \quad (\xi, h) \in \mathbb{R}^3 \times 2^{\mathbb{Z}}$$
on $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is called a deformation in $\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}$. Let a set $G \subset \mathcal{L}(\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}})$ be composed of all deformations, where $\mathcal{L}(\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}})$ denotes a whole of bounded linear operator on $\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}$.

Remark 3.31. G is a group with the functional composition as a binary operation. The identity element Id is $T(0)D(1) \in G$. For any $\mathcal{G} = T(\xi)D(h)$, the inverse element is $\mathcal{G}^{-1} = T(-\frac{\xi}{h})D(\frac{1}{h}) \in G$. We check that G forms a group.

Next, we introduce a class of families of deformations.

Definition 3.32 (A vanishing family)**.** Let $d = 3$. We say that a family of deformations $\{\mathcal{G}_n = T(\xi_n)D(h_n)\}_n \subset G$ is vanishing if $|\xi_n| + |\log h_n| \longrightarrow \infty$ as $n \to \infty$ holds.

Lemma 3.33. *Let $d = 3$. A family $\{\mathcal{G}_n\}_n \subset G$ is vanishing if and only if a family of inverse elements $\{\mathcal{G}_n^{-1}\}_n$ is vanishing.*

Proof. We set $\mathcal{G}_n = T(\xi_n)D(h_n)$. Let $\mathcal{G}_n$ be vanishing. If $|\log h_n| \longrightarrow \infty$ as $n \to \infty$, then
$$\left|\frac{\xi_n}{h_n}\right| + \left|\log \frac{1}{h_n}\right| \geq |\log h_n| \longrightarrow \infty$$
as $n \to \infty$. Let $|\xi_n| \longrightarrow \infty$ as $n \to \infty$. If
$$\left|\frac{\xi_n}{h_n}\right| + \left|\log \frac{1}{h_n}\right| \not\longrightarrow \infty,$$
then there exists $M > 0$ such that, for any $k \in \mathbb{N}$, there exists $n_k \geq k$ such that
$$\left|\frac{\xi_{n_k}}{h_{n_k}}\right| + \left|\log \frac{1}{h_{n_k}}\right| \leq M. \tag{3.19}$$
This inequality implies
$$|\xi_{n_k}| + |\log h_{n_k}| \leq M(|h_{n_k}| + 1).$$
Combining this inequality and $|\xi_n| + |\log h_n| \longrightarrow \infty$ as $n \to \infty$, we have $|h_{n_k}| \longrightarrow \infty$ as $k \to \infty$. However, this contradicts (3.19). Therefore, $\mathcal{G}_n^{-1}$ is vanishing. The other direction follows from the same argument by the relation $(\mathcal{G}_n^{-1})^{-1} = \mathcal{G}_n$. ∎

The following characterization of the vanishing family is useful.

Proposition 3.34. *Let $d = 3$. For a family $\{\mathcal{G}_n\}_n \subset G$ of deformations, the following three statements are equivalent.*

(1) *$\{\mathcal{G}_n\}_n$ is vanishing.*
(2) *For any $(\phi, \psi) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, $\mathcal{G}_n(\phi, \psi) \rightharpoonup (0,0)$ in $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ as $n \to \infty$.*

(3) *For any subsequence* $\{\mathcal{G}_{n_k}\}_k$*, there exist a subsequence* $\{\mathcal{G}_{n_{k_l}}\}_l$ *and a bounded sequence* $\{(f_l, g_l)\}_l \subset \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ *such that* $(f_l, g_l) \rightharpoonup (0,0)$ *and* $\mathcal{G}_{n_{k_l}}^{-1}(f_l, g_l) \rightharpoonup (\phi, \psi) \neq (0,0)$ *in* $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ *as* $l \to \infty$*.*

Proof. We mimic the argument in [90, 92]. (2) $\Longrightarrow$ (3) holds by taking $k_l = k$ and $(f_k, g_k) = \mathcal{G}_{n_k}(\phi, \psi)$ for some $(\phi, \psi) \neq (0,0)$.

Next, we prove the contraposition of (3) $\Longrightarrow$ (1). If $\mathcal{G}_n$ is not vanishing, then the corresponding sequence of parameters is bounded. Hence, there exists a subsequence $\{\mathcal{G}_{n_k}\}_k$ and $\mathcal{G} \in G$ such that $\mathcal{G}_{n_k} \longrightarrow \mathcal{G}$ in $\mathcal{L}(\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}})$ as $k \to \infty$. Then, for any subsequence $\{\mathcal{G}_{n_{k_l}}\}_l$ and for any bounded sequence $\{(f_l, g_l)\}_l$ such that $(f_l, g_l) \rightharpoonup (0,0)$ in $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ as $l \to \infty$, one has

$$\begin{aligned}
\left|\langle \mathcal{G}_{n_{k_l},1}^{-1} f_l, \phi \rangle_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\right| &= \left|\langle f_l, \mathcal{G}_{n_{k_l},1}\phi \rangle_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\right| \\
&\leq \left|\langle f_l, \mathcal{G}_1\phi \rangle_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\right| + \left|\langle f_l, \mathcal{G}_{n_{k_l},1}\phi - \mathcal{G}_1\phi \rangle_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\right| \\
&\leq \left|\langle f_l, \mathcal{G}_1\phi \rangle_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\right| + \|f_l\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \|\mathcal{G}_{n_{k_l},1}\phi - \mathcal{G}_1\phi\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \\
&\longrightarrow 0 \quad \text{as} \quad l \to \infty
\end{aligned}$$

for any $\phi \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Similarly, $\left|\langle \mathcal{G}_{n_k,2} g_{k_l}, \psi \rangle_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\right| \longrightarrow 0$ as $k \to \infty$ for any $\psi \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, that is, $(\mathcal{G}_{n_k})^{-1}(f_{k_l}, g_{k_l}) \rightharpoonup (0,0)$ in $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ as $k \to \infty$. Hence, (3) fails.

Finally, we prove (1) $\Longrightarrow$ (2). We take any $(f,g) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and any $(\phi, \psi) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. By the density argument, we assume $(f,g) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$ and $(\phi, \psi) \in C_c^\infty(\mathbb{R}^3) \times C_c^\infty(\mathbb{R}^3)$. If $|\log h_n| \longrightarrow \infty$, then for any subsequence $\{h_{n_k}\}$, there exists a subsequence $\{h_{n_{k_l}}\}$ such that $h_{n_{k_l}} \longrightarrow 0$ as $l \to \infty$ or $h_{n_{k_l}} \longrightarrow \infty$ as $l \to \infty$. The inequality

$$\left|\langle \mathcal{G}_{n_{k_l}}(f,g), (\phi, \psi) \rangle_{\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}}\right| \leq \|\mathcal{G}_{n_{k_l}}(f,g)\|_{L_x^r \times L_x^r} \|(|\cdot|\phi, |\cdot|\psi)\|_{L_x^{r'} \times L_x^{r'}} \sim (h_{n_{k_l}})^{2 - \frac{3}{r}}.$$

deduces the desired result by taking $r > \frac{3}{2}$ if $h_{n_{k_l}} \longrightarrow 0$ and $r < \frac{3}{2}$ if $h_{n_{k_l}} \longrightarrow \infty$. Next, we consider the case: $|\log h_n|$ is bounded and $|\xi_n| \longrightarrow \infty$ as $n \to \infty$. If we take a subsequence $\{h_{n_{k_l}}\} \subset \{h_{n_k}\}$ for any subsequence $\{h_{n_k}\} \subset \{h_n\}$ satisfying $h_{n_{k_l}} \longrightarrow h_0 \in 2^{\mathbb{Z}}$ as $l \to \infty$, then

$$\begin{aligned}
\left|\langle \mathcal{G}_{n_{k_l},1} f, \phi \rangle_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\right| &\leq \left\|h_{n_{k_l}}^2 f(h_{n_{k_l}}\cdot) - h_0^2 f(h_0\cdot)\right\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \|\phi\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + \left|\langle e^{ix\cdot\xi_{n_{k_l}}} h_0^2 f(h_0\cdot), \phi \rangle_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\right| \\
&\longrightarrow 0 \quad \text{as} \quad l \to \infty,
\end{aligned}$$

where the second term converge to 0 as $l \to \infty$ by Lemma 3.2. $\square$

We now introduce a notion of orthogonality.

Definition 3.35 (Orthogonality)**.** We say two families of deformations $\{\mathcal{G}_n\}$, $\{\widetilde{\mathcal{G}}_n\} \subset G$ are orthogonal if $\{\mathcal{G}_n^{-1}\widetilde{\mathcal{G}}_n\}$ is vanishing.

Remark 3.36. Let $\{\mathcal{G}_n^j = T(\xi_n^j)D(h_n^j)\} \subset G$ $(j = 1,2)$ be two families of deformations. $\{\mathcal{G}_n^1\}$ and $\{\mathcal{G}_n^2\}$ are orthogonal if and only if

$$\frac{h_n^1}{h_n^2} + \frac{h_n^2}{h_n^1} + \frac{|\xi_n^1 - \xi_n^2|}{h_n^1} \longrightarrow \infty$$

as $n \to \infty$. This equivalence holds from the identity $(\mathcal{G}_n^1)^{-1}\mathcal{G}_n^2 = T(\frac{\xi_n^2 - \xi_n^1}{h_n^1})D(\frac{h_n^2}{h_n^1})$.

Proposition 3.37. *Let* $d = 3$ *and* $\{\mathcal{G}_n\}, \{\widetilde{\mathcal{G}}_n\} \subset G$*. Define the relation* $\sim$ *as follows: If* $\{\mathcal{G}_n\}$ *and* $\{\widetilde{\mathcal{G}}_n\}$ *are not orthogonal then* $\{\mathcal{G}_n\} \sim \{\widetilde{\mathcal{G}}_n\}$*. Then,* $\sim$ *is an equivalent relation.*

Proof. The reflexivity of $\sim$ follows from a sequence of the identity $\{\mathcal{G}_n = Id\}$ is not vanishing. The symmetry of $\sim$ follows from Lemma 3.33. The transitivity of $\sim$ holds by Proposition 3.34. If $\{\mathcal{G}_n^1\} \sim \{\mathcal{G}_n^2\}$ and $\{\mathcal{G}_n^2\} \sim \{\mathcal{G}_n^3\}$, then there exists a subsequence n_k such that

$$(\mathcal{G}_{n_k}^1)^{-1}\mathcal{G}_{n_k}^2 \longrightarrow \mathcal{G} \in G, \quad (\mathcal{G}_{n_k}^2)^{-1}\mathcal{G}_{n_k}^3 \longrightarrow \widetilde{\mathcal{G}} \in G$$

in $\mathcal{L}(\mathcal{F}\dot{H}^{\frac12}\times\mathcal{F}\dot{H}^{\frac12})$ as $k\to\infty$. Then, we have

$$(\mathcal{G}^1_{n_k})^{-1}\mathcal{G}^3_{n_k}=[(\mathcal{G}^1_{n_k})^{-1}\mathcal{G}^2_{n_k}][(\mathcal{G}^2_{n_k})^{-1}\mathcal{G}^3_{n_k}]\longrightarrow \mathcal{G}\widetilde{\mathcal{G}}\in G,$$

in $\mathcal{L}(\mathcal{F}\dot{H}^{\frac12}\times\mathcal{F}\dot{H}^{\frac12})$ as $k\to\infty$. This implies that the sequence $\{((\mathcal{G}^1_n)^{-1}\mathcal{G}^3_n)^{-1}\}=\{(\mathcal{G}^3_n)^{-1}\mathcal{G}^1_n\}$ does not satisfy the third assertion of Proposition 3.34, that is, it follows that

$$(f_{k_l},g_{k_l})\not\rightharpoonup(0,0)\quad\text{or}\quad(\mathcal{G}^3_{n_k})^{-1}\mathcal{G}^1_{n_k}(f_{k_l},g_{k_l})\not\rightharpoonup(\phi,\psi)\neq(0,0)$$

for any $\{(f_{n_k},g_{n_k})\}_k$ and $\{n_{k_l}\}\subset\{n_k\}$. Indeed, if $(f_{k_l},g_{k_l})\rightharpoonup(0,0)$, then

$$\begin{aligned}\left|\langle(\mathcal{G}^3_{n_k,1})^{-1}\mathcal{G}^1_{n_k,1}f_{k_l},\phi\rangle_{\mathcal{F}\dot{H}^{\frac12}}\right|&\le\left|\langle f_{k_l},\mathcal{G}_1\widetilde{\mathcal{G}}_1\phi\rangle_{\mathcal{F}\dot{H}^{\frac12}}\right|+\|f_{k_l}\|_{\mathcal{F}\dot{H}^{\frac12}}\left\|(\mathcal{G}^1_{n_k,1})^{-1}\mathcal{G}^3_{n_k,1}\phi-\mathcal{G}_1\widetilde{\mathcal{G}}_1\phi\right\|_{\mathcal{F}\dot{H}^{\frac12}}\\&\longrightarrow0\quad\text{as}\quad k\to\infty\end{aligned}$$

for any $\phi\in\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$ and

$$\left|\langle(\mathcal{G}^3_{n_k,2})^{-1}\mathcal{G}^1_{n_k,2}g_{k_l},\psi\rangle_{\mathcal{F}\dot{H}^{\frac12}}\right|\longrightarrow0\quad\text{as}\quad k\to\infty$$

for any $\psi\in\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$. Thus, we have $(\mathcal{G}^3_{n_k})^{-1}\mathcal{G}^1_{n_k}(f_{k_l},g_{k_l})\rightharpoonup(0,0)$ in $\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$ as $k\to\infty$. Therefore, we obtain $\{\mathcal{G}^1_n\}\sim\{\mathcal{G}^3_n\}$. □

Let us now state the linear profile decomposition result.

Theorem 3.38 (Linear profile decomposition)**.** *Let $d=3$ and $\{(f_n,g_n)\}\subset\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$ be a bounded sequence. Passing to a sequence if necessary, there exist profile $\{(f^j,g^j)\}\subset\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$, $\{(R^J_n,L^J_n)\}\subset\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$, and pairwise orthogonal families of deformations $\{\mathcal{G}^j_n=T(\xi^j_n)D(h^j_n)\}_n\subset G$ $(j=1,2,\ldots)$ such that for each $J\ge1$,*

$$(f_n,g_n)=\sum_{j=1}^J\mathcal{G}^j_n(f^j,g^j)+(R^J_n,L^J_n)$$

for any $n\ge1$. Moreover, $\{(R^J_n,L^J_n)\}$ satisfies

$$(\mathcal{G}^j_n)^{-1}(R^J_n,L^J_n)\rightharpoonup\begin{cases}(f^j,g^j)&(J<j),\\(0,0)&(J\ge j)\end{cases}$$

in $\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$ as $n\to\infty$ for any $j\ge0$, where we use the convention $(R^0_n,L^0_n)=(f_n,g_n)$, and

$$\limsup_{n\to\infty}\|(e^{it\Delta}R^J_n,e^{\frac12it\Delta}L^J_n)\|_{L^{q,\infty}_tL^r_x\times L^{q,\infty}_tL^r_x}\longrightarrow0\tag{3.20}$$

as $J\to\infty$ for any $1<q,r<\infty$ satisfying $\frac1q\in(\frac12,1)$ and $\frac2q+\frac3r=2$. Furthermore, we have Pythagorean decomposition:

$$\begin{aligned}\|f_n\|^2_{\mathcal{F}\dot{H}^{\frac12}}&=\sum_{j=1}^J\|f^j\|^2_{\mathcal{F}\dot{H}^{\frac12}}+\|R^J_n\|^2_{\mathcal{F}\dot{H}^{\frac12}}+o_n(1),\\\|g_n\|^2_{\mathcal{F}\dot{H}^{\frac12}}&=\sum_{j=1}^J\|g^j\|^2_{\mathcal{F}\dot{H}^{\frac12}}+\|L^J_n\|^2_{\mathcal{F}\dot{H}^{\frac12}}+o_n(1),\end{aligned}\tag{3.21}$$

where $o_n(1)$ goes to 0 as $n\to\infty$.

Proof. We define

$$\nu(\{(f_n,g_n)\}):=\left\{(f,g)\in\mathcal{F}\dot{H}^{\frac12}\times\mathcal{F}\dot{H}^{\frac12}\;\middle|\;\begin{array}{l}\text{There exist }\xi_n\in\mathbb{R}^3\text{ and }h_n\in2^{\mathbb{Z}}\text{ such that}\\(\mathcal{G}^j_n)^{-1}(f_n,g_n)\rightharpoonup(f,g)\text{ in }\mathcal{F}\dot{H}^{\frac12}\times\mathcal{F}\dot{H}^{\frac12}\\\text{as }n\to\infty,\text{ up to subsequence.}\end{array}\right\}.$$

and

$$\eta(\{(f_n,g_n)\}) := \sup_{(f,g)\in\nu(\{(f_n,g_n)\})} \|(f,g)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}\times\mathcal{F}\dot{H}^{\frac{1}{2}}}.$$

It is obvious by definition that

$$\eta(\{(f_n,g_n)\}) \le \limsup_{n\to\infty} \|(f_n,g_n)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}\times\mathcal{F}\dot{H}^{\frac{1}{2}}}.$$

Let $J=1$.
If $\eta(\{(f_n,g_n)\})=0$, then this theorem holds by taking $(f^j,g^j)=(0,0)$ for any $1\le j\le J$. Hence, we assume $\eta(\{(f_n,g_n)\})>0$. Then, we can take $\{\xi_n^1\}\subset\mathbb{R}^3$, $\{h_n^1\}\subset 2^{\mathbb{Z}}$, and $(f^1,g^1)\in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ satisfying

$$\frac{1}{2}\eta(\{(f_n,g_n)\}) \le \|(f^1,g^1)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}\times\mathcal{F}\dot{H}^{\frac{1}{2}}}$$

and

$$(\mathcal{G}_n^1)^{-1}(f_n,g_n) \rightharpoonup (f^1,g^1)$$

in $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Here, we define

$$(R_n^1,L_n^1) := (f_n,g_n) - \mathcal{G}_n^1(f^1,g^1). \tag{3.22}$$

We remark that

$$(f_n,g_n) = \mathcal{G}_n^1(f^1,g^1) + (R_n^1,L_n^1)$$

and

$$(\mathcal{G}_n^1)^{-1}(R_n^1,L_n^1) \rightharpoonup (f^1,g^1)-(f^1,g^1)=(0,0) \tag{3.23}$$

in $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ holds. By the decomposition (3.22) and (3.23), we have

$$\|f_n\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \|f^1\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + \|R_n^1\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + 2\mathrm{Re}\langle f^1,(\mathcal{G}_{1,n}^1)^{-1}R_n^1\rangle_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \longrightarrow \|f^1\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + \|R_n^1\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$$

as $n\to\infty$. Therefore, we obtain (3.21) with $J=1$.
Let $J=2$.
If $\eta(\{(R_n^1,L_n^1)\})=0$, then this theorem holds by taking $(f^j,g^j)=(0,0)$ for any $2\le j\le J$. Hence, we assume $\eta(\{(R_n^1,L_n^1)\})>0$. Then, we can take $\{\xi_n^2\}\subset\mathbb{R}^3$, $\{h_n^2\}\subset 2^{\mathbb{Z}}$, and $(f^2,g^2)\in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ satisfying

$$\frac{1}{2}\eta(\{(R_n^1,L_n^1)\}) \le \|(f^2,g^2)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}\times\mathcal{F}\dot{H}^{\frac{1}{2}}}$$

and

$$(\mathcal{G}_n^2)^{-1}(R_n^1,L_n^1) \rightharpoonup (f^2,g^2)$$

in $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. Here, we define

$$(R_n^2,L_n^2) := (R_n^1,L_n^1) - \mathcal{G}_n^2(f^2,g^2).$$

We remark that

$$(f_n,g_n) = \sum_{j=1}^{2} \mathcal{G}_n^j(f^j,g^j) + (\Phi_n^2,\Psi_n^2)$$

and

$$(\mathcal{G}_n^2)^{-1}(R_n^2,L_n^2) \rightharpoonup (f^2,g^2)-(f^2,g^2)=(0,0)$$

in $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ holds. By the same argument with $J=1$, we obtain

$$\|R_n^1\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \|f^2\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + \|R_n^2\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + o_n(1) \tag{3.24}$$

and

$$\|L_n^1\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \|g^2\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + \|L_n^2\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + o_n(1). \tag{3.25}$$

Combining (3.21) with $J = 1$, (3.24), and (3.25), we have (3.21) with $J = 2$. Here, we prove that $\{\mathcal{G}_n^1\}_n, \{\mathcal{G}_n^2\}_n \subset G$ are orthogonal by using Lemma 3.34. For any subsequence $(\mathcal{G}_{n_k}^1)^{-1}\mathcal{G}_{n_k}^2$ of $(\mathcal{G}_n^1)^{-1}\mathcal{G}_n^2$, we have $(\mathcal{G}_{n_k}^1)^{-1}(R_{n_k}^1, L_{n_k}^1) \rightharpoonup (0,0)$ and

$$\left((\mathcal{G}_{n_k}^1)^{-1}\mathcal{G}_{n_k}^2\right)^{-1}(\mathcal{G}_{n_k}^1)^{-1}(R_{n_k}^1, L_{n_k}^1) = (\mathcal{G}_{n_k}^2)^{-1}(R_{n_k}^1, L_{n_k}^1) \rightharpoonup (f^2, g^2) \neq (0,0).$$

Therefore, $\{\mathcal{G}_n^1\}_n, \{\mathcal{G}_n^2\}_n \subset G$ are orthogonal.
Let $J \geq 3$.
We can construct $\{\xi_n^j\} \subset \mathbb{R}^3$, $\{h_n^j\} \subset \mathbb{R}$, (f^j, g^j), $(R_n^j, L_n^j) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ for any $1 \leq j \leq J$ inductively. When there exists $1 \leq j \leq J$ such that $\eta(\{(R_n^j, L_n^j)\}) = 0$, then this theorem holds. Thus, we assume that $\eta(\{(R_n^j, L_n^j)\}) > 0$ for any $1 \leq j \leq J$. We remark that $\{\xi_n^j\} \subset \mathbb{R}^3$, $\{h_n^j\} \subset \mathbb{R}$, (f^j, g^j), $(R_n^j, L_n^j) \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ satisfy

$$\frac{1}{2}\eta(\{(R_n^{j-1}, R_n^{j-1})\}) \leq \|(f^j, g^j)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}}, \tag{3.26}$$

$$(\mathcal{G}_n^j)^{-1}(R_n^{j-1}, L_n^{j-1}) \rightharpoonup (f^j, g^j), \tag{3.27}$$

$$(R_n^j, L_n^j) = (R_n^{j-1}, L_n^{j-1}) - (\mathcal{G}_n^j)(f^j, g^j), \tag{3.28}$$

and

$$(\mathcal{G}_n^j)^{-1}(R_n^j, L_n^j) \rightharpoonup (0,0) \tag{3.29}$$

for any $1 \leq j \leq J$. We also remark

$$(f_n, g_n) = \sum_{j=1}^{J} \mathcal{G}_n^j(f^j, g^j) + (R_n^J, L_n^J). \tag{3.30}$$

We prove (3.21) by induction. We assume

$$\|f_n\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \sum_{j=1}^{J-1} \|f^j\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + \|R_n^{J-1}\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + o_n(1). \tag{3.31}$$

Combining (3.28) and (3.29), we have

$$\|R_n^{J-1}\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \|f^J\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + \|R_n^J\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + o_n(1). \tag{3.32}$$

From (3.31) and (3.32), we obtain (3.21). By the same argument as $J = 2$, $\{\mathcal{G}_n^j\}_n, \{\mathcal{G}_n^{j+1}\}_n \subset G$ are orthogonal for any $1 \leq j \leq J-1$. For any $i, k \in \mathbb{N}$ with $1 \leq i < k \leq J$, we prove that $\{\mathcal{G}_n^i\}_n, \{\mathcal{G}_n^k\}_n \subset G$ are orthogonal by induction, that is, if $\{\mathcal{G}_n^i\}_n, \{\mathcal{G}_n^k\}_n \subset G$ are orthogonal for any $1 \leq k-i \leq K$ $(1 \leq K \leq J-2)$, then the same result satisfies for $k - i = K+1$. Subtracting (3.30) with $J = i$ from (3.30) with $J = k-1$, we have

$$(R_n^i, L_n^i) = \sum_{j=i+1}^{k-1} \mathcal{G}_n^j(f^j, g^j) + (R_n^{k-1}, L_n^{k-1}). \tag{3.33}$$

Operating $(\mathcal{G}_n^k)^{-1}$ to (3.33) and taking $n \to \infty$,

$$\begin{aligned}
\left((\mathcal{G}_n^i)^{-1}\mathcal{G}_n^k\right)^{-1}(\mathcal{G}_n^i)^{-1}(R_n^i, L_n^i) &= (\mathcal{G}_n^k)^{-1}(R_n^i, L_n^i) \\
&= \sum_{j=i+1}^{k-1} (\mathcal{G}_n^k)^{-1}\mathcal{G}_n^j(f^j, g^j) + (\mathcal{G}_n^k)^{-1}(R_n^{k-1}, L_n^{k-1}) \\
&\rightharpoonup (0,0) + (f^k, g^k) \neq (0,0)
\end{aligned}$$

in $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ by the assumption of induction and (3.27). On the other hand,

$$(\mathcal{G}_n^i)^{-1}(R_n^i, L_n^i) \rightharpoonup (0,0)$$

in $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ as $n \to \infty$. Therefore, $\{\mathcal{G}_n^i\}_n, \{\mathcal{G}_n^k\}_n \subset G$ are orthogonal by Lemma 3.34. We prove (3.20). By (3.21), we get the following estimate:

$$\sum_{j=1}^{J} \left\{ \|f^j\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 + \|g^j\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 \right\} \leq \limsup_{n\to\infty} \left\{ \|f_n\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 + \|g_n\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 \right\} < \infty.$$

Taking supremum in J for this estimate, we obtain

$$\sum_{j=1}^{\infty} \left\{ \|f^j\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 + \|g^j\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 \right\} \leq \limsup_{n\to\infty} \left\{ \|f_n\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 + \|g_n\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 \right\} < \infty.$$

Thus, we get $\lim_{j\to\infty} \|(f^j, g^j)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}} = 0$. Using (3.26), we have

$$\lim_{J\to\infty} \eta(\{(R_n^J, L_n^J)\}) = 0. \tag{3.34}$$

The proof is completed if we show that $\lim_{J\to\infty} \eta(\{(R_n^J, L_n^J)\}) = 0$ implies the desired smallness property (3.20). This part is established by the forthcoming Proposition 3.39. We assume that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|(e^{it\Delta} R_n^J, e^{\frac{1}{2}it\Delta} L_n^J)\|_{L_t^{q,\infty} L_x^r \times L_t^{q,\infty} L_x^r} > 0$$

for contradiction. Then, there exist $\varepsilon_0 > 0$, subsequences $\{J_k\} \subset \{J\}$ and $\{n_l\} \subset \{n\}$ such that

$$\|(e^{it\Delta} R_{n_l}^{J_k}, e^{\frac{1}{2}it\Delta} L_{n_l}^{J_k})\|_{L_t^{q,\infty} L_x^r \times L_t^{q,\infty} L_x^r} > \varepsilon_0$$

for any J_k and n_l. The definition of η and Proposition 3.39 deduces

$$\eta(\{(R_n^{J_k}, L_n^{J_k})\}) \geq \eta(\{(R_{n_l}^{J_k}, L_{n_l}^{J_k})\}) \gtrsim_{M,\varepsilon_0,q,r} 1 > 0,$$

where a positive constant $M > 0$ satisfying $\|(R_n^J, L_n^J)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \|(f_n, g_n)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}} \leq M$. This contradicts (3.34). □

3.8. **Control of vanishing.** To complete the proof of Theorem 3.38, we show the following in this subsection.

Proposition 3.39 (Control of vanishing)**.** *Let $d = 3$. If a sequence $\{(R_n, L_n)\}_n \subset \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ satisfies*

$$\|(R_n, L_n)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}} \leq M$$

and

$$\|(e^{it\Delta} R_n, e^{\frac{1}{2}it\Delta} L_n)\|_{L_t^{q,\infty} L_x^r \times L_t^{q,\infty} L_x^r} \geq \varepsilon_0$$

for some $M > 0$, $\varepsilon_0 > 0$, and $1 < q, r < \infty$ with $\frac{1}{q} \in (\frac{1}{2}, 1)$ and $\frac{2}{q} + \frac{3}{r} = 2$, then

$$\eta(\{(R_n, L_n)\}) \gtrsim_{M,\varepsilon_0,q,r} 1.$$

To prove the proposition, we need the following lemma.

Lemma 3.40 (Improved Strichartz estimate)**.** *Let $d = 3$ and $I \subset \mathbb{R}$ be a time interval. It holds that*

$$\|e^{it\Delta} f\|_{L_t^3(I;L_x^3)} \lesssim \|f\|_{\mathcal{F}\dot{H}^{\frac{1}{6}}}^{\frac{2}{3}} \sup_{N\in 2^{\mathbb{Z}}} \|e^{it\Delta} \psi_N f\|_{L_t^3(I;L_x^3)}^{\frac{1}{3}},$$

where ψ_N is defined as (3.1).

Proof. By Lemma 3.7, we have

$$\|e^{it\Delta}f\|_{L^3_x} = \|\mathcal{M}_{\frac{1}{2}}(-t)e^{it\Delta}f\|_{L^3_x} \sim \Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}}\big|P_{\frac{N}{2t}}\mathcal{M}_{\frac{1}{2}}(-t)e^{it\Delta}f\big|^2\Big)^{\frac{1}{2}}\Big\|_{L^3_x}$$

for each $t>0$, where the implicit constant is independent of t by virtue of the scaling. Denote $g_N = P_{\frac{N}{2t}}\mathcal{M}_{\frac{1}{2}}(-t)e^{it\Delta}f$ for simplicity. By a convexity argument, one has

$$\begin{aligned}\Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}}|g_N|^2\Big)^{\frac{1}{2}}\Big\|^3_{L^3_t(I;L^3_x)} &= \int_{I\times\mathbb{R}^3}\Big(\sum_{N\in 2^{\mathbb{Z}}}|g_N(t,x)|^2\Big)^{\frac{3}{4}}\Big(\sum_{M\in 2^{\mathbb{Z}}}|g_M(t,x)|^2\Big)^{\frac{3}{4}}dxdt\\ &\lesssim \sum_{M,N\in 2^{\mathbb{Z}},N\le M}\int_{I\times\mathbb{R}^3}|g_N(t,x)|^{\frac{3}{2}}|g_M(t,x)|^{\frac{3}{2}}dxdt,\end{aligned}$$

where we have used the symmetry in the last line to reduce the matter to the case $N\le M$. Take r_1 and r_2 so that $\frac{8}{3}<r_1<3<r_2<\frac{10}{3}$ and $\frac{2}{3}=\frac{1}{r_1}+\frac{1}{r_2}$. By Lemma 2.2, we have

$$\int_{I\times\mathbb{R}^3}|g_N(t,x)|^{\frac{3}{2}}|g_M(t,x)|^{\frac{3}{2}}dxdt \le \|g_N\|_{L^{r_1}_t(I;L^{r_1}_x)}\|g_N\|^{\frac{1}{2}}_{L^3_t(I;L^3_x)}\|g_M\|^{\frac{1}{2}}_{L^3_t(I;L^3_x)}\|g_M\|_{L^{r_2}_t(I;L^{r_2}_x)}.$$

Hence,

$$\|e^{it\Delta}f\|^3_{L^3_t(I;L^3_x)} \lesssim \sup_{N\in 2^{\mathbb{Z}}}\|g_N\|_{L^3_t(I;L^3_x)}\cdot\sum_{M,N\in 2^{\mathbb{Z}},N\le M}\|g_N\|_{L^{r_1}_t(I;L^{r_1}_x)}\|g_M\|_{L^{r_2}_t(I;L^{r_2}_x)}$$

Remark that

$$g_N = \mathcal{F}\psi_{\frac{N}{2t}}\mathcal{F}^{-1}D(t)\mathcal{F}\mathcal{M}_{\frac{1}{2}}(t)f = D(t)\mathcal{F}\mathcal{M}_{\frac{1}{2}}(t)\psi_N f = \mathcal{M}_{\frac{1}{2}}(t)^{-1}e^{it\Delta}\psi_N f.$$

By Lemma 3.14 and 3.5, we have

$$\begin{aligned}\|g_N\|_{L^r_t(I;L^r_x)} = \|e^{it\Delta}\psi_N f\|_{L^r_t(I;L^r_x)} &\lesssim \|e^{it\Delta}\psi_N f\|_{L_t^{\frac{2r}{3r-8},r}(I;\dot{X}^{\frac{10-3r}{2r},\frac{6r}{16-3r}}_{1/2})}\\ &\lesssim \|\psi_N f\|_{\mathcal{F}\dot{H}^{\frac{10-3r}{2r}}} = \||x|^{\frac{5}{r}-\frac{5}{3}}\psi_N|x|^{\frac{1}{6}}f\|_{L^2_x} \lesssim N^{\frac{5}{r}-\frac{5}{3}}\|\psi_N|x|^{\frac{1}{6}}f\|_{L^2_x}\end{aligned}$$

for $\frac{8}{3}<r<\frac{10}{3}$. Thus, we obtain

$$\begin{aligned}&\sum_{M,N\in 2^{\mathbb{Z}},N\le M}\|g_N\|_{L^{r_1}_t(I;L^{r_1}_x)}\|g_M\|_{L^{r_2}_t(I;L^{r_2}_x)}\\ &\quad\le \sum_{R\ge 1}\sum_{N\in 2^{\mathbb{Z}}}\|g_N\|_{L^{r_1}_t(I;L^{r_1}_x)}\|g_{NR}\|_{L^{r_2}_t(I;L^{r_2}_x)}\\ &\quad\lesssim \sum_{R\ge 1}R^{-\frac{5}{r_1}+\frac{5}{3}}\sum_{N\in 2^{\mathbb{Z}}}\|\psi_N|x|^{\frac{1}{6}}f\|_{L^2_x}\|\psi_{NR}|x|^{\frac{1}{6}}f\|_{L^2_x}\\ &\quad\le \sum_{R\ge 1}R^{-\frac{5}{r_1}+\frac{5}{3}}\Big(\sum_{N\in 2^{\mathbb{Z}}}\|\psi_N|x|^{\frac{1}{6}}f\|^2_{L^2_x}\Big)^{\frac{1}{2}}\Big(\sum_{N\in 2^{\mathbb{Z}}}\|\psi_{NR}|x|^{\frac{1}{6}}f\|^2_{L^2_x}\Big)^{\frac{1}{2}}\\ &\quad= \sum_{R\ge 1}R^{-\frac{5}{r_1}+\frac{5}{3}}\Big(\int_{\mathbb{R}^3}\sum_{N\in 2^{\mathbb{Z}}}|\psi_N(x)|x|^{\frac{1}{6}}f(x)|^2dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}\sum_{N\in 2^{\mathbb{Z}}}|\psi_{NR}(x)|x|^{\frac{1}{6}}f(x)|^2dx\Big)^{\frac{1}{2}}\\ &\quad\lesssim \|f\|^2_{\mathcal{F}\dot{H}^{\frac{1}{6}}}\sum_{R\ge 1}R^{-\frac{5}{r_1}+\frac{5}{3}} \lesssim \|f\|^2_{\mathcal{F}\dot{H}^{\frac{1}{6}}}\end{aligned}$$

from Lemma 3.7. This completes the proof. □

Proof of Proposition 3.39. In what follows, we denote various subsequences of n again by n. By the pigeon hole principle,

$$\|e^{it\Delta}R_n\|_{L^{q,\infty}_t(I_\pm;L^r_x)} \ge \frac{\varepsilon_0}{4} \quad\text{or}\quad \|e^{\frac{1}{2}it\Delta}L_n\|_{L^{q,\infty}_t(I_\pm;L^r_x)} \ge \frac{\varepsilon_0}{4}$$

holds for infinitely many n, where $I_+ := [0,\infty)$ and $I_- := (-\infty,0]$. We only consider the case

$$\|e^{it\Delta}R_n\|_{L_t^{q,\infty}([0,\infty);L_x^r)} \geq \frac{\varepsilon_0}{4} \tag{3.35}$$

holds for infinitely many n. The proof for the other case is similar. By the interpolation and boundedness lemma, there exists $\theta = \theta(q,r) > 0$ such that

$$\|e^{it\Delta}R_n\|_{L_t^{q,\infty}([0,\infty);L_x^r)} \lesssim \|e^{it\Delta}R_n\|^{\theta}_{L_t^3([0,\infty);X^{\frac{1}{3},3}_{1/2})}\|R_n\|^{1-\theta}_{\mathcal{F}\dot{H}^{\frac{1}{2}}}.$$

Indeed, if we take $q_1, q_2, q_3, r_1, r_2, r_3, \theta$ satisfying $\frac{1}{q} = \frac{1}{q_1}+\frac{1}{q_2}$, $\frac{1}{r} = \frac{1}{r_1}+\frac{1}{r_2}$, $\frac{1}{r_1\theta} = \frac{1}{3}-\frac{1}{9}$, $\frac{1}{q_1\theta} = \frac{1}{3}+\frac{1}{3}$, $\frac{1}{r_2(1-\theta)} = \frac{1}{r_3}-\frac{1}{6}$, and $\frac{1}{q_2(1-\theta)} = \frac{1}{2}+\frac{1}{q_3}$, then it follows from Lemma 3.14 that

$$\begin{aligned}
\|e^{it\Delta}R_n\|_{L_t^{q,\infty}([0,\infty);L_x^r)} &= \|\mathcal{M}_{\frac{1}{2}}(-t)e^{it\Delta}R_n\|_{L_t^{q,\infty}([0,\infty);L_x^r)}\\
&\lesssim \|\mathcal{M}_{\frac{1}{2}}(-t)e^{it\Delta}R_n\|^{\theta}_{L_t^{q_1\theta,\infty}([0,\infty);L_x^{r_1\theta})}\|\mathcal{M}_{\frac{1}{2}}(-t)e^{it\Delta}R_n\|^{1-\theta}_{L_t^{q_2(1-\theta),\infty}([0,\infty);L_x^{r_2(1-\theta)})}\\
&\lesssim \|e^{it\Delta}R_n\|^{\theta}_{L_t^{3,\infty}([0,\infty);\dot{X}^{\frac{1}{3},3}_{1/2})}\|e^{it\Delta}R_n\|^{1-\theta}_{L_t^{q_3}([0,\infty);\dot{X}^{\frac{1}{2},r_3}_{1/2})}\\
&\lesssim \|e^{it\Delta}R_n\|^{\theta}_{L_t^{3}([0,\infty);\dot{X}^{\frac{1}{3},3}_{1/2})}\|e^{it\Delta}R_n\|^{1-\theta}_{L_t^{q_3,2}([0,\infty);\dot{X}^{\frac{1}{2},r_3}_{1/2})}\\
&\lesssim \|e^{it\Delta}R_n\|^{\theta}_{L_t^{3}([0,\infty);\dot{X}^{\frac{1}{3},3}_{1/2})}\|R_n\|^{1-\theta}_{\mathcal{F}\dot{H}^{\frac{1}{2}}},
\end{aligned} \tag{3.36}$$

where the last inequality is used Lemma 3.5 ($\frac{2}{q_3}+\frac{3}{r_3} = \frac{3}{2}$). By the definition of $J^{1/3}_{1/2}$, we have $(-4t^2\Delta)^{\frac{1}{6}}\mathcal{M}_{\frac{1}{2}}(-t)e^{it\Delta} = \mathcal{M}_{\frac{1}{2}}(-t)e^{it\Delta}|x|^{\frac{1}{3}}$. Therefore, it follows that

$$\begin{aligned}
\|e^{it\Delta}R_n\|_{L_t^3([0,\infty);\dot{X}^{\frac{1}{3},3}_{1/2})} &= \|(-4t^2\Delta)^{\frac{1}{6}}\mathcal{M}_{\frac{1}{2}}(-t)e^{it\Delta}R_n\|_{L_t^3([0,\infty);L_x^3)}\\
&= \|\mathcal{M}_{\frac{1}{2}}(-t)e^{it\Delta}|x|^{\frac{1}{3}}R_n\|_{L_t^3([0,\infty);L_x^3)} = \|e^{it\Delta}|x|^{\frac{1}{3}}R_n\|_{L_t^3([0,\infty);L_x^3)}.
\end{aligned}$$

Applying (3.35), (3.36), this identity, an assumption in this proposition, and Lemma 3.40, we obtain

$$\begin{aligned}
\left(\frac{\varepsilon_0}{4}\right)^{\frac{3}{\theta}} &\leq \|e^{it\Delta}R_n\|^{\frac{3}{\theta}}_{L_t^{q,\infty}L_x^r} \lesssim \|e^{it\Delta}R_n\|^3_{L_t^3([0,\infty);\dot{X}^{\frac{1}{3},3}_{1/2})}\|R_n\|^{\frac{3(1-\theta)}{\theta}}_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\\
&\leq M^{\frac{3(1-\theta)}{\theta}}\|e^{it\Delta}|x|^{\frac{1}{3}}R_n\|^3_{L_t^3([0,\infty);L_x^3)}\\
&\lesssim M^{\frac{3(1-\theta)}{\theta}}\||x|^{\frac{1}{3}}R_n\|^2_{\mathcal{F}\dot{H}^{\frac{1}{6}}}\sup_{N\in 2^{\mathbb{Z}}}\|e^{it\Delta}\psi_N|x|^{\frac{1}{3}}R_n\|_{L_t^3([0,\infty);L_x^3)}\\
&= M^{\frac{3(1-\theta)}{\theta}}\|R_n\|^2_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\sup_{N\in 2^{\mathbb{Z}}}\|e^{it\Delta}\psi_N|x|^{\frac{1}{3}}R_n\|_{L_t^3([0,\infty);L_x^3)}\\
&\leq M^{\frac{3(1-\theta)}{\theta}+2}\sup_{N\in 2^{\mathbb{Z}}}\|e^{it\Delta}\psi_N|x|^{\frac{1}{3}}R_n\|_{L_t^3([0,\infty);L_x^3)},
\end{aligned}$$

that is,

$$\sup_{N\in 2^{\mathbb{Z}}}\|e^{it\Delta}|x|^{\frac{1}{3}}\psi_N R_n\|_{L_t^3([0,\infty);L_x^3)} \gtrsim_{M,\varepsilon_0,q,r} 1.$$

One can choose a sequence N_n so that

$$\|e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|_{L_t^3([0,\infty);L_x^3)} \gtrsim 1. \tag{3.37}$$

Since the scaling property and Lemma 3.4 give us

$$\begin{aligned}
\|e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|_{L_t^3([0,\tau N_n^2];L_x^3)} &= N_n^{\frac{5}{3}}\|e^{it\Delta}(N_n|x|)^{\frac{1}{3}}\psi R_n(N_n\cdot)\|_{L_t^3([0,\tau];L_x^3)}\\
&\leq N_n^{\frac{5}{3}}\|1\|_{L_t^{12}([0,\tau])}\|e^{it\Delta}(N_n|x|)^{\frac{1}{3}}\psi R_n(N_n\cdot)\|_{L_t^4([0,\tau];L_x^3)}
\end{aligned}$$

$$\lesssim N_n^{\frac{5}{3}}\tau^{\frac{1}{12}}\|(N_n|x|)^{\frac{1}{3}}\psi R_n(N_n\cdot)\|_{L_x^2}$$
$$\lesssim \tau^{\frac{1}{12}}\|R_n\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\lesssim \tau^{\frac{1}{12}},$$

one can choose $\tau_0=\tau_0(M,\varepsilon_0,q,r)>0$ small so that (3.37) is improved as

$$\|e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|_{L_t^3([\tau_0N_n^2,\infty);L_x^3)}\gtrsim 1$$

for all $n\geq 1$. Lemma 2.2 gives us

$$\begin{aligned}&\|e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|_{L_t^3([\tau_0N_n^2,\infty);L_x^3)}\\&\quad\leq \||t|^{\frac{3}{2}}e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|^{\frac{1}{18}}_{L_t^\infty([\tau_0N_n^2,\infty);L_x^\infty)}\||t|^{-\frac{3}{34}}e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|^{\frac{17}{18}}_{L_t^{\frac{17}{6}}([\tau_0N_n^2,\infty);L_x^{\frac{17}{6}})}.\end{aligned}$$

Using Lemma 3.8, 3.14, and Proposition 3.5,

$$\begin{aligned}\||t|^{-\frac{3}{34}}e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|_{L_t^{\frac{17}{6}}L_x^{\frac{17}{6}}}&\lesssim \||t|^{-\frac{3}{34}}\|_{L^{\frac{34}{3},\infty}}\|e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|_{L_t^{\frac{34}{9},\frac{17}{6}}L_x^{\frac{17}{6}}}\\&\lesssim \|e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|_{L_t^{\frac{17}{3},\frac{17}{6}}\dot{X}^{\frac{3}{34},\frac{34}{13}}_{1/2}}\\&\lesssim \|e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|_{L_t^{\frac{17}{3},2}\dot{X}^{\frac{3}{34},\frac{34}{13}}_{1/2}}\\&\lesssim \||x|^{\frac{1}{3}+\frac{3}{34}}\psi_{N_n}R_n\|_{L_x^2}\\&\lesssim N_n^{-\frac{4}{51}},\end{aligned}$$

we reach to the estimate

$$N_n^{-\frac{4}{3}}\||t|^{\frac{3}{2}}e^{it\Delta}|x|^{\frac{1}{3}}\psi_{N_n}R_n\|_{L_t^\infty([\tau_0N_n^2,\infty);L_x^\infty)}\gtrsim 1$$

for all $n\geq 1$. There exist $t_n\geq \tau_0N_n^2$ and $y_n\in\mathbb{R}^3$ such that

$$N_n^{-\frac{4}{3}}|t_n^{\frac{3}{2}}e^{it_n\Delta}(|x|^{\frac{1}{3}}\psi_{N_n}R_n)(y_n)|\gtrsim 1. \tag{3.38}$$

By the integral representation of the Schrödinger group, we obtain

$$\begin{aligned}&N_n^{-\frac{4}{3}}|t_n^{\frac{3}{2}}e^{it_n\Delta}(|x|^{\frac{1}{3}}\psi_{N_n}R_n)(y_n)|\\&\quad= N_n^{-\frac{4}{3}}\left|t_n^{\frac{3}{2}}(4\pi it_n)^{-\frac{3}{2}}\int_{\mathbb{R}^3}e^{\frac{i|x-y_n|^2}{4t_n}}|x|^{\frac{1}{3}}\psi_{N_n}(x)R_n(x)dx\right|\\&\quad\lesssim N_n^{-\frac{3}{2}}\left|\int_{\mathbb{R}^3}e^{-i\frac{y_n}{2t_n}\cdot x}e^{i\frac{|x|^2}{4t_n}}(N_n^{\frac{1}{6}}|x|^{-\frac{1}{6}}\psi_{N_n}(x))|x|^{\frac{1}{2}}R_n(x)dx\right|\\&\quad=\left|\int_{\mathbb{R}^3}e^{i\frac{N_n^2|x|^2}{4t_n}}(|x|^{-\frac{1}{6}}\psi(x))|x|^{\frac{1}{2}}(e^{-i\frac{N_ny_n}{2t_n}\cdot x}N_n^2R_n(N_nx))dx\right|.\end{aligned}\tag{3.39}$$

Let

$$\xi_n:=-\frac{N_ny_n}{2t_n}\in\mathbb{R}^3,\quad h_n:=N_n\in 2^{\mathbb{Z}}.$$

Define a deformation $\mathcal{G}_n\in G$ so that $\mathcal{G}_n^{-1}=T(\xi_n)D(h_n)$. Since $\{\mathcal{G}_n(R_n,L_n)\}_n$ is a bounded sequence in $\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, it weakly converges to a pair $(\widetilde{R},\widetilde{L})\in\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\times\mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ along a subsequence. It is obvious that $(\widetilde{R},\widetilde{L})\in\nu(\{(R_n,L_n)\})$. Notice that $0<\frac{N_n^2}{4t_n}\leq\frac{1}{4\tau_0}$. Hence, by extracting a subsequence if necessary, one has

$$\int_{\mathbb{R}^3}e^{i\frac{N_n^2|x|^2}{4t_n}}(|x|^{-\frac{1}{6}}\psi(x))|x|^{\frac{1}{2}}(e^{-i\frac{N_ny_n}{2t_n}\cdot x}N_n^2R_n(N_nx))dx\longrightarrow\int_{\mathbb{R}^3}e^{i\frac{N_\infty^2|x|^2}{4t_\infty}}(|x|^{-\frac{1}{6}}\psi(x))|x|^{\frac{1}{2}}\widetilde{R}(x)dx$$

as $n \to \infty$, where $\frac{N_\infty^2}{4t_\infty} \in \mathbb{R}$ is the limit of $\frac{N_n^2}{4t_n}$ along the (sub)sequence. Plugging this with (3.38) and (3.39), we conclude that

$$1 \lesssim \left| \int_{\mathbb{R}^3} e^{i\frac{N_\infty^2 |x|^2}{4t_\infty}} (|x|^{-\frac{1}{6}}\psi(x))|x|^{\frac{1}{2}} \widetilde{R}(x)dx \right| \lesssim_\psi \|(\widetilde{R}, \widetilde{L})\|_{\mathcal{F}\dot{H}^{\frac{1}{2}} \times \mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \eta(\{(R_n, L_n)\}).$$

This is the desired estimate. □

3.9. **Proof of Main theorems 1.39, 1.41, and 1.42.** In this subsection, we prove Main theorem 1.39, 1.41, and 1.42. The following proof shows all these theorems.

Proof of Main theorem 1.39, 1.41, and 1.42. Fix $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. First, we consider the case $\ell_{v_0}^\dagger = \infty$. In this case, we can obtain $\ell_{v_0}^\dagger = \ell_{v_0} = \ell_0 = \infty$. Indeed, we have $\infty = \ell_{v_0}^\dagger \leq \ell_{v_0}$ by Lemma 3.27. On the other hand, we have $\infty = \ell_{v_0}^\dagger \leq \ell_0^\dagger \leq \ell_0$ by Proposition 3.28 and Lemma 3.27.

From now on, we assume $\ell_{v_0}^\dagger < \infty$. By the definition of $\ell_{v_0}^\dagger$, we have $L_{v_0}(\ell_{v_0}^\dagger - \frac{1}{n}) < \infty$ for each $n \in \mathbb{N}$, that is,

$$\sup \left\{ \|(u,v)\|_{W_1([0,\infty)) \times W_2([0,\infty))} \;\middle|\; \begin{array}{l} (u,v) \text{ is the solution to (NLS) on } [0,\infty), \\ v(0) = v_0, \; \|u(0)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \ell_{v_0}^\dagger - \frac{1}{n} \end{array} \right\} < \infty.$$

We note that $T_{\max} = \infty$ because of Proposition 3.19. Since $L_{v_0}(\ell) < \infty$ for any $0 \leq \ell < \ell_{v_0}^\dagger$, $L_{v_0}(\ell_{v_0}^\dagger) = \infty$, and L_{v_0} is non-decreasing, we can take a sequence $\{m_n\}$ of $\mathbb{N}$ such that

$$L_{v_0}\left(\ell_{v_0}^\dagger - \frac{1}{m_n}\right) < L_{v_0}\left(\ell_{v_0}^\dagger - \frac{1}{m_{n+1}}\right)$$

for each $n \in \mathbb{N}$. We take a sequence $\{u_{0,n}\} \subset \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ satisfying

$$\ell_{v_0}^\dagger - \frac{1}{m_n} < \|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \ell_{v_0}^\dagger - \frac{1}{m_{n+1}} \tag{3.40}$$

and

$$L_{v_0}\left(\ell_{v_0}^\dagger - \frac{1}{m_n}\right) < \|(u_n, v_n)\|_{W_1([0,\infty)) \times W_2([0,\infty))} \leq L_{v_0}\left(\ell_{v_0}^\dagger - \frac{1}{m_{n+1}}\right),$$

where (u_n, v_n) is the solution to (NLS) with initial data $(u_{0,n}, v_0)$. We notice that

$$\lim_{n\to\infty} \|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{v_0}^\dagger \quad \text{and} \quad \lim_{n\to\infty} = \|(u_n, v_n)\|_{W_1([0,\infty)) \times W_2([0,\infty))} = \infty \tag{3.41}$$

from $m_n \longrightarrow \infty$ as $n \to \infty$, (3.40), and the continuity of L_{v_0} (Lemma 3.25). Since $\{(u_{0,n}, v_0)\} \subset \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is a bounded sequence, we apply Theorem 3.38 to this sequence. Then, there exist profile $\{(\phi^j, \psi^j)\} \subset \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, remainder $\{(R_n^J, L_n^J)\} \subset \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \times \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, and pairwise orthogonal families of deformations $\{\mathcal{G}_n^j = T(\xi_n^j)D(h_n^j)\}_n \subset G$ ($j = 1, 2, \ldots$) such that

$$(u_{0,n}, v_0) = \sum_{j=1}^{J} \mathcal{G}_n^j(\phi^j, \psi^j) + (R_n^J, L_n^J) \tag{3.42}$$

for any $J \geq 1$. Since v_0 is independent of n, there exists unique j_0 such that $\psi^{j_0} = v_0$ and $\mathcal{G}_n^{j_0} = \mathrm{Id}$. Furthermore, the remainder for v-component is zero: $L_n^J = 0$. Rearranging the profile (ϕ^j, ψ^j), we may let $j_0 = 1$. Then, the above decomposition reads as

$$(u_{0,n}, v_0) = (\phi^1, v_0) + \sum_{j=2}^{J} \mathcal{G}_n^j(\phi^j, 0) + (R_n^J, 0).$$

From Theorem 3.38, we have Pythagorean decomposition:

$$\|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 = \sum_{j=1}^{J} \|\phi^j\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 + \|R_n^J\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 + o_n(1) \tag{3.43}$$

for each $J \geq 1$. The parameters are asymptotically orthogonal: if $j \neq k$, then

$$\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} + \frac{|\xi_n^j - \xi_n^k|}{h_n^j} \longrightarrow \infty \quad \text{as} \quad n \to \infty. \tag{3.44}$$

The remainders satisfy

$$(\mathcal{G}_n^j)^{-1} R_n^J \longrightarrow 0 \quad \text{in} \quad \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \quad \text{as} \quad n \to \infty$$

for any $1 \leq j \leq J$.

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|e^{it\Delta} R_n^J\|_{L_t^{q,\infty} L_x^r} = 0 \tag{3.45}$$

for any $1 < q, r < \infty$ with $\frac{1}{q} \in (\frac{1}{2}, 1)$ and $\frac{2}{q} + \frac{3}{r} = 2$.

We will prove that there exists only one j_1 satisfying $\phi^{j_1} \neq 0$ and it satisfies $\|\phi^{j_1}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{v0}^{\dagger}$. From (3.43), we have

$$\limsup_{n\to\infty} \|R_n^J\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 + \sum_{j=1}^{J} \|\phi^j\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}^2 = (\ell_{v0}^{\dagger})^2, \tag{3.46}$$

and hence,

$$\limsup_{n\to\infty} \|R_n^J\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \ell_{v0}^{\dagger} \quad \text{and} \quad \|\phi^j\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \leq \ell_{v0}^{\dagger} \tag{3.47}$$

holds for any $j \geq 1$. Let (Φ_j, Ψ_j) be the solution to (NLS) with initial data (ϕ^j, ψ^j). We assume for contradiction that all (Φ_j, Ψ_j) scatter, that is,

$$\|(\Phi_j, \Psi_j)\|_{W_1([0,\infty)) \times W_2([0,\infty))} < \infty$$

is true for any $j \geq 1$. We set

$$(\widetilde{w}_n^J, \widetilde{z}_n^J) := \sum_{j=1}^{J} \big((\Phi_j)_{[h_n^j,\xi_n^j]}(t,x), (\Psi_j)_{[h_n^j,\xi_n^j]}(t,x)\big)$$

and

$$(\widetilde{u}_n^J, \widetilde{v}_n^J) := (\widetilde{w}_n^J, \widetilde{z}_n^J) + (e^{it\Delta} R_n^J, 0),$$

where

$$(\Phi_j)_{[h_n^j,\xi_n^j]}(t,x) := {h_n^j}^2 e^{ix\cdot\xi_n^j} e^{-it|\xi_n^j|^2} \Phi_j({h_n^j}^2 t, h_n^j(x - 2t\xi_n^j)),$$

$$(\Psi_j)_{[h_n^j,\xi_n^j]}(t,x) := {h_n^j}^2 e^{2ix\cdot\xi_n^j} e^{-2it|\xi_n^j|^2} \Psi_j({h_n^j}^2 t, h_n^j(x - 2t\xi_n^j)).$$

We note that $\big((\Phi_j)_{[h_n^j,\xi_n^j]}, (\Psi_j)_{[h_n^j,\xi_n^j]}\big)$ is a solution to (NLS) with initial data $\mathcal{G}_n^j(\phi^j, \psi^j)$ from (1.11) and (1.14). Then, $(\widetilde{u}_n^J, \widetilde{v}_n^J)$ solves

$$i\partial_t \widetilde{u}_n^J + \Delta \widetilde{u}_n^J = \sum_{j=1}^{J} \big(i\partial_t (\Phi_j)_{[h_n^j,\xi_n^j]} + \Delta (\Phi_j)_{[h_n^j,\xi_n^j]}\big) = -2 \sum_{j=1}^{J} (\Psi_j)_{[h_n^j,\xi_n^j]} \overline{(\Phi_j)_{[h_n^j,\xi_n^j]}},$$

$$i\partial_t \widetilde{v}_n^J + \frac{1}{2}\Delta \widetilde{v}_n^J = \sum_{j=1}^{J} \big(i\partial_t (\Psi_j)_{[h_n^j,\xi_n^j]} + \frac{1}{2}\Delta (\Psi_j)_{[h_n^j,\xi_n^j]}\big) = -\sum_{j=1}^{J} (\Phi_j)_{[h_n^j,\xi_n^j]}^2.$$

We also set

$$\widetilde{e}_{1,n}^J := i\partial_t \widetilde{u}_n^J + \Delta \widetilde{u}_n^J + 2\widetilde{v}_n^J \overline{\widetilde{u}_n^J} \quad \text{and} \quad \widetilde{e}_{2,n}^J := i\partial_t \widetilde{v}_n^J + \frac{1}{2}\Delta \widetilde{v}_n^J + (\widetilde{u}_n^J)^2.$$

Here, we introduce the following two lemmas, which are proved later.

Lemma 3.41. *For any $\varepsilon > 0$, there exists $J_1 = J_1(\varepsilon)$ such that*

$$\limsup_{n\to\infty} \|(\widetilde{w}_n^J, \widetilde{z}_n^J) - (\widetilde{w}_n^{J_1}, \widetilde{z}_n^{J_1})\|_{W_1([0,\infty))\times W_2([0,\infty))} \le \varepsilon$$

for any $J \ge J_1$.

Lemma 3.42. *It follows that*

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|(\widetilde{e}_{1,n}^J, \widetilde{e}_{2,n}^J)\|_{N_1([0,\infty))\times N_2([0,\infty))} = 0.$$

Using Lemma 3.41 with $\varepsilon = \frac{1}{2}$ and (3.47), it follows that there exists J_1 such that for any $J \ge J_1$, there exists $n_1 = n_1(J)$ such that

$$\begin{aligned}
&\|(\widetilde{u}_n^J, \widetilde{v}_n^J)\|_{W_1([0,\infty))\times W_2([0,\infty))} \\
&\quad \le \|(\widetilde{w}_n^{J_1}, \widetilde{z}_n^{J_1})\|_{W_1([0,\infty))\times W_2([0,\infty))} \\
&\qquad\qquad + \|(\widetilde{w}_n^J, \widetilde{z}_n^J) - (\widetilde{w}_n^{J_1}, \widetilde{z}_n^{J_1})\|_{W_1([0,\infty))\times W_2([0,\infty))} + \|e^{it\Delta} R_n^J\|_{W_1([0,\infty))} \\
&\quad \le \sum_{j=1}^{J_1} \|(\Phi_j, \Psi_j)\|_{W_1([0,\infty))\times W_2([0,\infty))} + c\,\|R_n^J\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} + 1 \\
&\quad \le \sum_{j=1}^{J_1} \|(\Phi_j, \Psi_j)\|_{W_1([0,\infty))\times W_2([0,\infty))} + c\ell_{v_0}^{\dagger} + 1 =: M
\end{aligned} \tag{3.48}$$

for any $n \ge n_1$. Let $\varepsilon_1 = \varepsilon_1(M)$ be given in Proposition 3.23. Then,

$$\|(u_{0,n} - \widetilde{u}_n^J(0), v_0 - \widetilde{v}_n^J(0))\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}\times\mathcal{F}\dot{H}^{\frac{1}{2}}} = 0. \tag{3.49}$$

Lemma 3.42 implies that there exists J_2 such that for any $J \ge J_2$, there exists $n_2 = n_2(J)$ such that

$$\|(\widetilde{e}_{1,n}^J, \widetilde{e}_{2,n}^J)\|_{N_1([0,\infty))\times N_2([0,\infty))} \le \frac{\varepsilon_1}{2}. \tag{3.50}$$

for any $n \ge n_2$. We set $J_0 := \max\{J_1, J_2\}$ and $n_0 := \max\{n_1, n_2\}$. By (3.48), (3.49), (3.50), and Proposition 3.23, we deduce that (u_n, v_n) satisfies

$$\|(u_n, v_n)\|_{W_1([0,\infty))\times W_2([0,\infty))} \lesssim_M \varepsilon_1 < \infty$$

for any $n \ge n_0$. However, this contradicts (3.41). Therefore, there exists $j_1 \ge 1$ such that

$$\|(\Phi_{j_1}, \Psi_{j_1})\|_{W_1([0,T_{\max}))\times W_2([0,T_{\max}))} = \infty.$$

By (3.46), (3.47), Proposition 3.26, and 3.28, we have $\|\phi^{j_1}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{v_0}^{\dagger}$ and $\phi^j = 0$ for all $j \ne j_1$. Indeed, if $j_1 = 1$, then $\ell_{v_0}^{\dagger} \le \|\phi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \ell_{v_0}^{\dagger}$ and if $j_1 \ne 1$, then $\ell_{v_0}^{\dagger} \le \ell_0^{\dagger} \le \|\phi^{j_1}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \ell_{v_0}^{\dagger}$. We encounter a dichotomy, $j_1 = 1$ or $j_1 = 2$.

Now, we suppose that $j_1 = 1$ (, which corresponds to Theorem 1.41). Since a solution (Φ_1, Ψ_1) to (NLS) with initial data (ϕ^1, v_0) does not scatter, we have $\ell_{v_0} \le \|\phi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{v_0}^{\dagger}$ by the definition of ℓ_{v_0}. Combining this inequality and Lemma 3.27, we obtain $\ell_{v_0} = \ell_{v_0}^{\dagger} = \|\phi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$. This shows that ϕ^1 is a minimizer to ℓ_{v_0}. Moreover, it follows from Proposition 3.28 and Lemma 3.27 that $\ell_{v_0} = \ell_{v_0}^{\dagger} \le \ell_0^{\dagger} \le \ell_0$. Therefore, we have the identity $\ell_{v_0}^{\dagger} = \min\{\ell_0, \ell_{v_0}\}$.

Let us move on to the case $j_1 = 2$ (, which corresponds to Theorem 1.42). In this case, it follows that $(\phi^1, \psi^1) = (0, v_0)$ and $(\phi^2, \psi^2) = (\phi^2, 0)$. Since (Φ_2, Ψ_2) does not scatter, we have $\ell_0 \le \|\phi^2\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{v_0}^{\dagger}$ by the definition of ℓ_0. Using Proposition 3.28 and Lemma 3.27, we obtain $\ell_0 \le \|\phi^2\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{v_0}^{\dagger} \le \ell_0^{\dagger} \le \ell_0$. In particular, we have $\ell_{v_0}^{\dagger} = \ell_0 = \|\phi^2\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$. This shows that ϕ^2 is a minimizer to ℓ_0. In addition, we have

$$(u_{0,n}, v_0) = \sum_{j=1,2} \mathcal{G}_n^j(\phi^j, \psi^j) + (R_n^2, 0) = (0, v_0) + \mathcal{G}_n^2(\phi^2, 0) + (R_n^2, 0),$$

$$\lim_{n\to\infty} \|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac12}} = \ell^\dagger_{v0}, \quad \text{and} \quad \lim_{n\to\infty} \|R_n^2\|_{\mathcal{F}\dot{H}^{\frac12}} = 0$$

by (3.41), (3.42), (3.43), and $\|\phi^2\|_{\mathcal{F}\dot{H}^{\frac12}} = \ell^\dagger_{v0}$. Remark that we have also the identity $\ell^\dagger_{v0} = \min\{\ell_0, \ell_{v0}\}$ in this case. Let $T_{\max}$ denote the maximal existence time of a solution to (NLS) with initial data $(\phi^2, 0)$. Fix $0 \le \tau < T_{\max}$. Recall that (Φ_j, Ψ_j) denotes the solution to (NLS) with initial data (ϕ^j, ψ^j) and $\big((\Phi_j)_{[h_n^j,\xi_n^j]}, (\Psi_j)_{[h_n^j,\xi_n^j]}\big)$ denotes the solution to (NLS) with initial data $\mathcal{G}_n^j(\phi^j, \psi^j)$. We set

$$(\widetilde{u}_n, \widetilde{v}_n) := \sum_{j=1,2} \big((\Phi_j)_{[h_n^j,\xi_n^j]}, (\Psi_j)_{[h_n^j,\xi_n^j]}\big) = (0, e^{\frac12 it\Delta}v_0) + \big((\Phi_2)_{[h_n^2,\xi_n^2]}, (\Psi_2)_{[h_n^2,\xi_n^2]}\big).$$

Then, $(\widetilde{u}_n, \widetilde{v}_n)$ solves

$$i\partial_t \widetilde{u}_n + \Delta \widetilde{u}_n = \sum_{j=1,2} \big(i\partial_t (\Phi_j)_{[h_n^j,\xi_n^j]} + \Delta(\Phi_j)_{[h_n^j,\xi_n^j]}\big) = -2(\Psi_2)_{[h_n^2,\xi_n^2]}\overline{(\Phi_2)_{[h_n^2,\xi_n^2]}},$$

$$i\partial_t \widetilde{v}_n + \frac12 \Delta \widetilde{v}_n = \sum_{j=1,2} \big(i\partial_t (\Psi_j)_{[h_n^j,\xi_n^j]} + \frac12\Delta(\Psi_j)_{[h_n^j,\xi_n^j]}\big) = -(\Phi_2)^2_{[h_n^2,\xi_n^2]}.$$

We also set

$$\widetilde{e}_{1,n} := i\partial_t \widetilde{u}_n + \Delta \widetilde{u}_n + 2\widetilde{v}_n\overline{\widetilde{u}_n} = 2(\Psi_1)_{[h_n^1,\xi_n^1]}\overline{(\Phi_2)_{[h_n^2,\xi_n^2]}},$$

$$\widetilde{e}_{2,n} := i\partial_t \widetilde{v}_n + \frac12 \Delta \widetilde{v}_n + (\widetilde{u}_n)^2 = 0.$$

We check the assumptions of Proposition 3.23. One has

$$\begin{aligned}
&\|(\widetilde{u}_n, \widetilde{v}_n)\|_{W_1([0,\tau/(h_n^2)^2))\times W_2([0,\tau/(h_n^2)^2))} \\
&\qquad \le \|(0, e^{\frac12 it\Delta}v_0)\|_{W_1([0,\infty))\times W_2([0,\infty))} + \|(\Phi_2,\Psi_2)\|_{W_1([0,\tau))\times W_2([0,\tau))} =: M < \infty,
\end{aligned}$$

$$\|(u_{0,n}, v_0) - (\widetilde{u}_n(0), \widetilde{v}_n(0))\|_{\mathcal{F}\dot{H}^{\frac12}\times\mathcal{F}\dot{H}^{\frac12}} = \|(R_n^2, 0)\|_{\mathcal{F}\dot{H}^{\frac12}\times\mathcal{F}\dot{H}^{\frac12}} \longrightarrow 0 \quad \text{as} \quad n\to\infty,$$

and

$$\|(\widetilde{e}_{1,n}, \widetilde{e}_{2,n})\|_{N_1([0,\tau/(h_n^2)^2))\times N_2([0,\tau/(h_n^2)^2))} = \|\widetilde{e}_{1,n}\|_{N_1([0,\tau/(h_n^2)^2))} \longrightarrow 0 \ \text{ as } \ n\to\infty,$$

where the last limit is shown as in the same spirit of Lemma 3.42 with a help of the first estimate. Therefore, we obtain

$$(u_n, v_n) - (0, e^{\frac12 it\Delta}v_0) - \big((\Phi_2)_{[h_n^2,\xi_n^2]}, (\Psi_2)_{[h_n^2,\xi_n^2]}\big) \longrightarrow 0$$

in $L_t^\infty([0,\tau/(h_n^2)^2); \dot{X}^{1/2}_{1/2}) \times L_t^\infty([0,\tau/(h_n^2)^2); \dot{X}^{1/2}_1)$ as $n\to\infty$.

In both cases, we have the identity $\ell^\dagger_{v0} = \min\{\ell_0, \ell_{v0}\}$, hence we have Theorem 1.39. If we assume that $\ell_0 > \ell^\dagger_{v0}$ then the second case is precluded. This is nothing but Theorem 1.41.

Similarly, the assumption $\ell_{v0} > \ell^\dagger_{v0}$ precludes the case $j_1 = 1$. This shows Theorem 1.42. Indeed, the above argument applies to the minimizing sequence satisfying the assumption of Theorem 1.42 and leads us to the same conclusion in the case $j_1 = 2$. □

To prove Lemma 3.41 and 3.42, we first prepare the following claims.

Lemma 3.43. *Let $d = 3$. Let $\{(h_n^j, \xi_n^j)\}_{(j,n)\in[1,J]\times\mathbb{N}} \subset 2^{\mathbb{Z}} \times \mathbb{R}^3$ satisfy (3.44) and let $(\Phi_j, \Psi_j) \in W_1([0,\infty)) \times W_2([0,\infty))$. We consider $e^J_{1,n}$ and $e^J_{2,n}$ defined by*

$$e^J_{1,n} := F_1\Big(\sum_{j=1}^J (\Phi_j)_{[h_n^j,\xi_n^j]}, \sum_{j=1}^J (\Psi_j)_{[h_n^j,\xi_n^j]}\Big) - \sum_{j=1}^J F_1\big((\Phi_j)_{[h_n^j,\xi_n^j]}, (\Psi_j)_{[h_n^j,\xi_n^j]}\big),$$

and

$$e^J_{2,n} := F_2\Big(\sum_{j=1}^J (\Phi_j)_{[h_n^j,\xi_n^j]}\Big) - \sum_{j=1}^J F_2\big((\Phi_j)_{[h_n^j,\xi_n^j]}\big),$$

where $F_1(w,z) := 2z\overline{w}$ *and* $F_2(w) = w^2$. *Then,*

$$\|(e_{1,n}^J, e_{2,n}^J)\|_{N_1([0,\infty))\times N_2([0,\infty))} \longrightarrow 0 \quad as \quad n \to \infty.$$

Proof. Thanks to Proposition 3.16, we only have to consider the case: $\operatorname{supp}(\Phi_j, \Psi_j) \subset [m, M] \times B_R(0)$ holds for some $m, M, R > 0$ and for any $j \in [1, J]$, where $B_r(y)$ denotes a ball in $\mathbb{R}^3$ with radius $r > 0$ and center $y \in \mathbb{R}^3$. By the definition of $(e_{1,n}^J, e_{2,n}^J)$, we have

$$\begin{aligned}\|e_{1,n}^J\|_{N_1([0,\infty))} &= \Big\|(-4t^2\Delta)^{\frac14}\Big(2\sum_{\substack{1\le j,k\le J\\ j\neq k}} \mathcal{M}_1(-t)(\Psi_j)_{[h_n^j,\xi_n^j]}\overline{\mathcal{M}_{\frac12}(-t)(\Phi_k)_{[h_n^k,\xi_n^k]}}\Big)\Big\|_{L_t^{\frac65,2}([0,\infty);L_x^{\frac{18}{11}})}\\ &\le c\sum_{\substack{1\le j,k\le J\\ j\neq k}} \Big\|(-4t^2\Delta)^{\frac14}\mathcal{M}_1(-t)(\Psi_j)_{[h_n^j,\xi_n^j]}\overline{\mathcal{M}_{\frac12}(-t)(\Phi_k)_{[h_n^k,\xi_n^k]}}\Big\|_{L_t^{\frac65,2}([0,\infty);L_x^{\frac{18}{11}})}\end{aligned}$$

and

$$\begin{aligned}\|e_{2,n}^J\|_{N_2([0,\infty))} &= \Big\|(-t^2\Delta)^{\frac14}\Big(\sum_{\substack{1\le j,k\le J\\ j\neq k}} \mathcal{M}_{\frac12}(-t)(\Phi_j)_{[h_n^j,\xi_n^j]}\mathcal{M}_{\frac12}(-t)(\Phi_k)_{[h_n^k,\xi_n^k]}\Big)\Big\|_{L_t^{\frac65,2}([0,\infty);L_x^{\frac{18}{11}})}\\ &\le c\sum_{\substack{1\le j,k\le J\\ j\neq k}} \Big\|(-t^2\Delta)^{\frac14}\mathcal{M}_{\frac12}(-t)(\Phi_j)_{[h_n^j,\xi_n^j]}\mathcal{M}_{\frac12}(-t)(\Phi_k)_{[h_n^k,\xi_n^k]}\Big\|_{L_t^{\frac65,2}([0,\infty);L_x^{\frac{18}{11}})},\end{aligned}$$

When $\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} \ge \sqrt{2 + 2\frac{M}{m}}$, time supports of $(\Phi_j)_{[h_n^j,\xi_n^j]}$ and $(\Psi_j)_{[h_n^j,\xi_n^j]}$ with different index do not intersect and hence,

$$\Big\|(-4t^2\Delta)^{\frac14}\mathcal{M}_1(-t)(\Psi_j)_{[h_n^j,\xi_n^j]}\overline{\mathcal{M}_{\frac12}(-t)(\Phi_k)_{[h_n^k,\xi_n^k]}}\Big\|_{L_t^{\frac65,2}([0,\infty);L_x^{\frac{18}{11}})} = 0$$

and

$$\Big\|(-t^2\Delta)^{\frac14}\mathcal{M}_{\frac12}(-t)(\Phi_j)_{[h_n^j,\xi_n^j]}\mathcal{M}_{\frac12}(-t)(\Phi_k)_{[h_n^k,\xi_n^k]}\Big\|_{L_t^{\frac65,2}([0,\infty);L_x^{\frac{18}{11}})} = 0$$

hold for such n and (j,k). Therefore, it suffices to prove under an additional assumption $\sup_{n\in\mathbb{N}}(\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j}) \le \sqrt{2+2\frac{M}{m}}$ for any $1 \le j,k \le J$. Changing notations if necessary, $h_n^j \equiv h_n$ and $\operatorname{supp}\big((\Phi_j)_{[h_n,\xi_n^j]}, (\Psi_j)_{[h_n,\xi_n^j]}\big) \subset \big[\frac{m^2}{2M}\frac{1}{h_n^2}, \frac{2M^2}{m}\frac{1}{h_n^2}\big] \times B_{\frac{2MR}{mh_n}}(2t\xi_n^j)$. If $t(h_n)^2 \in [\frac{m^2}{2M}, \frac{2M^2}{m}]$, then (3.44) deduces

$$|h_n(x - 2t\xi_n^j) - h_n(x - 2t\xi_n^k)| = 2th_n|\xi_n^j - \xi_n^k| \ge \frac{m^2}{M}\left|\frac{\xi_n^j - \xi_n^k}{h_n}\right| \longrightarrow \infty \quad \text{as} \quad n \to \infty.$$

Since the spatial support of Φ_j and Ψ_j are contained in $B_R(0)$, there exists $n_0 \in \mathbb{N}$ such that

$$\Big|(-4t^2\Delta)^{\frac14}\mathcal{M}_1(-t)(\Psi_j)_{[h_n,\xi_n^j]}\overline{\mathcal{M}_{\frac12}(-t)(\Phi_k)_{[h_n,\xi_n^k]}}\Big| = 0$$

and

$$\Big|(-t^2\Delta)^{\frac14}\mathcal{M}_{\frac12}(-t)(\Phi_j)_{[h_n,\xi_n^j]}\mathcal{M}_{\frac12}(-t)(\Phi_k)_{[h_n,\xi_n^k]}\Big| = 0$$

for any $(t,x) \in [0,\infty) \times \mathbb{R}^3$ and any $n \ge n_0$. $\square$

Proof of Lemma 3.41. Set

$$(\widetilde{u}_n^{J,J_1}, \widetilde{v}_n^{J,J_1}) := (\widetilde{w}_n^J, \widetilde{z}_n^J) - (\widetilde{w}_n^{J_1}, \widetilde{z}_n^{J_1}) = \Big(\sum_{j=J_1+1}^J (\Phi_j)_{[h_n^j,\xi_n^j]}, \sum_{j=J_1+1}^J (\Psi_j)_{[h_n^j,\xi_n^j]}\Big).$$

Then, we have

$$(\widetilde{u}_n^{J,J_1}(0), \widetilde{v}_n^{J,J_1}(0)) = \sum_{j=J_1+1}^J \mathcal{G}_n^j(\phi^j, \psi^j).$$

It follows from (3.44) that

$$\lim_{n\to\infty}\|\widetilde{u}_n^{J,J_1}(0)\|_{\mathcal{F}\dot{H}^{\frac12}}^2=\sum_{j=J_1+1}^{J}\|\phi^j\|_{\mathcal{F}\dot{H}^{\frac12}}^2,\quad \lim_{n\to\infty}\|\widetilde{v}_n^{J,J_1}(0)\|_{\mathcal{F}\dot{H}^{\frac12}}^2=\sum_{j=J_1+1}^{J}\|\psi^j\|_{\mathcal{F}\dot{H}^{\frac12}}^2=0$$

for any $J>J_1\geq 1$. Here, we have

$$\sum_{j=J_1+1}^{J}\|\phi^j\|_{\mathcal{F}\dot{H}^{\frac12}}^2<\infty$$

by (3.43). Hence, for any $\varepsilon>0$, there exists J_1 such that

$$\lim_{n\to\infty}\|(\widetilde{u}_n^{J,J_1}(0),\widetilde{v}_n^{J,J_1}(0))\|_{\mathcal{F}\dot{H}^{\frac12}\times\mathcal{F}\dot{H}^{\frac12}}\leq\varepsilon \tag{3.51}$$

for any $J>J_1$. $(\widetilde{u}_n^{J,J_1},\widetilde{v}_n^{J,J_1})$ satisfies the following equation:

$$\begin{cases} i\partial_t\widetilde{u}_n^{J,J_1}+\Delta\widetilde{u}_n^{J,J_1}=-\displaystyle\sum_{j=J_1+1}^{J}F_1\big((\Phi_j)_{[h_n^j,\xi_n^j]},(\Psi_j)_{[h_n^j,\xi_n^j]}\big),\\ i\partial_t\widetilde{v}_n^{J,J_1}+\dfrac12\Delta\widetilde{v}_n^{J,J_1}=-\displaystyle\sum_{j=J_1+1}^{J}F_2\big((\Phi_j)_{[h_n^j,\xi_n^j]}\big),\end{cases}$$

where F_1 and F_2 are defined in Lemma 3.43. We define functions $\widetilde{e}_{1,n}^{J,J_1}$ and $\widetilde{e}_{2,n}^{J,J_1}$ as

$$\begin{cases} \widetilde{e}_{1,n}^{J,J_1}:=F_1\Big(\displaystyle\sum_{j=J_1+1}^{J}(\Phi_j)_{[h_n^j,\xi_n^j]},\sum_{j=J_1+1}^{J}(\Psi_j)_{[h_n^j,\xi_n^j]}\Big)-\sum_{j=J_1+1}^{J}F_1\big((\Phi_j)_{[h_n^j,\xi_n^j]},(\Psi_j)_{[h_n^j,\xi_n^j]}\big),\\ \widetilde{e}_{2,n}^{J,J_1}:=F_2\Big(\displaystyle\sum_{j=J_1+1}^{J}(\Phi_j)_{[h_n^j,\xi_n^j]}\Big)-\sum_{j=J_1+1}^{J}F_2\big((\Phi_j)_{[h_n^j,\xi_n^j]}\big).\end{cases}$$

The above equation satisfied by $(\widetilde{u}_n^{J,J_1},\widetilde{v}_n^{J,J_1})$ can be rewritten as

$$\begin{cases} i\partial_t\widetilde{u}_n^{J,J_1}+\Delta\widetilde{u}_n^{J,J_1}+F_1(\widetilde{u}_n^{J,J_1},\widetilde{v}_n^{J,J_1})=\widetilde{e}_{1,n}^{J,J_1},\\ i\partial_t\widetilde{v}_n^{J,J_1}+\dfrac12\Delta\widetilde{v}_n^{J,J_1}+F_2(\widetilde{u}_n^{J,J_1})=\widetilde{e}_{2,n}^{J,J_1}.\end{cases}$$

We also rewrite the differential equation to a integral equation.

$$\begin{cases} \widetilde{u}_n^{J,J_1}(t)=e^{it\Delta}\widetilde{u}_n^{J,J_1}(0)+i\displaystyle\int_0^t e^{i(t-s)\Delta}\big\{F_1(\widetilde{u}_n^{J,J_1}(s),\widetilde{v}_n^{J,J_1}(s))-\widetilde{e}_{1,n}^{J,J_1}(s)\big\}ds,\\ \widetilde{v}_n^{J,J_1}(t)=e^{\frac12 it\Delta}\widetilde{v}_n^{J,J_1}(0)+i\displaystyle\int_0^t e^{\frac12 i(t-s)\Delta}\big\{F_2(\widetilde{u}_n^{J,J_1}(s))-\widetilde{e}_{2,n}^{J,J_1}(s)\big\}ds.\end{cases}$$

Using Proposition 3.5 and 3.12,

$$\|\widetilde{u}_n^{J,J_1}\|_{W_1([0,\infty))}\leq c\|\widetilde{u}_n^{J,J_1}(0)\|_{\mathcal{F}\dot{H}^{\frac12}}+c\|\widetilde{u}_n^{J,J_1}\|_{W_1([0,\infty))}\|\widetilde{v}_n^{J,J_1}\|_{W_2([0,\infty))}+c\|\widetilde{e}_{1,n}^{J,J_1}\|_{N_1([0,\infty))},$$

$$\|\widetilde{v}_n^{J,J_1}\|_{W_2([0,\infty))}\leq c\|\widetilde{v}_n^{J,J_1}(0)\|_{\mathcal{F}\dot{H}^{\frac12}}+c\|\widetilde{u}_n^{J,J_1}\|_{W_1([0,\infty))}^2+c\|\widetilde{e}_{2,n}^{J,J_1}\|_{N_2([0,\infty))}.$$

Combining these inequalities,

$$\begin{aligned}\|(\widetilde{u}_n^{J,J_1},\widetilde{v}_n^{J,J_1})\|_{W_1([0,\infty))\times W_2([0,\infty))}&\leq c\|(\widetilde{u}_n^{J,J_1}(0),\widetilde{v}_n^{J,J_1}(0))\|_{\mathcal{F}\dot{H}^{\frac12}\times\mathcal{F}\dot{H}^{\frac12}}\\ &\quad+c\|(\widetilde{u}_n^{J,J_1},\widetilde{v}_n^{J,J_1})\|_{W_1([0,\infty))\times W_2([0,\infty))}^2+c\|(\widetilde{e}_{1,n}^{J,J_1},\widetilde{e}_{2,n}^{J,J_1})\|_{N_1([0,\infty))\times N_2([0,\infty))}.\end{aligned} \tag{3.52}$$

Lemma 3.43 deduces

$$\lim_{n\to\infty}\|(\widetilde{e}_{1,n}^{J,J_1},\widetilde{e}_{2,n}^{J,J_1})\|_{N_1([0,\infty))\times N_2([0,\infty))}=0 \tag{3.53}$$

for any $J\geq J_1$. Combining (3.51), (3.52), and (3.53),

$$\|(\widetilde{u}_n^{J,J_1},\widetilde{v}_n^{J,J_1})\|_{W_1([0,\infty))\times W_2([0,\infty))}\leq c\varepsilon+c\|(\widetilde{u}_n^{J,J_1},\widetilde{v}_n^{J,J_1})\|_{W_1([0,\infty))\times W_2([0,\infty))}^2 \tag{3.54}$$

for any $J > J_1$ and $n \geq n_0(J, J_1, \varepsilon)$. It follows that there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ then the inequality (3.54) implies

$$\|(\widetilde{u}_n^{J,J_1}, \widetilde{v}_n^{J,J_1})\|_{W_1([0,\infty))\times W_2([0,\infty))} \leq 2c\varepsilon,$$

which completes the proof. □

Proof of Lemma 3.42. By the definitions of $\widetilde{e}_{1,n}^J$ and $\widetilde{e}_{2,n}^J$,

$$\begin{aligned}
\|\widetilde{e}_{1,n}^J\|_{N_1([0,\infty))} &= \Big\|F_1(\widetilde{w}_n^J + e^{it\Delta}R_n^J, \widetilde{z}_n^J) - \sum_{j=1}^{J} F_1\big((\Phi_j)_{[h_n^j,\xi_n^j]}, (\Psi_j)_{[h_n^j,\xi_n^j]}\big)\Big\|_{N_1([0,\infty))} \\
&\leq \|F_1(\widetilde{w}_n^{J_0} + e^{it\Delta}R_n^J, \widetilde{z}_n^{J_0}) - F_1(\widetilde{w}_n^{J_0}, \widetilde{z}_n^{J_0})\|_{N_1([0,\infty))} \\
&\qquad + \|F_1(\widetilde{w}_n^J + e^{it\Delta}R_n^J, \widetilde{z}_n^J) - F_1(\widetilde{w}_n^{J_0} + e^{it\Delta}R_n^J, \widetilde{z}_n^{J_0})\|_{N_1([0,\infty))} \\
&\qquad + \|F_1(\widetilde{w}_n^J, \widetilde{z}_n^J) - F_1(\widetilde{w}_n^{J_0}, \widetilde{z}_n^{J_0})\|_{N_1([0,\infty))} \\
&\qquad + \Big\|F_1(\widetilde{w}_n^J, \widetilde{z}_n^J) - \sum_{j=1}^{J} F_1\big((\Phi_j)_{[h_n^j,\xi_n^j]}, (\Psi_j)_{[h_n^j,\xi_n^j]}\big)\Big\|_{N_1([0,\infty))}, \\
\|\widetilde{e}_{2,n}^J\|_{N_2([0,\infty))} &\leq \|F_2(\widetilde{w}_n^{J_0} + e^{it\Delta}R_n^J) - F_2(\widetilde{w}^{J_0})\|_{N_2([0,\infty))} \\
&\qquad + \|F_2(\widetilde{w}_n^J + e^{it\Delta}R_n^J) - F_2(\widetilde{w}_n^{J_0} + e^{it\Delta}R_n^J)\|_{N_2([0,\infty))} \\
&\qquad + \|F_2(\widetilde{w}_n^J) - F_2(\widetilde{w}_n^{J_0})\|_{N_2([0,\infty))} \\
&\qquad + \Big\|F_2(\widetilde{w}_n^J) - \sum_{j=1}^{J} F_2\big((\Phi_j)_{[h_n^j,\xi_n^j]}\big)\Big\|_{N_2([0,\infty))}.
\end{aligned}$$

The last term of the right hand side goes to zero as $n \to \infty$ for all J by Lemma 3.43. Moreover, the second and the third terms become small if we take J_0 sufficiently large by Lemma 3.41 and 3.12. Indeed, we have

$$\begin{aligned}
&\|F_1(\widetilde{w}_n^J + e^{it\Delta}R_n^J, \widetilde{z}_n^J) - F_1(\widetilde{w}_n^{J_0} + e^{it\Delta}R_n^J, \widetilde{z}_n^{J_0})\|_{N_1([0,\infty))} \\
&\leq 2\|\widetilde{z}_n^J\overline{(\widetilde{w}_n^J - \widetilde{w}_n^{J_0})}\|_{N_1([0,\infty))} + 2\|\overline{\widetilde{w}_n^{J_0}}(\widetilde{z}_n^J - \widetilde{z}_n^{J_0})\|_{N_1([0,\infty))} + 2\|(\widetilde{z}_n^J - \widetilde{z}_n^{J_0})\overline{e^{it\Delta}R_n^J}\|_{N_1([0,\infty))} \\
&\lesssim \|\widetilde{z}_n^J\|_{W_2([0,\infty))}\|\widetilde{w}_n^J - \widetilde{w}_n^{J_0}\|_{W_1([0,\infty))} + \big\{\|\widetilde{w}_n^{J_0}\|_{W_1([0,\infty))} + \|e^{it\Delta}R_n^J\|_{W_1([0,\infty))}\big\}\|\widetilde{z}_n^J - \widetilde{z}_n^{J_0}\|_{W_2([0,\infty))},
\end{aligned}$$

$$\begin{aligned}
&\|F_1(\widetilde{w}_n^J, \widetilde{z}_n^J) - F_1(\widetilde{w}_n^{J_0}, \widetilde{z}_n^{J_0})\|_{N_1([0,\infty))} \\
&\qquad \lesssim \|\widetilde{z}_n^J\|_{W_2([0,\infty))}\|\widetilde{w}_n^J - \widetilde{w}_n^{J_0}\|_{W_1([0,\infty))} + \|\widetilde{w}_n^{J_0}\|_{W_1([0,\infty))}\|\widetilde{z}_n^J - \widetilde{z}_n^{J_0}\|_{W_2([0,\infty))},
\end{aligned}$$

$$\begin{aligned}
&\|F_2(\widetilde{w}_n^J + e^{it\Delta}R_n^J) - F_2(\widetilde{w}_n^{J_0} + e^{it\Delta}R_n^J)\|_{N_2([0,\infty))} \\
&\qquad = \|(\widetilde{w}_n^J + \widetilde{w}_n^{J_0} + 2e^{it\Delta}R_n^J)(\widetilde{w}_n^J - \widetilde{w}_n^{J_0})\|_{N_2([0,\infty))} \\
&\qquad \lesssim \|\widetilde{w}_n^J + \widetilde{w}_n^{J_0} + 2e^{it\Delta}R_n^J\|_{W_1([0,\infty))}\|\widetilde{w}_n^J - \widetilde{w}_n^{J_0}\|_{W_1([0,\infty))} \\
&\qquad \lesssim \big\{\|\widetilde{w}_n^J\|_{W_1([0,\infty))} + \|\widetilde{w}_n^{J_0}\|_{W_1([0,\infty))} + \|e^{it\Delta}R_n^J\|_{W_1([0,\infty))}\big\}\|\widetilde{w}_n^J - \widetilde{w}_n^{J_0}\|_{W_1([0,\infty))},
\end{aligned}$$

and

$$\|F_2(\widetilde{w}_n^J) - F_2(\widetilde{w}_n^{J_0})\|_{N_2([0,\infty))} \lesssim \big\{\|\widetilde{w}_n^J\|_{W_1([0,\infty))} + \|\widetilde{w}_n^{J_0}\|_{W_1([0,\infty))}\big\}\|\widetilde{w}_n^J - \widetilde{w}_n^{J_0}\|_{W_1([0,\infty))}.$$

Thus, we have to estimate the first term. We may assume by Proposition 3.16 that $\operatorname{supp}(\Phi_j, \Psi_j) \subset [m, M] \times B_0(R)$ holds for some $m, M, R > 0$ and for all $j \in [1, J_0]$. We set

$$I_n^j := [(h_n^j)^{-2}m, (h_n^j)^{-2}M].$$

If $t \notin I_n^j$ then $\big((\Phi_j)_{[h_n^j,\xi_n^j]}, (\Psi_j)_{[h_n^j,\xi_n^j]}\big) = (0, 0)$ by the assumption for time support of (Φ_j, Ψ_j). Thus, we have

$$\|F_1(\widetilde{w}_n^{J_0} + e^{it\Delta}R_n^J, \widetilde{z}_n^{J_0}) - F_1(\widetilde{w}_n^{J_0}, \widetilde{z}_n^{J_0})\|_{N_1([0,\infty)\setminus\cup_{j=1}^{J_0} I_n^j)} = 0$$

and

$$\|F_2(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J)-F_2(\widetilde{w}^{J_0})\|_{N_2([0,\infty)\setminus(\cup_{j=1}^{J_0}I_n^j))}=\|F_2(e^{it\Delta}R_n^J)\|_{N_2([0,\infty)\setminus\cup_{j=1}^{J_0}I_n^j)}.$$

By Lemma 3.12 and (3.45),

$$\begin{aligned}\limsup_{n\to\infty}\|F_2(e^{it\Delta}R_n^J)\|_{N_2([0,\infty)\setminus\cup_{j=1}^{J_0})}&\le\limsup_{n\to\infty}\|F_2(e^{it\Delta}R_n^J)\|_{N_2([0,\infty))}\\&\lesssim\limsup_{n\to\infty}\|e^{it\Delta}R_n^J\|_{W_1([0,\infty))}\|e^{it\Delta}R_n^J\|_{S^{\mathrm{weak}}([0,\infty))}\\&\longrightarrow 0\ \text{ as }\ J\to\infty.\end{aligned}$$

We shall consider

$$\|F_1(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J,\widetilde{z}_n^{J_0})-F_1(\widetilde{w}_n^{J_0},\widetilde{z}_n^{J_0})\|_{N_1(\cup_{j=1}^{J_0}I_n^j)}$$

and

$$\|F_2(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J)-F_2(\widetilde{w}^{J_0})\|_{N_2(\cup_{j=1}^{J_0}I_n^j)}.$$

We only have to treat the case: $\frac{h_n^j}{h_n^k}+\frac{h_n^k}{h_n^j}$ is bounded for any n and $1\le j,k\le J_0$ since $\frac{h_n^j}{h_n^k}+\frac{h_n^k}{h_n^j}\ge\frac{m}{M}+\frac{M}{m}$ implies $I_n^j\cap I_n^k=\emptyset$. Changing scales and notations if necessary, we may assume $h_n^j\equiv 1$. Then, orthogonality (3.44) becomes $|\xi_n^j-\xi_n^k|\longrightarrow\infty$ as $n\to\infty$ for any $j\ne k$. We want to estimate

$$\|F_1(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J,\widetilde{z}_n^{J_0})-F_1(\widetilde{w}_n^{J_0},\widetilde{z}_n^{J_0})\|_{N_1([m',M'])}$$

and

$$\|F_2(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J)-F_2(\widetilde{w}^{J_0})\|_{N_2([m',M'])},$$

where $m':=\frac{m^2}{M}<m$ and $M':=\frac{M^2}{m}>M$. We set $\chi_n^j(t,x)=\mathbf{1}_{[m',M']}(t)\mathscr{Y}_R(x-2t\xi_n^j)$, where $\mathscr{Y}_R$ is defined as (2.2). We recall

$$(\Phi_j)_{[1,\xi_n^j]}(t,x)=e^{ix\cdot\xi_n^j}e^{-it|\xi_n^j|^2}\Phi_j(t,x-2t\xi_n^j)$$

and

$$(\Psi_j)_{[1,\xi_n^j]}(t,x)=e^{2ix\cdot\xi_n^j}e^{-2it|\xi_n^j|^2}\Psi_j(t,x-2t\xi_n^j).$$

Hence,

$$\operatorname{supp}\big((\Phi_j)_{[1,\xi_n^j]},(\Psi_j)_{[1,\xi_n^j]}\big)\subset\operatorname{supp}\chi_n^j\subset\bigcup_{m'\le t\le M'}(\{t\}\times B_{2R}(2t\xi_n^j))=:\Sigma_n^j.$$

By the orthogonality (3.44), Σ_n^j $(1\le j\le J_0)$ are pairwise disjoint for large n. For such n, we have

$$\chi_n^k(\Phi_j)_{[1,\xi_n^j]}=\begin{cases}(\Phi_j)_{[1,\xi_n^j]}, & (k=j),\\ 0, & (k\ne j),\end{cases}\quad\text{and}\quad\chi_n^k(\Psi_j)_{[1,\xi_n^j]}=\begin{cases}(\Psi_j)_{[1,\xi_n^j]}, & (k=j),\\ 0, & (k\ne j).\end{cases}$$

Let $\widetilde{\chi}_n^j=(\chi_n^j)^2$. Then,

$$\widetilde{\chi}_n^jF_1(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J,\widetilde{z}_n^{J_0})=F_1(\chi_n^j\widetilde{w}_n^{J_0}+\chi_n^je^{it\Delta}R_n^J,\chi_n^j\widetilde{z}_n^{J_0})=F_1\big((\Phi_j)_{[1,\xi_n^j]}+\chi_n^je^{it\Delta}R_n^J,(\Psi_j)_{[1,\xi_n^j]}\big)$$

and

$$\widetilde{\chi}_n^jF_2(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J)=F_2(\chi_n^j\widetilde{w}_n^{J_0}+\chi_n^je^{it\Delta}R_n^J)=F_2\big((\Phi_j)_{[1,\xi_n^j]}+\chi_n^je^{it\Delta}R_n^J\big)$$

for each $1\le j\le J_0$ and provided sufficiently large n. Similarly, we have

$$\widetilde{\chi}_n^jF_1(\widetilde{w}_n^{J_0},\widetilde{z}_n^{J_0})=F_1(\chi_n^j\widetilde{w}_n^{J_0},\chi_n^j\widetilde{z}_n^{J_0})=F_1\big((\Phi_j)_{[1,\xi_n^j]},(\Psi_j)_{[1,\xi_n^j]}\big)$$

and

$$\widetilde{\chi}_n^jF_2(\widetilde{w}_n^{J_0})=F_2(\chi_n^j\widetilde{w}_n^{J_0})=F_2\big((\Phi_j)_{[1,\xi_n^j]}\big)$$

for large n. Further, we may see that $1-\sum_{j=1}^{J}\widetilde{\chi}_n^j \equiv 0$ on $\cup_{j=1}^{J_0}\operatorname{supp}(\Phi_j)_{[1,\xi_n^j]}(t,x)$. Therefore,

$$\Big(1-\sum_{j=1}^{J}\widetilde{\chi}_n^j\Big)F_1(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J,\widetilde{z}_n^{J_0})=0,$$

$$\Big(1-\sum_{j=1}^{J}\widetilde{\chi}_n^j\Big)F_2(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J)=\Big(1-\sum_{j=1}^{J}\widetilde{\chi}_n^j\Big)F_2(e^{it\Delta}R_n^J),$$

$$\Big(1-\sum_{j=1}^{J}\widetilde{\chi}_n^j\Big)F_1(\widetilde{w}_n^{J_0},\widetilde{z}_n^{J_0})\equiv 0,\quad \text{and}\quad \Big(1-\sum_{j=1}^{J}\widetilde{\chi}_n^j\Big)F_2(\widetilde{w}_n^{J_0})\equiv 0.$$

Thus, it follows for large n that

$$\begin{aligned}
&\|F_1(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J,\widetilde{z}_n^{J_0})-F_1(\widetilde{w}_n^{J_0},\widetilde{z}_n^{J_0})\|_{N_1([m',M'])}\\
&\quad=\Big\|\sum_{j=1}^{J_0}F_1\big((\Phi_j)_{[1,\xi_n^j]}+\chi_n^je^{it\Delta}R_n^J,(\Psi_j)_{[1,\xi_n^j]}\big)-\sum_{j=1}^{J_0}F_1\big((\Phi_j)_{[1,\xi_n^j]},(\Psi_j)_{[1,\xi_n^j]}\big)\Big\|_{N_1([m',M'])}\\
&\quad\le\sum_{j=1}^{J_0}\big\|F_1\big((\Phi_j)_{[1,\xi_n^j]}+\chi_n^je^{it\Delta}R_n^J,(\Psi_j)_{[1,\xi_n^j]}\big)-F_1\big((\Phi_j)_{[1,\xi_n^j]},(\Psi_j)_{[1,\xi_n^j]}\big)\big\|_{N_1([m',M'])}\\
&\quad=\sum_{j=1}^{J_0}\big\|F_1\big(\chi_n^je^{it\Delta}R_n^J,(\Psi_j)_{[1,\xi_n^j]}\big)\big\|_{N_1([m',M'])}=:I
\end{aligned}$$

and

$$\begin{aligned}
&\|F_2(\widetilde{w}_n^{J_0}+e^{it\Delta}R_n^J)-F_2(\widetilde{w}^{J_0})\|_{N_2([m',M'])}\\
&=\Big\|\Big(1-\sum_{j=1}^{J_0}\widetilde{\chi}_n^j\Big)F_2(e^{it\Delta}R_n^J)+\sum_{j=1}^{J_0}F_2\big((\Phi_j)_{[1,\xi_n^j]}+\chi_n^je^{it\Delta}R_n^J\big)-\sum_{j=1}^{J_0}F_2\big((\Phi_j)_{[1,\xi_n^j]}\big)\Big\|_{N_2([m',M'])}\\
&\le\sum_{j=1}^{J_0}\big\|F_2\big((\Phi_j)_{[1,\xi_n^j]}+\chi_n^je^{it\Delta}R_n^J\big)-F_2\big((\Phi_j)_{[1,\xi_n^j]}\big)\big\|_{N_2([m',M'])}\\
&\qquad\qquad+\|F_2(e^{it\Delta}R_n^J)\|_{N_2([m',M'])}+\sum_{j=1}^{J_0}\|F_2(\chi_n^je^{it\Delta}R_n^J)\|_{N_2([m',M'])}\\
&=:II+III+IV.
\end{aligned}$$

First, we estimate III. By Lemma 3.12, we have

$$III\lesssim\|e^{it\Delta}R_n^J\|_{W_1([m',M'])}\|e^{it\Delta}R_n^J\|_{S^{\mathrm{weak}}([m',M'])}\lesssim\|R_n^J\|_{\mathcal{F}\dot{H}^{\frac12}}\|e^{it\Delta}R_n^J\|_{S^{\mathrm{weak}}([m',M'])}.$$

Since R_n^J is uniformly bounded in $\mathcal{F}\dot{H}^{\frac12}(\mathbb{R}^3)$, Lemma 3.12 and (3.45) deduce

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|F_2(e^{it\Delta}R_n^J)\|_{N_2([m',M'])}=0.$$

Next, we estimate IV. We change of variable $x-2t\xi_n^j=y$ and apply (1.13). Then, we have

$$\|F_2(\chi_n^je^{it\Delta}R_n^J)\|_{N_2([m',M'])}=\|F_2(\mathscr{Y}_Re^{it\Delta}e^{-i\xi_n^j\cdot x}R_n^J)\|_{N_2([m',M'])}.$$

Lemma 3.11 and Proposition 3.5 imply $\|\mathscr{Y}_Re^{it\Delta}R_n^J\|_{W_1([m',M'])}$ is uniformly bounded. Thus, using (1.13) and (3.45), it follows that

$$\begin{aligned}
\|F_2(\chi_n^je^{it\Delta}R_n^J)\|_{N_2([m',M'])}&\lesssim\|\mathscr{Y}_Re^{it\Delta}e^{-i\xi_n^j\cdot x}R_n^J\|_{S^{\mathrm{weak}}([m',M'])}\|\mathscr{Y}_Re^{it\Delta}e^{-i\xi_n^j\cdot x}R_n^J\|_{W_1([m',M'])}\\
&\lesssim\|e^{it\Delta}e^{-i\xi_n^j\cdot x}R_n^J\|_{S^{\mathrm{weak}}([m',M'])}\\
&=\|e^{-it|\xi_n^j|^2-ix\cdot\xi_n^j}(e^{it\Delta}R_n^J)(x+2t\xi_n^j)\|_{S^{\mathrm{weak}}([m',M'])}
\end{aligned}$$

$$= \|e^{it\Delta}R_n^J\|_{S^{\mathrm{weak}}([m',M'])} \longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

Finally, we estimate I and II. We consider only the case $j=1$, that is, we deal with

$$\big\|F_1\big(\chi_n^1 e^{it\Delta}R_n^J, (\Psi_1)_{[1,\xi_n^1]}\big)\big\|_{N_1([m',M'])}$$

and

$$\big\|F_2\big((\Phi_1)_{[1,\xi_n^1]} + \chi_n^1 e^{it\Delta}R_n^J\big) - F_2\big((\Phi_1)_{[1,\xi_n^1]}\big)\big\|_{N_2([m',M'])}.$$

By the same argument with the proof of Lemma 3.12, we have

$$\begin{aligned}
&\|F_1(\chi_n^1 e^{it\Delta}R_n^J, (\Psi_1)_{[1,\xi_n^1]})\|_{N_1([m',M'])}\\
&\quad = 2\|(-4t^2\Delta)^{\frac14}\overline{\mathcal{M}_{\frac12}(-t)\chi_n^1 e^{it\Delta}R_n^J}\mathcal{M}_1(-t)(\Psi_1)_{[1,\xi_n^1]}\|_{L_t^{\frac65,2}([m',M'];L_x^{\frac{18f}{11}})}\\
&\quad \lesssim \|(-4t^2\Delta)^{\frac14}\mathcal{M}_{\frac12}(-t)\chi_n^1 e^{it\Delta}R_n^J\|_{L_t^{4,\infty}([m',M'];L_x^2)}\|(\Psi_1)_{[1,\xi_n^1]}\|_{L_t^{\frac{12}{7},2}([m',M'];L_x^9)}\\
&\qquad + \|\chi_n^1 e^{it\Delta}R_n^J\|_{L_t^{\frac32,\infty}([m',M'];L_x^{\frac92})}\|(-t^2\Delta)^{\frac14}\mathcal{M}_1(-t)(\Psi_1)_{[1,\xi_n^1]}\|_{L_t^{6,2}([m',M'];L_x^{\frac{18}{7}})}\\
&\quad \lesssim \|\chi_n^1 e^{it\Delta}R_n^J\|_{L_t^{4,\infty}([m',M'];\dot X^{\frac12}_{1/2})}\|(\Psi_1)_{[1,\xi_n^1]}\|_{L_t^{\frac{12}{7},2}([m',M'];L_x^9)}\\
&\qquad + \|\chi_n^1 e^{it\Delta}R_n^J\|_{S^{\mathrm{weak}}([m',M'])}\|(\Psi_1)_{[1,\xi_n^1]}\|_{W_1([m',M'])}.
\end{aligned}$$

$\|(\Phi_1)_{[1,\xi_n^1]}\|_{W_1([m',M'])}$ is uniformly bounded in n. From (3.45), we have

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|\chi_n^1 e^{it\Delta}R_n^J\|_{S^{\mathrm{weak}}([m',M'])} \le \lim_{J\to\infty}\limsup_{n\to\infty}\|e^{it\Delta}R_n^J\|_{S^{\mathrm{weak}}([m',M'])} = 0.$$

By Lemma 3.8,

$$\|(\Psi_1)_{[1,\xi_n^1]}\|_{L_t^{\frac{12}{7},2}([m',M'];L_x^9)} \lesssim \|(\Psi_1)_{[1,\xi_n^1]}\|^{\frac12}_{L_t^{\frac65,2}([m',M'];L_x^9)}\|(\Psi_1)_{[1,\xi_n^1]}\|^{\frac12}_{L_t^{3,2}([m',M'];L_x^9)}.$$

Lemma 2.3 and (3.3) deduce

$$\begin{aligned}
\|(\Psi_1)_{[1,\xi_n^1]}\|_{L_x^9} &= \|\mathcal{M}_{\frac12}(-t)(\Psi_1)_{[1,\xi_n^1]}\|_{L_x^9} \lesssim \||\nabla|^{\frac12}\mathcal{M}_{\frac12}(-t)(\Psi_1)_{[1,\xi_n^1]}\|_{L_x^{\frac{18}{5}}}\\
&= |t|^{-\frac12}\||t|^{\frac12}|\nabla|^{\frac12}\mathcal{M}_{\frac12}(-t)(\Psi_1)_{[1,\xi_n^1]}\|_{L_x^{\frac{18}{5}}} \lesssim |t|^{-\frac12}\|(\Psi_1)_{[1,\xi_n^1]}\|_{\dot X^{\frac12,\frac{18}{5}}_{1/2}}
\end{aligned}$$

and hence,

$$\begin{aligned}
\|(\Psi_1)_{[1,\xi_n^1]}\|_{L_t^{3,2}([m',M'];L_x^9)} &\lesssim \||t|^{-\frac12}\|_{L_t^\infty([m',M'])}\|(\Psi_1)_{[1,\xi_n^1]}\|_{L_t^{3,2}([m',M'];\dot X^{\frac12,\frac{18}{5}}_{1/2})}\\
&= (m')^{-\frac12}\|(\Psi_1)_{[1,\xi_n^1]}\|_{L_t^{3,2}([m',M'];\dot X^{\frac12,\frac{18}{5}}_{1/2})}.
\end{aligned}$$

On the other hand, we have

$$\|(\Psi_1)_{[1,\xi_n^1]}\|_{L_t^{\frac65,2}([m',M'];L_x^9)} \lesssim \|(\Psi_1)_{[1,\xi_n^1]}\|_{L_t^{3,2}([m',M'];\dot X^{\frac12,\frac{18}{5}}_{1/2})} = \|\Psi_1\|_{L_t^{3,2}([m',M'];\dot X^{\frac12,\frac{18}{5}}_{1/2})}$$

by Lemma 3.14, where we note that $(3,\frac{18}{5})$ is an admissible pair. Therefore, to give desired estimate on $[m',M']$, it suffices to show that

$$\limsup_{n\to\infty}\|\mathscr{Y}_R e^{it\Delta}e^{-i\xi_n^1\cdot x}R_n^J\|_{L_t^2([m',M'];\dot X^{\frac12}_{1/2})} \longrightarrow 0 \quad \text{as} \quad J\to\infty. \tag{3.55}$$

Since the multiplication by $\mathscr{Y}_R$ is a bounded operator on $L^\infty([m',M'];\dot X^{\frac12}_{1/2})$ from Lemma 3.11, if we obtain (3.55), then

$$\begin{aligned}
\|\mathscr{Y}_R e^{it\Delta}e^{-i\xi_n^1\cdot x}R_n^J\|_{L_t^{4,\infty}([m',M'];\dot X^{\frac12}_{1/2})} &\le \|\mathscr{Y}_R e^{it\Delta}e^{-i\xi_n^1\cdot x}R_n^J\|_{L_t^{4}([m',M'];\dot X^{\frac12}_{1/2})}\\
&\lesssim \|\mathscr{Y}_R e^{it\Delta}e^{-i\xi_n^1\cdot x}R_n^J\|^{\frac12}_{L_t^2([m',M'];\dot X^{\frac12}_{1/2})}\|\mathscr{Y}_R e^{it\Delta}e^{-i\xi_n^1\cdot x}R_n^J\|^{\frac12}_{L_t^\infty([m',M'];\dot X^{\frac12}_{1/2})}
\end{aligned}$$

$$\lesssim \|\mathscr{Y}_R e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_t^2([m',M'];\dot{X}_{1/2}^{\frac12})}^{\frac12} \|R_n^J\|_{\mathcal{F}\dot{H}^{\frac12}}^{\frac12} \longrightarrow 0 \quad \text{as} \quad J\to\infty.$$

Let us check (3.55). It follows that

$$\|\mathscr{Y}_R e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{\dot{X}_{1/2}^{\frac12}} \sim |t|^{\frac12} \||\nabla|^{\frac12} \mathcal{M}_{\frac12}(-t)\mathscr{Y}_R e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^2}$$

for $t\neq 0$ by (3.3). We take $0<\delta<\frac13$, $\frac13<\theta<1$, $\frac{18}{5}<q_1<\frac92$, and $\frac{18}{5}<q_2<\frac92$ satisfying

$$\frac{1-2\delta}{2} = \frac{2(1-\theta)}{3} + \frac{\theta}{6},\quad \frac{1}{q_1} = \frac29 + \frac{\theta}{6} = \frac{2(1-\theta)}{9} + \frac{7\theta}{18},\quad \frac12 = \frac{1}{q_1} + \frac{1}{q_2}.$$

Using Lemma 3.9,

$$\begin{aligned}
&\|\mathscr{Y}_R e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{\dot{X}_{1/2}^{\frac12}} \\
&\sim |t|^{\frac12} \||\nabla|^{\frac12} \mathcal{M}_{\frac12}(-t)\mathscr{Y}_R e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^2} \\
&\lesssim |t|^{\frac12} \|\mathscr{Y}_R |\nabla|^{\frac12} \mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^2} + |t|^{\frac12} \|\mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J |\nabla|^{\frac12} \mathscr{Y}_R\|_{L_x^2} \\
&\qquad + |t|^{\frac12} \||\nabla|^{\frac{\theta}{2}} \mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^{q_1}} \||\nabla|^{\frac12(1-\theta)} \mathscr{Y}_R\|_{L_x^{q_2}} \\
&\leq |t|^{\frac12} \|\mathscr{Y}_R |\nabla|^{\frac12} \mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^2} + |t|^{\frac12} \|\mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^{\frac92}} \||\nabla|^{\frac12} \mathscr{Y}_R\|_{L_x^{\frac{18}{5}}} \\
&\qquad + |t|^{\frac12} \||\nabla|^{\frac{\theta}{2}} \mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^{q_1}} \||\nabla|^{\frac12(1-\theta)} \mathscr{Y}_R\|_{L_x^{q_2}} \\
&\leq |t|^{\frac12} \|\mathscr{Y}_R |\nabla|^{\frac12} \mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^2} + C_{\mathscr{Y}_R} |t|^{\frac12} \|\mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^{\frac92}} \\
&\qquad + C_{\mathscr{Y}_R} |t|^{\frac12} \||\nabla|^{\frac{\theta}{2}} \mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^{q_1}}.
\end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned}
|t|^{\frac12} \|\mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_x^{\frac92}} &\lesssim |t|^{\frac12(1-\theta)} \||t|^{\frac{\theta}{2}} |\nabla|^{\frac{\theta}{2}} \mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L^{q_1}} \\
&\sim |t|^{\frac12(1-\theta)} \|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{\dot{X}_{\frac12}^{\frac{\theta}{2},q_1}} \\
&\leq (M')^{\frac12(1-\theta)} \|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{\dot{X}_{\frac12}^{\frac{\theta}{2},q_1}}
\end{aligned}$$

for $m'\leq t\leq M'$.

$$\begin{aligned}
|t|^{\frac12} \||\nabla|^{\frac{\theta}{2}} \mathcal{M}_{\frac12}(-t) e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L^{q_1}} &\sim |t|^{\frac12(1-\theta)} \|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{\dot{X}_{1/2}^{\frac{\theta}{2},q_1}} \\
&\leq (M')^{\frac12(1-\theta)} \|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{\dot{X}_{1/2}^{\frac{\theta}{2},q_1}}
\end{aligned}$$

for $m'\leq t\leq M'$. Applying Lemma 3.8, 3.13, and Proposition 3.5,

$$\begin{aligned}
&\|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_t^2([m',M'];\dot{X}_{1/2}^{\frac{\theta}{2},q_1})} \\
&\qquad\lesssim \|1\|_{L_t^{\frac{1}{\delta},2}([m',M'])} \|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_t^{\frac{2}{1-2\delta},\infty}([m',M'];\dot{X}_{1/2}^{\frac{\theta}{2},q_1})} \\
&\qquad\lesssim (M')^{\delta} \|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_t^{\frac32,\infty}([m',M'];\dot{X}_{1/2}^{0,\frac92})}^{1-\theta} \|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_t^{6,\infty}([m',M'];\dot{X}_{1/2}^{\frac12,\frac{18}{7}})}^{\theta} \\
&\qquad\lesssim (M')^{\delta} \|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_t^{\frac32,\infty}([m',M'];L_x^{\frac92})}^{1-\theta} \|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{L_t^{6,2}([m',M'];\dot{X}_{1/2}^{\frac12,\frac{18}{7}})}^{\theta} \\
&\qquad\lesssim (M')^{\delta} \|e^{it\Delta} e^{-i\xi_n^1\cdot x} R_n^J\|_{S^{\mathrm{weak}}([m',M'])}^{1-\theta} \|R_n^J\|_{\mathcal{F}\dot{H}^{\frac12}}^{\theta}.
\end{aligned}$$

Since It follows from (3.45) that

$$\limsup_{n\to\infty} \|e^{it\Delta}e^{-i\xi_n^1\cdot x}R_n^J\|_{S^{\mathrm{weak}}([m',M'])} = \limsup_{n\to\infty} \|e^{it\Delta}R_n^J\|_{S^{\mathrm{weak}}([m',M'])} \longrightarrow 0 \quad \text{as} \quad J\to\infty,$$

we have

$$\limsup_{n\to\infty} \|e^{it\Delta}e^{-i\xi_n^1\cdot x}R_n^J\|_{L_t^2([m',M'];\dot{X}_{1/2}^{\frac{\theta}{2},q_1})} \longrightarrow 0 \quad \text{as} \quad J\to\infty,$$

where, we note that

$$\|1\|^2_{L_t^{\frac{1}{\delta},2}([m',M'])} = \frac{1}{2}(M'-m')^{2\delta} < \frac{1}{2}(M')^{2\delta}.$$

If we also use Lemma 3.10, then we get

$$\begin{aligned}
\|\mathscr{Y}_R|t|^{\frac12}|\nabla|^{\frac12}\mathcal{M}_{\frac12}(-t)e^{it\Delta}e^{-i\xi_n^1\cdot x}R_n^J\|_{L_t^2([m',M'];L_x^2)} &\sim \|\mathscr{Y}_R e^{it\Delta}|x|^{\frac12}e^{-i\xi_n^1\cdot x}R_n^J\|_{L_t^2([m',M'];L_x^2)}\\
&\lesssim \|e^{it\Delta}|x|^{\frac12}e^{-i\xi_n^1\cdot x}R_n^J\|_{L_t^2([m',M'];L_x^2(B_{2R}(0)))}\\
&\le \varepsilon\|R_n^J\|_{\mathcal{F}\dot{H}^{\frac12}} + C_\varepsilon\|e^{it\Delta}e^{-i\xi_n^1\cdot x}R_n^J\|_{L_t^{\frac32,\infty}([0,\infty);L_x^{\frac92})},
\end{aligned}$$

where the first equivalence is used (3.2). Therefore, we conclude from (3.45) that

$$\limsup_{n\to\infty} \|\mathscr{Y}_R|t|^{\frac12}|\nabla|^{\frac12}\mathcal{M}_{\frac12}(-t)e^{it\Delta}e^{-i\xi_n^1\cdot x}R_n^J\|_{L_t^2([m',M'];L_x^2)} \longrightarrow 0 \quad \text{as} \quad J\to\infty.$$

Moreover, we can estimate

$$\begin{aligned}
&\big\|F_2\big((\Phi_1)_{[1,\xi_n^1]} + \chi_n^1 e^{it\Delta}R_n^J\big) - F_2\big((\Phi_1)_{[1,\xi_n^1]}\big)\big\|_{N_2([m',M'])}\\
&\qquad = \big\|2(\Phi_1)_{[1,\xi_n^1]}\chi_n^1 e^{it\Delta}R_n^J + (\chi_n^1 e^{it\Delta}R_n^J)^2\big\|_{N_2([m',M'])}
\end{aligned}$$

by the same argument. □

3.10. **Study of related optimization problems.** We next consider the optimizing problem $\mathcal{B}(\rho)$ defined as (1.19).

Theorem 3.44. *Let $d=3$ and $\kappa=\frac12$. $\mathcal{B}(\rho)$ is non-increasing and right continuous. Suppose $\rho\ge 0$ is such that $\mathcal{B}(\rho)<\infty$ is true. Then, there exists a minimizer (u_ρ, v_ρ) to $\mathcal{B}(\rho)$ with the following properties:*

(1) $\|u_\rho\|_{\mathcal{F}\dot{H}^{\frac12}} = \mathcal{B}(\rho)$ *and* $\|v_\rho\|_{\mathcal{F}\dot{H}^{\frac12}} \le \rho$;
(2) $(u_\rho, v_\rho) \notin \mathcal{S}_+$.

Moreover, the identity

$$\mathcal{B}(\rho) = \inf\{\ell_{v_0} : \|v_0\|_{\mathcal{F}\dot{H}^{\frac12}} \le \rho\} = \inf\{\ell^\dagger_{v_0} : \|v_0\|_{\mathcal{F}\dot{H}^{\frac12}} \le \rho\}$$

holds and the minimizer satisfies $\ell_{v_\rho} = \ell^\dagger_{v_\rho} = \|u_\rho\|_{\mathcal{F}\dot{H}^{\frac12}}$. Furthermore,

$$\sup\{L_{v_0}(\ell) : \|v_0\|_{\mathcal{F}\dot{H}^{\frac12}} \le \rho\} \lesssim_{\rho,\ell} 1.$$

for any $\ell\in[0,\mathcal{B}(\rho))$.

Proof. Non-increasing property of $\mathcal{B}(\rho)$ follows from its definition. The identity

$$\mathcal{B}(\rho) = \inf\{\ell_{v_0} : \|v_0\|_{\mathcal{F}\dot{H}^{\frac12}} \le \rho\} \tag{3.56}$$

is also immediate by its definition.

Introduce $\mathcal{B}^\dagger(\rho)$ as follows:

$$L(\ell,\rho) := \sup\{L_{v_0}(\ell) : \|v_0\|_{\mathcal{F}\dot{H}^{\frac12}} \le \rho\},$$

and

$$\mathcal{B}^\dagger(\rho) := \sup\{\ell : L(\ell,\rho)<\infty\} = \inf\{\ell : L(\ell,\rho)=\infty\} \in (0,\infty].$$

By Proposition 3.24 and 3.5, one has $L(\ell,\rho)\lesssim_\rho 1$ for $\ell \lesssim_\rho 1$. Hence, $\mathcal{B}^\dagger(\rho)>0$ for any $\rho>0$. Mimicking the argument in Proposition 3.25, we see that, for each fixed $\rho\ge 0$, $L(\ell,\rho)$ is a non-decreasing continuous function of ℓ defined on $[0,\infty)$. Further, we see that $\mathcal{B}^\dagger(\rho)$ is right continuous by a standard argument.

We now claim that

$$\mathcal{B}^\dagger(\rho) \le \mathcal{B}(\rho). \tag{3.57}$$

Indeed, we have

$$\begin{aligned}\mathcal{B}^\dagger(\rho) &= \inf\{\ell : L(\ell,\rho) = \infty\}\\ &\le \inf\{\ell : \text{There exists } v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \text{ such that } L_{v_0}(\ell) = \infty \text{ and } \|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \rho\}\\ &= \inf\{\ell^\dagger_{v_0} : \|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \rho\}\\ &\le \inf\{\ell_{v_0} : \|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \rho\} = \mathcal{B}(\rho),\end{aligned}$$

where the first inequality follows from fact that the existence of $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ such that $L_{v_0}(\ell) = \infty$ and $\|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \rho$ implies $L(\ell,\rho) = \infty$.

Fix $\rho > 0$ satisfying $\mathcal{B}(\rho) < \infty$. Take an optimizing sequence $(u_n(t), v_n(t))$ to $\mathcal{B}^\dagger(\rho)$ satisfying

$$\mathcal{B}^\dagger(\rho) - \frac{1}{n} \le \|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \mathcal{B}^\dagger(\rho), \quad \|v_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \rho$$

and

$$n \le \|(u_n, v_n)\|_{W_1([0,\infty))\times W_2([0,\infty))} < \infty,$$

where $(u_{0,n}, v_{0,n}) = (u_n(0), v_n(0))$. Then, by a similar argument to the proof of Theorem 1.39, we obtain a minimizer $(u_\rho(t), v_\rho(t))$ to $\mathcal{B}^\dagger(\rho)$, which completes the proof of $\mathcal{B}^\dagger(\rho) = \mathcal{B}(\rho)$. We omit the details of the proof but point out different respects compared with an optimizing sequence for $\ell^\dagger_{v_0}$. The biggest difference is that the second component $v_{0,n}$ of the optimizing sequence may vary in n. As a result, we do not have a priori information about the second component in the profile decompositions, hence the decomposition takes the form

$$(u_{0,n}, v_{0,n}) = \sum_{j=1}^{J} \mathcal{G}_n^j(\phi^j, \psi^j) + (R_n^J, L_n^J).$$

A contradiction argument shows there exists at least one j such that $(\phi^j, \psi^j) \notin \mathcal{S}_+$. We may let $j = 1$. One has

$$\|\phi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \limsup_{n\to\infty} \|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \mathcal{B}^\dagger(\rho) \tag{3.58}$$

and

$$\|\psi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \limsup_{n\to\infty} \|v_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \rho$$

by the Pythagorean decomposition. Since $(\phi^1, \psi^1) \notin \mathcal{S}_+$ and $\|\psi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \rho$, one has

$$\|\phi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \ge \mathcal{B}(\rho)$$

by the definition of $\mathcal{B}(\rho)$. Together with (3.57) and (3.58), we see that $\mathcal{B}(\rho) = \mathcal{B}^\dagger(\rho) = \|\phi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$ holds and that the solution corresponds to the data (ϕ^1, ψ^1) is a minimizer to both of them. By (3.56) and $(\phi^1, \psi^1) \notin \mathcal{S}_+$,

$$\mathcal{B}(\rho) = \inf\{\ell_{v_0} : \|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le \rho\} \le \ell_{\psi^1} \le \|\phi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \mathcal{B}(\rho).$$

Hence, $\|\phi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{\psi^1} = \mathcal{B}(\rho)$. Similarly, we have $\|\phi^1\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell^\dagger_{\psi^1}$. □

We now turn to the study of optimizing problem (1.16). Let us formulate the problem in an abstract setting. Let $f(x,y)$ be a function on $[0,\infty)\times[0,\infty)$ satisfying the following three conditions:

- Non-decreasing with respect to the both variables, that is,

$$0 \le x_1 \le x_2,\ 0 \le y_1 \le y_2 \Longrightarrow f(x_1, y_1) \le f(x_2, y_2).$$

- Continuous, that is, for any $(x_0, y_0) \in [0, \infty) \times [0, \infty)$,

$$\lim_{[0,\infty)\times[0,\infty)\ni(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

- $f(0, 0) = 0$.
- No leaking to the infinity, that is,

$$\inf\left\{ f(\|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, \|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}) : (u_0, v_0) \notin \mathcal{S}_+ \right\} < \min\left\{ \lim_{x\to\infty} f(x, 0), \lim_{y\to\infty} f(0, y) \right\}. \tag{3.59}$$

Let

$$\ell_f := \inf\{f(\|u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, \|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}) : (u_0, v_0) \notin \mathcal{S}_+\}.$$

Theorem 3.45. *Let $d = 3$ and $\kappa = \frac{1}{2}$. Let f satisfy the condition above. Then, it follows that*

$$\ell_f = \inf_{v_0\in\mathcal{F}\dot{H}^{\frac{1}{2}}} f(\ell_{v_0}, \|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}) = \inf_{v_0\in\mathcal{F}\dot{H}^{\frac{1}{2}}} f(\ell^\dagger_{v_0}, \|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}).$$

Furthermore, there exists a minimizer $(u^{(f)}(t), v^{(f)}(t))$ to ℓ_f such that

(1) $f(\|u^{(f)}(0)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, \|v^{(f)}(0)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}) = \ell_f$;
(2) $(u^{(f)}(t), v^{(f)}(t))$ *does not scatter;*
(3) $\|u^{(f)}(0)\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{v^{(f)}(0)} = \ell^\dagger_{v^{(f)}(0)}$.

The minimizer is not a ground state.

Remark 3.46. The condition (3.59) is hard to check for general f since one does not know much about the set $\mathcal{S}_+$. Following two are examples of a sufficient condition for (3.59) that does not involve $\mathcal{S}_+$:

- $\lim_{x\to\infty} f(x, 0) = \lim_{y\to\infty} f(0, y) = +\infty$;
- there exists a solution $(\phi_\omega, \psi_\omega)$ to (SP$_\omega$) with $\omega = 1$ such that

$$f\left(\frac{3}{4}\|\phi_\omega\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, \frac{3}{4}\|\psi_\omega\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}\right) < \min\left\{ \lim_{x\to\infty} f(x, 0), \lim_{y\to\infty} f(0, y) \right\}.$$

The problem (1.16) corresponds to the choice $f(x, y) = (x^2 + \alpha y^2)^{1/2}$. The function satisfies the first sufficient condition.

Proof. Since $\mathcal{S}_+ \neq \emptyset$, we have $\ell_f < \infty$. Take a minimizing sequence for ℓ_f, that is, take a sequence of initial data such that $(u_{0,n}, v_{0,n}) \notin \mathcal{S}_+$ and

$$\ell_f \le f(\|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, \|v_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}) \le \ell_f + \frac{1}{n}.$$

We claim that $(\|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, \|v_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}})$ is bounded. If not then $\|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$ or $\|v_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$ is not bounded. Let us consider the case $\|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$ is not bounded. Take a subsequence so that $\|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \longrightarrow \infty$ as $n \to \infty$. Then, by the nondecreasing assumption,

$$f(\|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, 0) \le f(\|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, \|v_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}).$$

Letting $n \to \infty$, one obtains $\lim_{x\to\infty} f(x, 0) \le \ell_f$. This is contradiction. Hence, the claim is proved.

Take a subsequence so that $(\|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, \|v_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}})$ converges to a point, say (x_∞, y_∞). By the continuity of f, we have

$$f(x_\infty, y_\infty) = \ell_f.$$

On the other hand, $(u_{0,n}, v_{0,n}) \notin \mathcal{S}_+$ gives us $\mathcal{B}(\|v_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}) \le \|u_{0,n}\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}$. So, by the continuity of $\mathcal{B}$, we have $\mathcal{B}(y_\infty) \le x_\infty$. Let $(u_\infty, v_\infty) \notin \mathcal{S}_+$ be the initial data of a minimizer to $\mathcal{B}(y_\infty)$ given in Theorem 3.44. It satisfies

$$\|u_\infty\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \ell_{v_\infty} = \ell_{v_\infty}^\dagger = \mathcal{B}(y_\infty) \le x_\infty \quad \text{and} \quad \|v_\infty\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \le y_\infty.$$

The minimizer is what we desired because

$$\ell_f \le f(\|u_\infty\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, \|v_\infty\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}) \le f(x_\infty, y_\infty) = \ell_f,$$

which implies that $f(\|u_\infty\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}, \|v_\infty\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}) = \ell_f$. □

3.11. **Proof of corollaries of Theorem 1.44.** We have proven Theorem 1.44 in Subsection 3.4. Let us show its corollaries.

Proof of Corollary 1.45. For given $v_0 \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ with $v_0 \neq 0$, we take

$$u_0 = v_0(x)^{\frac{1}{2}} |v_0(x)|^{\frac{1}{2}} \in \mathcal{F}\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3).$$

Then, we have

$$E(c^{\frac{1}{2}} d u_0, c v_0) \le c d^2 \|\nabla v_0\|_{L^2}^2 + \frac{c^2}{2} \|\nabla v_0\|_{L^2}^2 - 2c^2 d^2 \|v_0\|_{L^3}^3$$

for $c > 0$ and $d = \|\nabla v_0\|_{L^2} \|v_0\|_{L^3}^{-3/2}$. There exists $c_0 = c_0(v_0) > 0$ such that the right side is negative for any $c \ge c_0$. For such c, the corresponding solution does not scatter by virtue of Theorem 1.44. This also shows the bound

$$\ell_{c v_0} \le \|c^{\frac{1}{2}} d u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = c^{\frac{1}{2}} \|v_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} \|\nabla v_0\|_{L^2} \|v_0\|_{L^3}^{-3/2}.$$

We have the desired result. □

Proof of Corollary 1.46. We have

$$-\Delta \varphi - 2\mathrm{Re}(e^{i\theta} v_0) \varphi = \widetilde{e} \varphi.$$

Remark that φ is real-valued. Multiplying this identity by φ, and integrating, we have

$$\langle -\Delta \varphi, \varphi \rangle_{L^2} - \langle 2\mathrm{Re}(e^{i\theta} v_0) \varphi, \varphi \rangle_{L^2} = \widetilde{e} \langle \varphi, \varphi \rangle_{L^2}.$$

This can be rearranged as

$$\|\nabla \varphi\|_{L^2}^2 - 2\mathrm{Re} \int_{\mathbb{R}^3} v_0(x) \{e^{i\theta} \varphi(x)^2\} dx = \widetilde{e} \|\varphi\|_{L^2}^2.$$

Here, we take $u_0 = e^{-i\theta/2} \varphi$. Then,

$$\begin{aligned} E(c u_0, v_0) &= c^2 \|\nabla \varphi\|_{L^2}^2 + \frac{1}{2} \|\nabla v_0\|_{L^2}^2 - 2\mathrm{Re} \int_{\mathbb{R}^3} v_0(x) \{c^2 e^{i\theta} \varphi(x)^2\} dx \\ &= c^2 \widetilde{e} \|\varphi\|_{L^2}^2 + \frac{1}{2} \|\nabla v_0\|_{L^2}^2. \end{aligned}$$

From $\widetilde{e} < 0$, the choice $c^2 = \frac{\|\nabla v_0\|_{L^2}^2}{2|\widetilde{e}| \|\varphi\|_{L^2}^2}$ gives us $E(u_0, v_0) = 0$. Therefore, $(c u_0, v_0) \notin \mathcal{S}_+$ by Theorem 1.44. This also implies the bound

$$\ell_{v_0} \le \|c u_0\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}} = \frac{\|\varphi\|_{\mathcal{F}\dot{H}^{\frac{1}{2}}}}{\sqrt{2|\widetilde{e}|} \|\varphi\|_{L^2}} \|\nabla v_0\|_{L^2}.$$

We complete the proof. □

4. Proof of theorems for NLS with a potential

4.1. **Some tools for Section 4.** We collect some standard tools, which are used in this paper.

Lemma 4.1 (Norm equivalence, [68]). *If $d = 3$, $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, and $\|V_-\|_{\mathcal{K}} < 4\pi$, then*

$$\|f\|_{\dot{W}_V^{s,r}} \sim \|f\|_{\dot{W}^{s,r}}, \quad \|f\|_{W_V^{s,r}} \sim \|f\|_{W^{s,r}},$$

where $1 < r < \frac{3}{s}$ and $0 \leq s \leq 2$.

Theorem 4.2 (Dispersive estimate, [68]). *If $d = 3$, $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, and $\|V_-\|_{\mathcal{K}} < 4\pi$, then*

$$\|e^{it\Delta_V} f\|_{L_x^p} \lesssim |t|^{-\frac{3}{2}(\frac{1}{p'}-\frac{1}{p})} \|f\|_{L_x^{p'}}.$$

Definition 4.3 ($\dot{H}^s$-admissible and Strichartz norm)**.** We say that a pair of exponents (q, r) is called $\dot{H}^s$-admissible in three dimensions if $2 \leq q, r \leq \infty$ and

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2} - s.$$

We define Strichartz norm by

$$\|u\|_{S(L^2)} := \sup_{\substack{(q,r):L^2\text{-admissible} \\ 2\leq q\leq\infty, 2\leq r\leq 6}} \|u\|_{L_t^q L_x^r}$$

and its dual norm by

$$\|u\|_{S'(L^2)} := \inf_{\substack{(q,r):L^2\text{-admissible} \\ 2\leq q\leq\infty, 2\leq r\leq 6}} \|u\|_{L_t^{q'} L_x^{r'}}.$$

Proposition 4.4 (Strichartz estimates, [39, 76]). *If $d = 3$, $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, and $\|V_-\|_{\mathcal{K}} < 4\pi$, then the following estimates hold.*

- *(Homogeneous estimates)*

$$\|e^{it\Delta_V} f\|_{S(L^2)} \lesssim \|f\|_{L_x^2}.$$

If (q, r) is $\dot{H}^{s_c}$-admissible and is in a set Λ_{s_c} defined as

$$\Lambda_{s_c} := \begin{cases} \left\{ (q,r) : 2 \leq q \leq \infty,\ \dfrac{6}{3-2s_c} \leq r \leq \dfrac{6}{1-2s_c} \right\} & \left(0 < s_c < \dfrac{1}{2}\right), \\ \left\{ (q,r) : \dfrac{4}{3-2s_c} < q \leq \infty,\ \dfrac{6}{3-2s_c} \leq r < \infty \right\} & \left(\dfrac{1}{2} \leq s_c < 1\right), \end{cases}$$

then

$$\|e^{it\Delta_V} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}_x^{s_c}}.$$

- *(Inhomogeneous estimates) Let $t_0 \subset I$.*

$$\left\| \int_{t_0}^t e^{i(t-s)\Delta_V} F(\cdot, s) ds \right\|_{S(L^2;I)} \lesssim \|F\|_{S'(L^2;I)}.$$

If (q, r) is $\dot{H}^{s_c}$-admissible and is in a set Λ_{s_c}, then

$$\left\| \int_0^t e^{i(t-s)\Delta_V} F(\cdot, s) ds \right\|_{L_t^q(I;L_x^r)} \lesssim \||\nabla|^{s_c} F\|_{S'(L^2;I)},$$

where implicit constants are independent of f and F.

Lemma 4.5 (Fractional calculus, [18]). *Suppose $G \in C^1(\mathbb{C})$ and $s \in (0, 1]$. Let $1 < r, r_2 < \infty$ and $1 < r_1 \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then, we have*

$$\||\nabla|^s G(f)\|_{L^r} \lesssim \|G'(f)\|_{L^{r_1}} \||\nabla|^s f\|_{L^{r_2}}.$$

Lemma 4.6 (Hardy's inequality, [123]). *Let $d \geq 1$, $1 < q < \infty$, and $0 < \mu < d$. Then, we have*

$$\int_{\mathbb{R}^d} \frac{1}{|x|^\mu} |f(x)|^q dx \leq_{q,\mu} \||\nabla|^{\frac{\mu}{q}} f\|_{L_x^q}.$$

Lemma 4.7 (Radial Sobolev embedding, [112, 124]). *Let $d \geq 3$. For $f \in H^1_{rad}(\mathbb{R}^d)$ and $\frac{d-2}{2} \leq s \leq \frac{d-1}{2}$, it follows that*

$$\||\cdot|^s f\|_{L_x^\infty} \lesssim \|f\|_{H_x^1}.$$

Proposition 4.8 (Virial identity, [13, 68]). *Let u be a solution given in Theorem 1.50 or Theorem 1.51. We assume that $x^{\mathfrak{a}}\partial^{\mathfrak{a}} V \in L^\eta(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for any $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^d$ with $|\mathfrak{a}| = 1$, some $\eta \geq 1$ if $d = 1$, some $\eta > 1$ if $d = 2$, and $\eta = \frac{d}{2}$ if $d \geq 3$. We define a function*

$$I(t) := \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 dx.$$

Then, the following identities hold:

$$I'(t) = 4Im \int_{\mathbb{R}^d} \overline{u(t,x)} x \cdot \nabla u(t,x) dx,$$

$$I''(t) = 8\|\nabla u(t)\|^2_{L_x^2} - 4\int_{\mathbb{R}^d} (x\cdot\nabla V)|u(t,x)|^2 dx - \frac{4d(p-1)}{p+1}\|u(t)\|^{p+1}_{L_x^{p+1}} \; (= 4K_V(u(t))).$$

Proposition 4.9 (Localized virial identity, [25, 68, 113]). *Let u be a solution given in Theorem 1.50 or Theorem 1.51. For a given suitable real-valued weight function w, we define a function*

$$I(t) := \int_{\mathbb{R}^d} w(x)|u(t,x)|^2 dx.$$

Then, the following identities hold:

$$I'(t) = 2Im \int_{\mathbb{R}^d} \overline{u(t,x)} \nabla w(x) \cdot \nabla u(t,x) dx.$$

If w is radial, then we can write

$$I'(t) = 2Im \int_{\mathbb{R}^d} \frac{w'(r)}{r} \overline{u(t,x)} x \cdot \nabla u(t,x) dx,$$

$$\begin{aligned} I''(t) = \int_{\mathbb{R}^d} F_1(w,r)|x\cdot\nabla u(t,x)|^2 dx + 4\int_{\mathbb{R}^d} \frac{w'(r)}{r}|\nabla u(t,x)|^2 dx - \int_{\mathbb{R}^d} F_2(w,r)|u(t,x)|^{p+1} dx \\ - \int_{\mathbb{R}^d} F_3(w,r)|u(t,x)|^2 dx - 2\int_{\mathbb{R}^d} \frac{w'(r)}{r}(x\cdot\nabla V)|u(t,x)|^2 dx. \end{aligned}$$

where $r = |x|$ and

$$F_1(w,r) := 4\left\{\frac{w''(r)}{r^2} - \frac{w'(r)}{r^3}\right\}, \quad F_2(w,r) := \frac{2(p-1)}{p+1}\left\{w''(r) + \frac{d-1}{r}w'(r)\right\}$$

$$F_3(w,r) := w^{(4)}(r) + \frac{2(d-1)}{r}w^{(3)}(r) + \frac{(d-1)(d-3)}{r^2}w''(r) + \frac{(d-1)(3-d)}{r^3}w'(r).$$

4.2. **Proof of Main theorem 1.56.** In this subsection, we prove Main theorem 1.56.

4.2.1. *Local well-posedness of* (NLS$_V$). In this subsubsection, we investigate local well-posedness of (NLS$_V$).

Proof of Theorem 1.51. We define a function space E, a metric d on E, and a map Φ_{u_0} respectively as follows

$$E := \{u \in C_t(I; H^1_x(\mathbb{R}^3)) : \|(1-\Delta_V)^{\frac{1}{2}} u\|_{S(L^2;I)} \leq 2c\,\|u_0\|_{H^1_x}\},$$

$$d(u,v) := \|u-v\|_{S(L^2;I)},$$

$$\Phi_{u_0}(u) := e^{it\Delta_V}u_0 + i\int_0^t e^{i(t-s)\Delta_V}(|u|^{p-1}u)(s)ds.$$

We take a constant β satisfying

$$\max\left\{2, \frac{4}{5-p}\right\} < \beta < \begin{cases} \dfrac{4}{3-p} & (1 < p < 3), \\ \infty & (3 \le p < 5). \end{cases}$$

Using Proposition 4.4, Lemma 4.1, 4.5, and 2.3, we have

$$\begin{aligned}
\|(1-\Delta_V)^{\frac{1}{2}}\Phi_{u_0}(u)\|_{S(L^2;I)} &\le c\,\|(1-\Delta_V)^{\frac{1}{2}}u_0\|_{L^2_x} + c\,\|(1-\Delta_V)^{\frac{1}{2}}(|u|^{p-1}u)\|_{L^2_t(I;L_x^{\frac{6}{5}})} \\
&\le c\,\|u_0\|_{H^1_x} + c\,\||u|^{p-1}u\|_{L^2_t(I;W_x^{1,\frac{6}{5}})} \\
&\le c\,\|u_0\|_{H^1_x} + c\,T^{\frac{1}{\beta}}\|u\|^{p-1}_{L_t^{\frac{2\beta(p-1)}{\beta-2}}(I;L_x^{3(p-1)})}\|(1-\Delta_V)^{\frac{1}{2}}u\|_{L^\infty_t(I;L^2_x)} \\
&\le c\,\|u_0\|_{H^1_x} + c\,T^{\frac{1}{\beta}}\|u\|^{p-1}_{L_t^{\frac{2\beta(p-1)}{\beta-2}}(I;W_x^{1,\frac{6(p-1)\beta}{\beta(3p-5)+4}})}\|(1-\Delta_V)^{\frac{1}{2}}u\|_{L^\infty_t(I;L^2_x)} \\
&\le c\,\|u_0\|_{H^1_x} + c\,T^{\frac{1}{\beta}}\|(1-\Delta_V)^{\frac{1}{2}}u\|^{p-1}_{L_t^{\frac{2\beta(p-1)}{\beta-2}}(I;L_x^{\frac{6(p-1)\beta}{\beta(3p-5)+4}})}\|(1-\Delta_V)^{\frac{1}{2}}u\|_{L^\infty_t(I;L^2_x)} \\
&\le c\,\|u_0\|_{H^1_x} + c\,T^{\frac{1}{\beta}}\|(1-\Delta_V)^{\frac{1}{2}}u\|^{p}_{S(L^2;I)} \\
&\le \left\{1+(2c)^pT^{\frac{1}{\beta}}\|u_0\|^{p-1}_{H^1_x}\right\}c\,\|u_0\|_{H^1_x}
\end{aligned}$$

and

$$\|\Phi_{u_0}(u)-\Phi_{u_0}(v)\|_{S(L^2;I)} \le (2c)^pT^{\frac{1}{\beta}}\|u_0\|^{p-1}_{H^1_x}\|u-v\|_{S(L^2;I)}.$$

If we take $T>0$ sufficiently small satisfying $(2c)^pT^{\frac{1}{\beta}}\|u_0\|^{p-1}_{H^1_x} < 1$, then Φ_{u_0} is a contraction map on E and hence, there exists a unique solution to (NLS$_V$) on E. □

Theorem 4.10 (Local well-posedness in $H^1 \cap |x|^{-1}L^2$)**.** *Let $d=3$, $1 \le p < 5$, $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, and $\|V_-\|_{\mathcal{K}} < 4\pi$. Let u be a solution to (NLS$_V$) given in Theorem 1.51. If $|\cdot|u_0 \in L^2(\mathbb{R}^3)$, then a map $t \mapsto |\cdot|u(t,\cdot)$ belongs to $C_t(I;L^2_x)$.*

The proof of this theorem is based on the argument in [13, Lemma 6.5.2].

Proof. We set $I=[0,T)$ with $0<T\le\infty$. Let $\varepsilon>0$. We define a function

$$f_\varepsilon(t) := \|e^{-\varepsilon|\cdot|^2}|\cdot|u(t)\|^2_{L^2_x}.$$

Then, we have

$$\begin{aligned}
f'_\varepsilon(t) &= \frac{d}{dt}\int_{\mathbb{R}^3} e^{-2\varepsilon|x|^2}|x|^2|u(t,x)|^2dt = 2\mathrm{Re}\int_{\mathbb{R}^3} e^{-2\varepsilon|x|^2}|x|^2\partial_t u(t,x)\overline{u(t,x)}dx \\
&= 2\mathrm{Re}\int_{\mathbb{R}^3} e^{-2\varepsilon|x|^2}|x|^2\{i\Delta u(t,x) - iV(x)u(t,x) + i|u(t,x)|^{p-1}u(t,x)\}\overline{u(t,x)}dx \\
&= 2\mathrm{Im}\int_{\mathbb{R}^3} \nabla(e^{-2\varepsilon|x|^2}|x|^2)\cdot\nabla u(t,x)\overline{u(t,x)}dx \\
&= 4\mathrm{Im}\int_{\mathbb{R}^3} (1-2\varepsilon|x|^2)e^{-2\varepsilon|x|^2}x\cdot\nabla u(t,x)\overline{u(t,x)}dx.
\end{aligned}$$

Integrating this identity on $[0,t]$,

$$f_\varepsilon(t) = f_\varepsilon(0) + 4\int_0^t \mathrm{Im}\int_{\mathbb{R}^3}\{e^{-\varepsilon|x|^2}(1-2\varepsilon|x|^2)\}e^{-\varepsilon|x|^2}x\cdot\nabla u(t,x)\overline{u(t,x)}dxdt.$$

Since $e^{-\varepsilon|x|^2}(1-2\varepsilon|x|^2)$ is bounded for x and ε, and $\|e^{-\varepsilon|x|^2}|x|u_0\|_{L^2_x} \le \|xu_0\|_{L^2_x}$, it follows that

$$f_\varepsilon(t) \le \|xu_0\|^2_{L^2_x} + C\int_0^t \|\nabla u(s)\|_{L^2_x}\sqrt{f_\varepsilon(s)}ds.$$

This inequality deduces

$$\sqrt{f_\varepsilon(t)} \le \|xu_0\|_{L^2_x} + \frac{C}{2}\int_0^t \|\nabla u(s)\|_{L^2_x} ds$$

for any $t \in I$. Taking a limit as $\varepsilon \searrow 0$ and using Fatou's lemma, we see that $xu(t) \in L^2_x$ for any $t \in I$. □

4.2.2. *Scattering part.* In this subsubsection, we prove the scattering part in Main theorem 1.56.

Proposition 4.11 (Coercivity I)**.** *Let $d = 3$ and $\frac{7}{3} < p < 5$, $V \ge 0$, and $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$. Let $Q_{1,0}$ be the ground state to ($\mathrm{SP}_{\omega,0}$) with $\omega = 1$. Assume that $u_0 \in H^1(\mathbb{R}^3)$ satisfies*

$$M(u_0)^{\frac{1-s_c}{s_c}} E_V(u_0) < (1-\delta)^{\frac{1}{s_c}} M(Q_{1,0})^{\frac{1-s_c}{s_c}} E_0(Q_{1,0}) \tag{4.1}$$

for some $\delta > 0$ and (1.4). Then, there exist $\delta' = \delta'(\delta) > 0$, $c = c(\delta, \|u_0\|_{L^2_x}) > 0$, and $R = R(\delta, \|u_0\|_{L^2_x}) > 0$ such that the solution u to (NLS_V) with (IC) satisfies

(1) $\|u(t)\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla u(t)\|_{L^2_x} < (1-2\delta')^{\frac{1}{s_c(p-1)}} \|Q_{1,0}\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L^2_x}$,

(2) $\|\nabla u(t)\|_{L^2_x}^2 - \frac{3(p-1)}{2(p+1)} \|u(t)\|_{L^{p+1}_x}^{p+1} \ge c\, \|u(t)\|_{L^{p+1}_x}^{p+1}$,

(3) $\|\mathscr{Y}_{\frac{R}{2}} u(t)\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla(\mathscr{Y}_{\frac{R}{2}} u(t))\|_{L^2_x} < (1-\delta')^{\frac{1}{s_c(p-1)}} \|Q_{1,0}\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L^2_x}$

hold, where $\mathscr{Y}_R$ is defined as (2.2). In particular, we can see global well-posedness in Main theorem 1.56 by this proposition (1).

Proof. We prove (1). By $V \ge 0$, Proposition 1.16, and (2.5), we have

$$\begin{aligned}
(1-\delta)^{\frac{1}{s_c}} M(Q_{1,0})^{\frac{1-s_c}{s_c}} E_0(Q_{1,0}) &> M(u_0)^{\frac{1-s_c}{s_c}} E_V(u_0) \\
&\ge \|u(t)\|_{L^2_x}^{\frac{2(1-s_c)}{s_c}} \left(\frac{1}{2}\|\nabla u(t)\|_{L^2_x}^2 - \frac{1}{p+1} C_{\mathrm{GN}} \|u(t)\|_{L^2_x}^{(p-1)(1-s_c)} \|\nabla u(t)\|_{L^2_x}^{(p-1)s_c+2} \right) \\
&= \frac{1}{2}\|u(t)\|_{L^2_x}^{\frac{2(1-s_c)}{s_c}} \|\nabla u(t)\|_{L^2_x}^2 - \frac{2}{3(p-1)} \cdot \frac{\|u(t)\|_{L^2_x}^{\frac{1-s_c}{s_c}\{(p-1)s_c+2\}} \|\nabla u(t)\|_{L^2_x}^{(p-1)s_c+2}}{\|Q_{1,0}\|_{L^2_x}^{(p-1)(1-s_c)} \|\nabla Q_{1,0}\|_{L^2_x}^{(p-1)s_c}}
\end{aligned}$$

and hence,

$$(1-\delta)^{\frac{1}{s_c}} \ge g\left(\frac{\|u(t)\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla u(t)\|_{L^2_x}}{\|Q_{1,0}\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L^2_x}} \right),$$

where $g(y) := \frac{3(p-1)}{3p-7} y^2 - \frac{4}{3p-7} y^{\frac{3(p-1)}{2}}$. Then, g has a local minimum at $y_0 = 0$ and a local maximum at $y_1 = 1$. Combining these facts and the assumption of Proposition 4.11 (1), there exists $\delta' = \delta'(\delta) > 0$ such that

$$\|u(t)\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla u(t)\|_{L^2_x} < (1-2\delta')^{\frac{1}{s_c(p-1)}} \|Q_{1,0}\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L^2_x},$$

which completes the proof of Proposition 4.11 (1).
We prove (2). Using Proposition 1.16, this proposition (1), and (2.5),

$$\begin{aligned}
E_0(u(t)) &\ge \frac{1}{2}\|\nabla u(t)\|_{L^2_x}^2 - \frac{1}{p+1} C_{GN} \|u(t)\|_{L^2_x}^{(p-1)(1-s_c)} \|\nabla u(t)\|_{L^2_x}^{(p-1)s_c+2} \\
&> \|\nabla u(t)\|_{L^2_x}^2 \left(\frac{1}{2} - \frac{1}{p+1} C_{GN} (1-2\delta') \|Q_{1,0}\|_{L^2_x}^{(p-1)(1-s_c)} \|\nabla Q_{1,0}\|_{L^2_x}^{(p-1)s_c} \right) \\
&= \left\{ \frac{3p-7}{6(p-1)} + \frac{4}{3(p-1)}\delta' \right\} \|\nabla u(t)\|_{L^2_x}^2.
\end{aligned}$$

This inequality deduces

$$\begin{aligned}\|\nabla u(t)\|_{L_x^2}^2 - \frac{3(p-1)}{2(p+1)}\|u(t)\|_{L_x^{p+1}}^{p+1} &= \frac{3(p-1)}{2}E_0(u(t)) + \frac{7-3p}{4}\|\nabla u(t)\|_{L_x^2}^2 \\ &> \frac{3(p-1)}{2}\left\{\frac{3p-7}{6(p-1)} + \frac{4}{3(p-1)}\delta'\right\}\|\nabla u(t)\|_{L_x^2}^2 + \frac{7-3p}{4}\|\nabla u(t)\|_{L_x^2}^2 \\ &= 2\delta'\|\nabla u(t)\|_{L_x^2}^2 > \frac{3(p-1)\delta'}{(p+1)(1-2\delta')}\|u(t)\|_{L_x^{p+1}}^{p+1},\end{aligned}$$

which completes the proof of Proposition 4.11 (2).
Finally, we prove (3). $\|\mathscr{Y}_{\frac{R}{2}}u(t)\|_{L_x^2} \leq \|u(t)\|_{L_x^2}$ holds clearly. Since

$$\begin{aligned}\|\nabla(\mathscr{Y}_{\frac{R}{2}}u(t))\|_{L_x^2}^2 &= \|\mathscr{Y}_{\frac{R}{2}}\nabla u(t)\|_{L_x^2}^2 - \frac{1}{R^2}\int_{\mathbb{R}^3}\mathscr{Y}_{\frac{R}{2}}(x)(\Delta\mathscr{Y})\left(\frac{2x}{R}\right)|u(t,x)|^2dx \\ &\leq \|\nabla u(t)\|_{L_x^2}^2 + \frac{c}{R^2}M(u_0),\end{aligned} \tag{4.2}$$

we have

$$\begin{aligned}\|\mathscr{Y}_{\frac{R}{2}}u(t)\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|\nabla(\mathscr{Y}_{\frac{R}{2}}u(t))\|_{L_x^2} &\leq \|u(t)\|_{L_x^2}^{\frac{1-s_c}{s_c}}\left\{\|\nabla u(t)\|_{L_x^2}^2 + \frac{c}{R^2}M(u_0)\right\}^{\frac{1}{2}} \\ &\leq \|u(t)\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|\nabla u(t)\|_{L_x^2} + \frac{c}{R}M(u_0)^{\frac{1}{2s_c}} \\ &< (1-2\delta')^{\frac{1}{s_c(p-1)}}\|Q_{1,0}\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|\nabla Q_{1,0}\|_{L_x^2} + \frac{c}{R}M(u_0)^{\frac{1}{2s_c}} \\ &< (1-\delta')^{\frac{1}{s_c(p-1)}}\|Q_{1,0}\|_{L_x^2}^{\frac{1-s_c}{s_c}}\|\nabla Q_{1,0}\|_{L_x^2}\end{aligned}$$

for sufficiently large $R = R(\delta, \|u_0\|_{L_x^2})$. □

We define the exponents

$$q_0 = \frac{5(p-1)}{2}, \qquad r_0 = \frac{30(p-1)}{15p-23}, \qquad \rho = \frac{5(p-1)}{2p}, \qquad \gamma = \frac{30(p-1)}{27p-35}.$$

We note that (q_0, q_0) is $\dot{H}^{s_c}$-admissible, (q_0, r_0) is L^2-admissible, and (ρ, γ) is dual L^2-admissible. These exponents appear in [88].

Lemma 4.12 (Small data global existence)**.** *Let $d = 3$, $\frac{7}{3} < p < 5$, $T > 0$, $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, and $\|u_T\|_{H^{s_c}} \leq A$. There exists $\varepsilon_0 = \varepsilon_0(A) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, if*

$$\|e^{i(t-T)\Delta_V}u_T\|_{L_t^{q_0}(T,\infty;L_x^{q_0})} < \varepsilon,$$

then (NLS$_V$) with initial data $u(T) = u_T$ has a unique solution u on $[T, \infty)$ and

$$\|u\|_{L_t^{q_0}(T,\infty;L_x^{q_0})} \lesssim \varepsilon.$$

Proof. Let $I := [T, \infty)$. We define a function space E, a distance d on E, and a map Φ as follows

$$E := \left\{ u \in C_t(I; H^{s_c}) \cap L_t^{q_0}(I; W_x^{s_c,r_0}) \;\middle|\; \begin{array}{l}\|u\|_{L_t^\infty(I;H_x^{s_c}) \cap L_t^{q_0}(I;W_x^{s_c,r_0})} \leq 2cA, \\ \|u\|_{L_t^{q_0}(I;L_x^{q_0})} \leq 2\varepsilon\end{array}\right\},$$

$$d(u_1, u_2) = \|u_1 - u_2\|_{L_t^{q_0}(I;L_x^{r_0})},$$

$$\Phi(u)(t) = e^{i(t-T)\Delta_V}u_T + i\int_T^t e^{i(t-s)\Delta_V}(|u|^{p-1}u)(s)ds.$$

Using Proposition 4.4, Lemma 4.1, and 4.5, we have

$$\begin{aligned}\|\Phi(u)\|_{L_t^{q_0}(I;L_x^{q_0})} &\leq \varepsilon + c\,\||u|^{p-1}u\|_{L_t^{\rho}(I;W_x^{s_c,\gamma})} \\ &\leq \varepsilon + c\,\|u\|_{L_t^{q_0}(I;L_x^{q_0})}^{p-1}\|u\|_{L_t^{q_0}(I;W_x^{s_c,r_0})} \leq (1 + 2^p c^2 A\varepsilon^{p-2})\varepsilon\end{aligned}$$

and

$$\|\Phi(u)\|_{L_t^\infty(I;H_x^{s_c})\cap L_t^{q_0}(I;W_x^{s_c,r_0})} \le c\|u_T\|_{H_x^{s_c}} + c\|u\|_{L_t^{q_0}(I;L_x^{q_0})}^{p-1}\|u\|_{L_t^{q_0}(I;W_x^{s_c,r_0})}$$
$$\le (1+2^p c\varepsilon^{p-1})cA.$$

Thus, if $\varepsilon > 0$ satisfies $\max\{2^p c^2 A\varepsilon^{p-2}, 2^p c\varepsilon^{p-1}\} \le \frac{1}{2}$, then

$$\|u\|_{L_t^\infty(I;H_x^{s_c})\cap L_t^{q_0}(I;W_x^{s_c,r_0})} \le 2cA \quad \text{and} \quad \|u\|_{L_t^{q_0}(I;L_x^{q_0})} \le 2\varepsilon.$$

Also, for $u, v \in E$,

$$\|\Phi(u)-\Phi(v)\|_{L_t^{q_0}(I;L_x^{r_0})} \le c\left\{\|u\|_{L_t^{q_0}(I;L_x^{q_0})}^{p-1} + \|v\|_{L_t^{q_0}(I;L_x^{q_0})}^{p-1}\right\}\|u-v\|_{L_t^{q_0}(I;L_x^{r_0})}$$
$$\le 2^p c\varepsilon^{p-1}\|u-v\|_{L_t^{q_0}(I;L_x^{r_0})}$$
$$\le \frac{1}{2}\|u-v\|_{L_t^{q_0}(I;L_x^{r_0})}$$

Therefore, Φ is a contraction map on E, and hence, there exists a unique solution u to (NLS$_V$) on E. □

Lemma 4.13 (Small data scattering). *Let $d = 3$, $\frac{7}{3} < p < 5$, $T > 0$, and $V \in L^{\frac{3}{2}}(\mathbb{R}^3)\cap\mathcal{K}_0(\mathbb{R}^3)$. $u \in L_t^\infty(T,\infty;H_x^1)$ is a solution to (NLS$_V$) satisfying $\|u\|_{L_t^\infty(T,\infty;H_x^1)} \le E$. Then, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, if*

$$\|e^{i(t-T)\Delta_V}u(T)\|_{L_t^{q_0}(T,\infty;L_x^{q_0})} < \varepsilon,$$

then u scatters in positive time.

Proof. We take $\varepsilon_0 > 0$ as in Lemma 4.12 with $A = E$. From Lemma 4.12, the unique solution u to (NLS$_V$) satisfies

$$\|u\|_{L_t^{q_0}(T,\infty;W_x^{s_c,r_0})} \le 2cE \quad \text{and} \quad \|u\|_{L_t^{q_0}(T,\infty;L_x^{q_0})} \le 2\varepsilon.$$

Here, we take exponents q_1, r_1, q_2, r_2, and r as follows.
Case $\frac{7}{3} < p \le 3$:
We choose

$$q_1 := 2(p-1)^+, \quad r_1 := \frac{6(p-1)}{3p-5}^-, \quad q_2 := \infty^-, \quad r_2 := 2^+, \quad r := 3(p-1)^-$$

satisfying that (q_1, r_1) and (q_2, r_2) are L^2-admissible pairs, the embedding $\dot{W}^{s_c,r_1} \hookrightarrow L^r$ holds, $\dot{W}_V^{s_c,r_1}$ and $\dot{W}^{s_c,r_1}$ are equivalent, and $\dot{W}_V^{1,r_2}$ and $\dot{W}^{1,r_2}$ are equivalent.
Case $3 < p < 5$:

$$q_1 := \frac{4(p-1)^2}{p+1}, \quad r_1 := \frac{6(p-1)^2}{3p^2-7p+2}, \quad q_2 := \frac{4(p-1)}{p-3}, \quad r_2 := \frac{3(p-1)}{p}, \quad r := \frac{6(p-1)^2}{3p-5}.$$

Then, (q_1, r_1) and (q_2, r_2) are L^2-admissible pairs, the embedding $\dot{W}^{s_c,r_1} \hookrightarrow L^r$ holds, $\dot{W}_V^{s_c,r_1}$ and $\dot{W}^{s_c,r_1}$ are equivalent, and $\dot{W}_V^{1,r_2}$ and $\dot{W}^{1,r_2}$ are equivalent. Then,

$$\|u\|_{L_t^{q_1}(T,\infty;W_x^{s_c,r_1})} \le c\|u(T)\|_{H_x^{s_c}} + c\|u\|_{L_t^{q_0}(T,\infty;L_x^{q_0})}^{p-1}\|u\|_{L_t^{q_0}(T,\infty;W_x^{s_c,r_0})} < \infty$$

and

$$\|u\|_{L_t^{q_2}(T,\infty;W_x^{1,r_2})} \le c\|u(T)\|_{H_x^1} + c\left\||u|^{p-1}u\right\|_{L_t^2(T,\infty;W_x^{1,\frac{6}{5}})}$$
$$\le c\|u(T)\|_{H_x^1} + c\|u\|_{L_t^{q_1}(T,\infty;L_x^r)}^{p-1}\|u\|_{L_t^{q_2}(T,\infty;W_x^{1,r_2})}$$
$$\le c\|u(T)\|_{H_x^1} + c\|u\|_{L_t^{q_1}(T,\infty;\dot{W}_x^{s_c,r_1})}^{p-1}\|u\|_{L_t^{q_2}(T,\infty;W_x^{1,r_2})} < \infty$$

hold. Thus, we have

$$u \in L_t^\infty(T,\infty;H_x^1)\cap L_t^{q_0}(T,\infty;W_x^{s_c,r_0})\cap L_t^{q_1}(T,\infty;W_x^{s_c,r_1})\cap L_t^{q_2}(T,\infty;W_x^{1,r_2}).$$

Therefore, we obtain

$$\|e^{-it\Delta_V}u(t) - e^{-i\tau\Delta_V}u(\tau)\|_{H^1_x} = \left\| \int_\tau^t e^{-is\Delta_V}(|u|^{p-1}u)(s)ds \right\|_{H^1_x} \le c \left\| |u|^{p-1}u \right\|_{L^2_t(\tau,t;W^{1,\frac{6}{5}}_x)}$$
$$\le c\,\|u\|^{p-1}_{L^{q_1}_t(\tau,t;W^{s_c,r_1}_x)}\|u\|_{L^{q_2}_t(\tau,t;W^{1,r_2}_x)} \longrightarrow 0 \quad \text{as} \quad t > \tau \to \infty.$$

Therefore, $\{e^{-it\Delta_V}u(t)\}$ is a Cauchy sequence in $H^1(\mathbb{R}^3)$. □

Theorem 4.14 (Scattering criterion)**.** *Let* $d = 3$, $\frac{7}{3} < p < 5$,, $V \ge 0$, $V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$, *and* V *is radially symmetric. Suppose that* $u : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{C}$ *is radially symmetric and a solution to* (NLS$_V$) *satisfying* $\|u\|_{L^\infty_t H^1_x} \le E$ *for some* $E > 0$. *Then, there exist* $\varepsilon = \varepsilon(E) > 0$ *and* $R = R(E) > 0$ *such that "if*

$$\liminf_{t\to\infty} \int_{|x|\le R} |u(t,x)|^2 dx \le \varepsilon^2,$$

then u *scatters in positive time".*

The proof of this theorem is based on the argument in [112, §4]. We have to change exponents of function spaces.

Proof. Set $0 < \varepsilon < 1$ and $R > 0$, which will be chosen later. Using Proposition 4.4, we have

$$\|e^{it\Delta_V}u_0\|_{L^{q_0}_t L^{q_0}_x} \le c\,\|u_0\|_{\dot{H}^{s_c}_x} < \infty.$$

Thus, there exists $T_0 > \varepsilon^{-1}$ such that

$$\|e^{it\Delta_V}u_0\|_{L^{q_0}_t(T_0,\infty;L^{q_0}_x)} < \varepsilon. \tag{4.3}$$

By the assumption of this theorem, there exists $T > T_0$ such that

$$\int_{|x|\le R} |u(T,x)|^2 dx \le 2\varepsilon^2. \tag{4.4}$$

By the integral equation, we have

$$e^{i(t-T)\Delta_V}u(T) = e^{it\Delta_V}u_0 + i\int_{I_1} e^{i(t-s)\Delta_V}(|u|^{p-1}u)(s)ds + i\int_{I_2} e^{i(t-s)\Delta_V}(|u|^{p-1}u)(s)ds$$
$$=: e^{it\Delta_V}u_0 + F_1(t) + F_2(t) \tag{4.5}$$

where $I_1 := [0, T - \varepsilon^{-\theta}]$ and $I_2 := [T - \varepsilon^{-\theta}, T]$. Here, we will choose $0 < \theta = \theta(p) < 1$ later. First, we estimate $\|F_1\|_{L^{q_0}_t(T,\infty;L^{q_0}_x)}$. By the integral equation, we have

$$F_1(t) = e^{i(t-T+\varepsilon^{-\theta})\Delta_V}u(T-\varepsilon^{-\theta}) - e^{it\Delta_V}u_0.$$

We take a positive constant μ satisfying

$$\frac{3p-7}{3(p-1)} < \mu < \min\left\{\frac{3p-7}{p-1}, \frac{5p-9}{5(p-1)}\right\}$$

and set

$$q_3 = \frac{20(p-1)}{15(1-\mu)p + 15\mu - 27}, \quad r_3 = \frac{4(p-1)}{-3(1-\mu)p - 3\mu + 7}.$$

Then, the following relations hold:

$$\frac{1}{q_0} = \frac{1}{q_3} + \frac{1}{r_3}, \qquad \frac{2}{q_3(1-\mu)} + \frac{3}{q_0(1-\mu)} = \frac{3}{2}, \qquad r_3\mu > 2, \qquad q_3(1-\mu) > 2.$$

Proposition 4.4 implies

$$\|F_1\|_{L^{q_3(1-\mu)}_t(T,\infty;L^{q_0(1-\mu)}_x)} \lesssim \|u(T-\varepsilon^{-\theta})\|_{L^2_x} + \|u_0\|_{L^2_x} = 2\|u_0\|_{L^2_x} \lesssim 1. \tag{4.6}$$

On the other hand,

$$\|F_1(t)\|_{L^\infty_x} \le \int_{I_1} \|e^{i(t-s)\Delta_V}(|u|^{p-1}u)(s)\|_{L^\infty_x} ds \lesssim \int_{I_1} (t-s)^{-\frac{3}{2}} \|u(s)\|^p_{L^p_x} ds$$
$$\lesssim \|u\|^p_{L^\infty_t H^1_x} \int_0^{T-\varepsilon^{-\theta}} (t-s)^{-\frac{3}{2}} ds \lesssim_E (t-T+\varepsilon^{-\theta})^{-\frac{1}{2}}$$

from Theorem 4.2 and Lemma 2.3. Thus, we have

$$\|F_1\|_{L^{r_3\mu}_t(T,\infty;L^\infty_x)} \lesssim \|(t-T+\varepsilon^{-\theta})^{-\frac{1}{2}}\|_{L^{r_3\mu}_t(T,\infty)} \sim \varepsilon^{(\frac{1}{2}-\frac{1}{r_3\mu})\theta}. \tag{4.7}$$

Combining Lemma 2.2, (4.6), and (4.7), we have

$$\|F_1\|_{L^{q_0}_t(T,\infty;L^{q_0}_x)} \le \|F_1\|^{1-\mu}_{L^{q_3(1-\mu)}_t(T,\infty;L^{q_0(1-\mu)}_x)} \|F_1\|^{\mu}_{L^{r_3\mu}_t(T,\infty;L^\infty_x)} \lesssim \varepsilon^{(\frac{\mu}{2}-\frac{1}{r_3})\theta}. \tag{4.8}$$

Next, we estimate $\|F_2\|_{L^{q_0}_t(T,\infty;L^{q_0}_x)}$. Applying Proposition 4.9 and the assumption of this theorem, we have

$$\left| \frac{d}{dt} \int_{\mathbb{R}^3} \mathscr{Y}_{\frac{R}{2}}(x)|u(t,x)|^2 dx \right| \le 2\|\nabla \mathscr{Y}_{\frac{R}{2}}\|_{L^\infty_x} \|u(t)\|_{L^2_x} \|\nabla u(t)\|_{L^2_x} \le \frac{c}{R},$$

where $\mathscr{Y}_{\frac{R}{2}}$ is defined as (2.2). Thus, we have

$$-\frac{c}{R} \le \frac{d}{dt} \int_{\mathbb{R}^3} \mathscr{Y}_{\frac{R}{2}}(x)|u(t,x)|^2 dx \le \frac{c}{R}.$$

Integrating each terms in this inequality on $[t,T]$,

$$-\frac{c}{R}(T-t) \le \int_{\mathbb{R}^3} \mathscr{Y}_{\frac{R}{2}}(x)|u(T,x)|^2 dx - \int_{\mathbb{R}^3} \mathscr{Y}_{\frac{R}{2}}(x)|u(t,x)| dx \le \frac{c}{R}(T-t).$$

The left inequality implies

$$\int_{\mathbb{R}^3} \mathscr{Y}_{\frac{R}{2}}(x)|u(t,x)|^2 dx \le \int_{\mathbb{R}^3} \mathscr{Y}_{\frac{R}{2}}(x)|u(T,x)|^2 dx + \frac{c}{R}(T-t). \tag{4.9}$$

Here, we choose $R > 0$ satisfying $R > \varepsilon^{-2-\theta}$. By taking supremum over I_2 for (4.9) and using (4.4), we have

$$\sup_{t\in I_2} \int_{\mathbb{R}^3} \mathscr{Y}_{\frac{R}{2}}(x)|u(t,x)|^2 dx \le 2\varepsilon^2 + c\varepsilon^{2+\theta}\varepsilon^{-\theta} \lesssim c\varepsilon^2,$$

that is,

$$\|\mathscr{Y}_{\frac{R}{2}} u\|_{L^\infty_t(I_2;L^2_x)} \le c\varepsilon. \tag{4.10}$$

By Lemma 2.2, (4.10), Lemma 4.7, and 2.3,

$$\begin{aligned}
\|u\|_{L^{\frac{10}{3}}_t(I_2;L^{\frac{10}{3}}_x)} &\le \|1\|_{L^{\frac{10}{3}}_t(I_2)} \|u\|_{L^\infty_t(I_2;L^{\frac{10}{3}}_x)} \le \varepsilon^{-\frac{3}{10}\theta} \|u\|_{L^\infty_t(I_2;L^{\frac{10}{3}}_x)} \\
&\le \varepsilon^{-\frac{3}{10}\theta} \left\{ \|\mathscr{Y}_{\frac{R}{2}} u\|_{L^\infty_t(I_2;L^{\frac{10}{3}}_x)} + \|(1-\mathscr{Y}_{\frac{R}{2}})u\|_{L^\infty_t(I_2;L^{\frac{10}{3}}_x)} \right\} \\
&\le \varepsilon^{-\frac{3}{10}\theta} \left\{ \|\mathscr{Y}_{\frac{R}{2}} u\|^{\frac{2}{5}}_{L^\infty_t(I_2;L^2_x)} \|u\|^{\frac{3}{5}}_{L^\infty_t(I_2;L^6_x)} + \|(1-\mathscr{Y}_{\frac{R}{2}})u\|^{\frac{2}{5}}_{L^\infty_t(I_2;L^\infty_x)} \|u\|^{\frac{3}{5}}_{L^\infty_t(I_2;L^2_x)} \right\} \\
&\le \varepsilon^{-\frac{3}{10}\theta} \left\{ c\,\varepsilon^{\frac{2}{5}} \|u\|^{\frac{3}{5}}_{L^\infty_t(I_2;\dot{H}^1_x)} + \||x|^{-\frac{1}{2}}|x|^{\frac{1}{2}}u\|^{\frac{2}{5}}_{L^\infty_t(I_2;L^\infty_x(|x|\ge R/2))} \|u\|^{\frac{3}{5}}_{L^\infty_t(I_2;L^2_x)} \right\} \\
&\le \varepsilon^{-\frac{3}{10}\theta} \left\{ c\,\varepsilon^{\frac{2}{5}} \|u\|^{\frac{3}{5}}_{L^\infty_t(I_2;\dot{H}^1_x)} + R^{-\frac{1}{5}} \|u\|^{\frac{2}{5}}_{L^\infty_t(I_2;H^1_x)} \|u\|^{\frac{3}{5}}_{L^\infty_t(I_2;L^2_x)} \right\} \\
&\le \varepsilon^{-\frac{3}{10}\theta} \left\{ c\,\varepsilon^{\frac{2}{5}} + c\,\varepsilon^{\frac{1}{5}(2+\theta)} \right\} \lesssim c\,\varepsilon^{\frac{2}{5}-\frac{3}{10}\theta}.
\end{aligned} \tag{4.11}$$

By using Proposition 4.4 and a continuity argument, we have

$$\||\nabla|^{s_c}u\|_{L_t^{10}(I_2;L_x^{\frac{30}{13}})}^{10} + \|u\|_{L_t^{10}(I_2;L_x^{10})}^{10} \lesssim 1 + |I_2|. \tag{4.12}$$

From Proposition 4.4, Lemma 4.5, 2.2, (4.12), and (4.11) it follows that

$$\begin{aligned}
\|F_2\|_{L_t^{q_0}(T,\infty;L_x^{q_0})} &\lesssim \||\nabla|^{s_c}(|u|^{p-1}u)\|_{L_t^2(I_2;L_x^{\frac{6}{5}})} \\
&\leq \|u\|_{L_t^{\frac{5}{2}(p-1)}(I_2;L_x^{\frac{5}{2}(p-1)})}^{p-1} \||\nabla|^{s_c}u\|_{L_t^{10}(I_2;L_x^{\frac{30}{13}})} \\
&\lesssim (1+|I_2|)^{\frac{1}{10}} \left\{ \|u\|_{L_t^{\frac{10}{3}}(I_2;L_x^{\frac{10}{3}})}^{1-s_c} \|u\|_{L_t^{10}(I_2;L_x^{10})}^{s_c} \right\}^{p-1} \\
&\lesssim |I_2|^{\frac{1}{10}} \left(\varepsilon^{\left(\frac{2}{5}-\frac{3}{10}\theta\right)(1-s_c)} |I_2|^{\frac{s_c}{10}} \right)^{p-1} \\
&= \varepsilon^{-\frac{1}{10}\theta - \frac{1}{10}s_c(p-1)\theta} \varepsilon^{\left(\frac{2}{5}-\frac{3}{10}\theta\right)(1-s_c)(p-1)} = \varepsilon^{\frac{5-p}{5}-\frac{1}{2}\theta}.
\end{aligned}$$

Thus, if we take $\theta = \frac{5-p}{5} \in (0,1)$, then

$$\|F_2\|_{L_t^{q_0}(T,\infty;L_x^{q_0})} \lesssim \varepsilon^{\frac{1}{2}\theta}. \tag{4.13}$$

Combining (4.5), (4.3), (4.8), and (4.13), we obtain

$$\|e^{i(t-T)\Delta_V}u(T)\|_{L_t^{q_0}(T,\infty;L_x^{q_0})} \lesssim \varepsilon + \varepsilon^{\frac{1}{2}\theta}.$$

From Lemma 4.12 and 4.13, the solution u scatters in positive time. □

Proposition 4.15 (Virial/Morawetz estimate)**.** *Let $d=3$, $\frac{7}{3}<p<5$, $T>0$, $V\geq 0$, $V\in L^{\frac{3}{2}}(\mathbb{R}^3)\cap\mathcal{K}_0(\mathbb{R}^3)$, $x\cdot\nabla V\leq 0$, $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V\in L^{\frac{3}{2}}(\mathbb{R}^3)$ for any $\mathfrak{a}\in(\mathbb{N}\cup\{0\})^3$ with $|\mathfrak{a}|=1$, and V be radially symmetric. We assume that u is a radial global solution to (NLS$_V$) satisfying (4.1) and (1.4). Then, it follows that*

$$\frac{1}{T}\int_0^T \int_{|x|\leq\frac{R}{2}} |u(t,x)|^{p+1}dxdt \lesssim \frac{R}{T} + \frac{1}{R^2} + \frac{1}{R^{p-1}}$$

for sufficiently large $R=R(\delta, M(u_0), Q_{1,0})$.

Proof. We set a radial function

$$w(x) = \begin{cases} |x|^2 & (|x|\leq 1), \\ \text{smooth} & (1<|x|<2), \\ 3|x|-4 & (2\leq|x|), \end{cases}$$

satisfying $\partial_r w\geq 0$ and $\partial_r^2 w\geq 0$. We define w_R as

$$w_R(x) = R^2 w\left(\frac{x}{R}\right)$$

for $R>0$. We define a function $M(t)$ as

$$M(t) := 2\mathrm{Im}\int_{\mathbb{R}^3} \overline{u(t,x)}\nabla w_R(x)\cdot\nabla u(t,x)dx.$$

By Lemma 2.2, we have

$$|M(t)| \lesssim \|u(t)\|_{L_x^2}\|\nabla u(t)\|_{L_x^2}R.$$

Since $\|u(t)\|_{H_x^1}$ is uniformly bounded from Proposition 4.11 (1), we have

$$\sup_{t\in\mathbb{R}}|M(t)| \lesssim_{Q_{1,0}} R. \tag{4.14}$$

Since $|x||\nabla u| = |x\cdot\nabla u|$ a.e. in $\mathbb{R}^3$ for $u\in H_{\mathrm{rad}}^1(\mathbb{R}^3)$ (see [100]), using Proposition 4.9,

$$\frac{d}{dt}M(t) = 4\int_{\mathbb{R}^3} w''\left(\frac{r}{R}\right)|\nabla u(t,x)|^2dx - \int_{\mathbb{R}^3} F_{2,R}(w,r)|u(t,x)|^{p+1}dx$$

$$\begin{aligned} &- \int_{\mathbb{R}^3} F_{3,R}(w,r)|u(t,x)|^2 dx - 2\int_{\mathbb{R}^3} \frac{R}{r} w'\left(\frac{r}{R}\right)(x\cdot\nabla V)|u(t,x)|^2 dx \\ =:\ & I_{|x|\le R} + I_{R\le|x|\le 2R} + I_{2R\le|x|}, \end{aligned} \tag{4.15}$$

where I_A is an integral restricted $\frac{d}{dt}M(t)$ to a set A, $r = |x|$, and

$$F_{2,R}(w,r) := \frac{2(p-1)}{p+1}\left\{w''\left(\frac{r}{R}\right) + \frac{2R}{r}w'\left(\frac{r}{R}\right)\right\}, \quad F_{3,R}(w,r) := \frac{1}{R^2}w^{(4)}\left(\frac{r}{R}\right) + \frac{4}{Rr}w^{(3)}\left(\frac{r}{R}\right).$$

For an integral $I_{|x|\le R}$, it follows from (4.2) and Proposition 4.11 (2), (3) that

$$\begin{aligned} I_{|x|\le R} &= 8\|\nabla u(t)\|^2_{L^2_x(|x|\le R)} - \frac{12(p-1)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}_x(|x|\le R)} - 4\int_{|x|\le R}(x\cdot\nabla V)|u(t,x)|^2 dx \\ &\ge 8\|\mathscr{Y}_{\frac{R}{2}}\nabla u(t)\|^2_{L^2_x} - \frac{12(p-1)}{p+1}\|\mathscr{Y}_{\frac{R}{2}}u(t)\|^{p+1}_{L^{p+1}_x(|x|\le R)} \\ &\qquad + \frac{12(p-1)}{p+1}\left\{\|\mathscr{Y}_{\frac{R}{2}}u(t)\|^{p+1}_{L^{p+1}_x(|x|\le R)} - \|u(t)\|^{p+1}_{L^{p+1}_x(|x|\le R)}\right\} \\ &\ge 8\|\nabla(\mathscr{Y}_{\frac{R}{2}}u(t))\|^2_{L^2_x} - \frac{c}{R^2}M(u_0) - \frac{12(p-1)}{p+1}\|\mathscr{Y}_{\frac{R}{2}}u(t)\|^{p+1}_{L^{p+1}_x} - \frac{12(p-1)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}_x(\frac{R}{2}\le|x|\le R)} \\ &\ge c\,\|\mathscr{Y}_{\frac{R}{2}}u(t)\|^{p+1}_{L^{p+1}_x} - \frac{c}{R^2}M(u_0) - \frac{12(p-1)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}_x(\frac{R}{2}\le|x|\le R)}, \end{aligned} \tag{4.16}$$

where $\mathscr{Y}_R$ is defined as (2.2). For $I_{R\le|x|\le 2R}$, we have

$$\begin{aligned} I_{R\le|x|\le 2R} &\ge -\int_{R\le|x|\le 2R} F_{2,R}(w,r)|u(t,x)|^{p+1}dx - \int_{R\le|x|\le 2R} F_{3,R}(w,r)|u(t,x)|^2 dx \\ &\ge -c\,\|u(t)\|^{p+1}_{L^{p+1}_x(R\le|x|\le 2R)} - \frac{c}{R^2}M(u_0). \end{aligned} \tag{4.17}$$

For $I_{2R\le|x|}$, we have

$$\begin{aligned} I_{2R\le|x|} &= -\frac{12(p-1)}{p+1}\int_{2R\le|x|}\frac{R}{r}|u(t,x)|^{p+1}dx - 6\int_{2R\le|x|}\frac{R}{r}(x\cdot\nabla V)|u(t,x)|^2 dx \\ &\ge -c\,\|u(t)\|^{p+1}_{L^{p+1}_x(2R\le|x|)}. \end{aligned} \tag{4.18}$$

Combining (4.15), (4.16), (4.17), (4.18), Lemma 2.4 and Proposition 4.11 (1), we obtain

$$\|\mathscr{Y}_{\frac{R}{2}}u(t)\|^{p+1}_{L^{p+1}_x} \lesssim_\delta \frac{d}{dt}M(t) + \frac{1}{R^2}M(u_0) + \|u(t)\|^{p+1}_{L^{p+1}_x(\frac{R}{2}\le|x|)} \lesssim_{\delta,Q_{1,0}} \frac{d}{dt}M(t) + \frac{1}{R^2}M(u_0) + \frac{1}{R^{p-1}}.$$

Integrating both sides of this inequality on $[0,T]$,

$$\int_0^T \|\mathscr{Y}_{\frac{R}{2}}u(t)\|^{p+1}_{L^{p+1}_x}dt \lesssim_{\delta,Q_{1,0}} \sup_{t\in[0,T]}|M(t)| + \frac{T}{R^2}M(u_0) + \frac{T}{R^{p-1}}.$$

Therefore,

$$\frac{1}{T}\int_0^T\int_{|x|\le\frac{R}{2}}|u(t,x)|^{p+1}dxdt \lesssim_{\delta,Q_{1,0}} \frac{R}{T} + \frac{1}{R^2} + \frac{1}{R^{p-1}}$$

from (4.14), which completes the proof of this proposition. □

Proposition 4.16 (Potential energy evacuation)**.** *Let u be a solution to* (NLS$_V$) *satisfying the conditions in Main theorem 1.56 (1). Then, there exist a time sequence $\{t_n\}$ with $t_n \longrightarrow \infty$ and a radius sequence $\{R_n\}$ with $R_n \longrightarrow \infty$ such that*

$$\liminf_{n\to\infty}\int_{|x|\le R_n}|u(t_n,x)|^{p+1}dx = 0. \tag{4.19}$$

Proof. Applying Proposition 4.15 with $T = R^3$ implies

$$\frac{1}{R^3}\int_0^{R^3}\int_{|x|\leq \frac{R}{2}} |u(t,x)|^{p+1}dxdt \lesssim \frac{1}{R^2} + \frac{1}{R^{p-1}} \longrightarrow 0 \quad \text{as} \quad R\to\infty. \tag{4.20}$$

By contradiction, we prove

$$\liminf_{t\to\infty}\int_{|x|\leq \frac{1}{2}t^{\frac{1}{3}}} |u(t,x)|^{p+1}dx = 0. \tag{4.21}$$

We assume that

$$\liminf_{t\to\infty}\int_{|x|\leq \frac{1}{2}t^{\frac{1}{3}}} |u(t,x)|^{p+1}dx =: C > 0.$$

Then, there exists $t_0 > 0$ such that

$$\inf_{s>t}\int_{|x|\leq \frac{1}{2}s^{\frac{1}{3}}} |u(s,x)|^{p+1}dx > \frac{C}{2} > 0$$

for any $t > t_0$. Therefore, we have

$$\begin{aligned}\frac{1}{R^3}\int_0^{R^3}\int_{|x|\leq \frac{1}{2}R} |u(t,x)|^{p+1}dxdt &\geq \frac{1}{R^3}\int_0^{R^3}\int_{|x|\leq \frac{1}{2}t^{\frac{1}{3}}} |u(t,x)|^{p+1}dxdt \\ &\geq \frac{1}{R^3}\int_{t_0}^{R^3}\int_{|x|\leq \frac{1}{2}t^{\frac{1}{3}}} |u(t,x)|^{p+1}dxdt \\ &> \frac{1}{R^3}\int_{t_0}^{R^3}\frac{C}{2}dt = \frac{R^3-t_0}{R^3}\cdot\frac{C}{2} \longrightarrow \frac{C}{2} > 0 \quad \text{as} \quad R\to\infty.\end{aligned}$$

This contradiction (4.20). Consequently, we can take sequences $\{t_n\} : t_n \longrightarrow \infty$ and $\{R_n\}$: $R_n = \frac{1}{2}t_n^{\frac{1}{3}} \longrightarrow \infty$ such that (4.19) holds from (4.21). □

Finally, we prove the scattering part in Main theorem (1).

Proof of Main theorem 1.56 (1). We recall that global well-posedness has been already shown in Proposition 4.11 (1). Fix ε and R as in Theorem 4.14. Then, take sequences $\{t_n\}$ and $\{R_n\}$ satisfying $t_n \to \infty$ and $R_n \to \infty$ as in Proposition 4.16. From Lemma 2.2 and Proposition 4.16, we obtain

$$\begin{aligned}\int_{|x|\leq R} |u(t_n,x)|^2dx &\leq \left(\int_{|x|\leq R} dx\right)^{\frac{p-1}{p+1}}\left(\int_{|x|\leq R_n} |u(t_n,x)|^{p+1}dx\right)^{\frac{2}{p+1}} \\ &\lesssim R^{\frac{3(p-1)}{p+1}}\left(\int_{|x|\leq R_n} |u(t_n,x)|^{p+1}dx\right)^{\frac{2}{p+1}} \longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Applying Theorem 4.14, the solution u to (NLS_V) scatters in positive time. □

4.2.3. *Blow-up or grow-up part.* In this subsubsection, we prove the blow-up or grow-up part in Main theorem 1.56 (2).

Proof of Proposition 1.54. The inequality holds by Proposition 1.16 and $V \geq 0$. We set the best constant $C_{\text{GN}}^{\dagger}$ and prove $C_{\text{GN}}^{\dagger} = C_{\text{GN}}$. We define a functional

$$L_V(f) := \frac{\|f\|_{L^{p+1}}^{p+1}}{\|f\|_{L^2}^{p+1-\frac{d(p-1)}{2}}\|(-\Delta_V)^{\frac{1}{2}}f\|_{L^2}^{\frac{d(p-1)}{2}}}$$

for $f \in H^1(\mathbb{R}^d)$. Proposition 1.16 and $V \geq 0$ imply $C_{\text{GN}} = L_0(Q_{1,0}) \geq L_0(f) \geq L_V(f)$ for any $f \in H^1(\mathbb{R}^d)\setminus\{0\}$, where $Q_{1,0}$ is the ground state to ($\text{SP}_{\omega,0}$) with $\omega = 1$. This inequality deduces $C_{\rm GN} \geq C_{\rm GN}^{\dagger}$. On the other hand, we consider a sequence $\{Q_{1,0}(n\,\cdot\,)\}$. Then, we have $L_V(Q_{1,0}(n\,\cdot\,)) \leq C_{\rm GN}^{\dagger}$ for each $n \in \mathbb{N}$. Thus, it follows that

$$C_{\rm GN}^{\dagger} \geq \lim_{n\to\infty} L_V(Q_{1,0}(n\,\cdot\,)) = L_0(Q_{1,0}) = C_{\rm GN}.$$

Therefore, we obtain $C_{\rm GN}^{\dagger} = C_{\rm GN}$. To finish the proof, we see that the inequality is not attained. If $L_0(f) = C_{\rm GN}$, then $f(x) = \lambda_0 Q_{1,0}(x+x_0)$ for some $\lambda_0 \in \mathbb{C}$ and $x_0 \in \mathbb{R}^d$. Since $Q_{1,0} > 0$ for any $x \in \mathbb{R}^d$, $V \geq 0$, and $V \neq 0$, we have $C_{\rm GN} = L_0(\lambda_0 Q_{1,0}(\,\cdot\,+x_0)) > L_V(\lambda_0 Q_{1,0}(\,\cdot\,+x_0))$, which implies the desired result. ☐

Lemma 4.17 (Coercivity II)**.** *Let $d=3$, $\frac{7}{3} < p < 5$, $V \geq 0$, and "$V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$ or $V \in L^{\sigma}(\mathbb{R}^3)$ for some $\frac{3}{2} < \sigma \leq \infty$". Let $Q_{1,0}$ be the ground state to (SP$_{\omega,0}$) with $\omega = 1$. Assume that u_0 satisfies (4.1) and (1.5). Then, there exists $\delta' = \delta'(\delta) > 0$ such that the solution u to (NLS$_V$) with (IC) satisfies*

$$\|u(t)\|_{L_x^2}^{\frac{1-s_c}{s_c}} \|(-\Delta_V)^{\frac{1}{2}} u(t)\|_{L_x^2}^{s_c} > (1+\delta')^{\frac{1}{s_c}} \|Q_{1,0}\|_{L_x^2}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L_x^2}$$

for any $t \in (T_{min}, T_{max})$.

Proof. By the similar argument in Proposition 4.11 (1) with Proposition 1.54, we have

$$\begin{aligned}
&(1-\delta)^{\frac{1}{s_c}} M(Q_{1,0})^{\frac{1-s_c}{s_c}} E_0(Q_{1,0}) \\
&> \frac{1}{2}\|u(t)\|_{L_x^2}^{\frac{2(1-s_c)}{s_c}} \|(-\Delta_V)^{\frac{1}{2}} u(t)\|_{L_x^2}^2 - \frac{2}{3(p-1)} \cdot \frac{\|u(t)\|_{L_x^2}^{\frac{1-s_c}{s_c}\{(p-1)s_c+2\}} \|(-\Delta_V)^{\frac{1}{2}} u(t)\|_{L_x^2}^{(p-1)s_c+2}}{\|Q_{1,0}\|_{L_x^2}^{(p-1)(1-s_c)} \|\nabla Q_{1,0}\|_{L_x^2}^{(p-1)s_c}}
\end{aligned}$$

and hence,

$$(1-\delta)^{\frac{1}{s_c}} \geq g\left(\frac{\|u(t)\|_{L_x^2}^{\frac{1-s_c}{s_c}} \|(-\Delta_V)^{\frac{1}{2}} u(t)\|_{L_x^2}}{\|Q_{1,0}\|_{L_x^2}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L_x^2}} \right),$$

where $g(y) := \frac{3(p-1)}{3p-7} y^2 - \frac{4}{3p-7} y^{\frac{3(p-1)}{2}}$. The rest of the proof is the same argument as Proposition 4.11 (1). ☐

Lemma 4.18. *Let $d \geq 1$ and $2 < p+1 < 2^*$. Let u be a solution to (NLS$_V$) given in Theorem 1.50 or Theorem 1.51. We assume that $u \in C_t([0,\infty); H_x^1(\mathbb{R}^d))$ and*

$$C_0 := \sup_{t\in[0,\infty)} \|\nabla u(t)\|_{L_x^2} < \infty.$$

Then, we have

$$\|u(t)\|_{L^2(R\leq|x|)}^2 \leq o_R(1) + \eta \tag{4.22}$$

for any $\eta > 0$, $R > 0$, and $t \in \left[0, \frac{\eta R}{4C_0 M(u)^{1/2}}\right]$

Proof. Let $\mathscr{X}_{\frac{R}{2}}$ be defined as (2.3). We note that $\|\nabla \mathscr{X}_{\frac{R}{2}}\|_{L_x^\infty} \leq \frac{4}{R}$. We define a function

$$I(t) := \int_{\mathbb{R}^d} \mathscr{X}_{\frac{R}{2}}(x) |u(t,x)|^2 dx.$$

Using Proposition 4.9, we have

$$\begin{aligned}
I(t) &= I(0) + \int_0^t \frac{d}{ds} I(s) ds \leq I(0) + \int_0^t |I'(s)| ds \\
&\leq I(0) + t\|\nabla \mathscr{X}_{\frac{R}{2}}\|_{L_x^\infty} \sup_{t\in[0,\infty)} \|\nabla u(t)\|_{L_x^2} \|u(t)\|_{L_x^2} \leq I(0) + \frac{4C_0 M(u)^{1/2} t}{R}
\end{aligned}$$

for any $t \in [0,\infty)$. Since $u_0 \in H^1_x(\mathbb{R}^d)$,

$$I(0) = \int_{\mathbb{R}^d} \mathscr{X}_{\frac{R}{2}}(x)|u_0(x)|^2 dx \leq \|u_0\|^2_{L^2_x(\frac{R}{2}\leq |x|)} = o_R(1).$$

The inequality $\|u(t)\|^2_{L^2_x(R\leq|x|)} \leq I(t)$ deduces (4.22) for any $t \in \left[0, \frac{\eta R}{4C_0 M(u)^{1/2}}\right]$. □

Lemma 4.19. *Let $d=3$, $1<p<5$, "$V \in L^{\frac{3}{2}}(\mathbb{R}^3)\cap \mathcal{K}_0(\mathbb{R}^3)$ or $V \in L^\sigma(\mathbb{R}^3)$ for some $\frac{3}{2}<\sigma\leq\infty$", $x^{\mathfrak{a}}\partial^{\mathfrak{a}} V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ for any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}|=1$, $V\geq 0$, and $x\cdot\nabla V + 2V \geq 0$. Let $u \in C([0,\infty);H^1)$ be a solution to (NLS$_V$). We define a function*

$$I(t) := \int_{\mathbb{R}^3} \mathscr{X}_R(x)|u(t,x)|^2 dx,$$

where $\mathscr{X}_R$ is defined as (2.1). Then, for any $p+1<q<6$, there exist constants $C = C(q,M(u),C_0)>0$ and $\theta_q>0$ such that for any $R>0$ and $t\in[0,\infty)$, the estimate

$$\begin{aligned} I''(t) \leq 8\left\{\|\nabla u(t)\|^2_{L^2_x} - \frac{3(p-1)}{2(p+1)}\|u(t)\|^{p+1}_{L^{p+1}_x} + \int_{\mathbb{R}^3} V(x)|u(t,x)|^2 dx\right\} \\ + C\|u(t)\|^{(p+1)\theta_q}_{L^2_x(R\leq|x|)} + \frac{C}{R^2}\|u(t)\|^2_{L^2_x(R\leq|x|)} + C\|x\cdot\nabla V\|_{L^{\frac{3}{2}}_x(R\leq|x|\leq 3R)} \end{aligned}$$

holds, where $\theta_q := \frac{2\{q-(p+1)\}}{(p+1)(q-2)} \in (0,\frac{2}{p+1}]$ and C_0 is given in Lemma 4.18.

Proof. Using Proposition 4.9, we have

$$I''(t) = 8\|\nabla u(t)\|^2_{L^2_x} - \frac{12(p-1)}{p+1}\|u(t)\|^{p+1}_{L^{p+1}_x} + 8\int_{\mathbb{R}^3} V(x)|u(t,x)|^2dx + \mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3+\mathcal{R}_4,$$

where $\mathcal{R}_k = \mathcal{R}_k(t)$ $(k=1,2,3,4)$ are defined as

$$\begin{aligned} \mathcal{R}_1 &:= 4\int_{\mathbb{R}^3}\left\{\frac{1}{r^2}\mathscr{X}''\left(\frac{r}{R}\right) - \frac{R}{r^3}\mathscr{X}'\left(\frac{r}{R}\right)\right\}|x\cdot\nabla u|^2 dx + 4\int_{\mathbb{R}^3}\left\{\frac{R}{r}\mathscr{X}'\left(\frac{r}{R}\right)-2\right\}|\nabla u(t,x)|^2dx,\\ \mathcal{R}_2 &:= -\frac{2(p-1)}{p+1}\int_{\mathbb{R}^3}\left\{\mathscr{X}''\left(\frac{r}{R}\right)+\frac{2R}{r}\mathscr{X}'\left(\frac{r}{R}\right)-6\right\}|u(t,x)|^{p+1}dx,\\ \mathcal{R}_3 &:= -\int_{\mathbb{R}^3}\left\{\frac{1}{R^2}\mathscr{X}^{(4)}\left(\frac{r}{R}\right)+\frac{4}{Rr}\mathscr{X}^{(3)}\left(\frac{r}{R}\right)\right\}|u(t,x)|^2dx,\\ \mathcal{R}_4 &:= -8\int_{\mathbb{R}^3}V(x)|u(t,x)|^2dx - 2\int_{\mathbb{R}^3}\frac{R}{r}\mathscr{X}'\left(\frac{r}{R}\right)(x\cdot\nabla V)|u(t,x)|^2dx. \end{aligned}$$

We set

$$\Omega := \left\{x\in\mathbb{R}^3 : \frac{1}{r^2}\mathscr{X}''\left(\frac{r}{R}\right) - \frac{R}{r^3}\mathscr{X}'\left(\frac{r}{R}\right)\geq 0\right\}.$$

By $\mathscr{X}'(\frac{r}{R}) \leq \frac{2r}{R}$, we have

$$\begin{aligned} \mathcal{R}_1 &\leq 4\int_\Omega\left\{\mathscr{X}''\left(\frac{r}{R}\right)-\frac{R}{r}\mathscr{X}'\left(\frac{r}{R}\right)\right\}|\nabla u(t,x)|^2dx + 4\int_\Omega\left\{\frac{R}{r}\mathscr{X}'\left(\frac{r}{R}\right)-2\right\}|\nabla u(t,x)|^2dx\\ &= 4\int_\Omega\left\{\mathscr{X}''\left(\frac{r}{R}\right)-2\right\}|\nabla u(t,x)|^2dx \leq 0. \end{aligned}$$

We estimate $\mathcal{R}_2$. Lemma 2.2 and 2.3 deduce

$$\begin{aligned} \mathcal{R}_2 &\lesssim \|u(t)\|^{p+1}_{L^{p+1}_x(R\leq|x|)} \leq \|u(t)\|^{(p+1)(1-\theta_q)}_{L^q_x(R\leq|x|)}\|u(t)\|^{(p+1)\theta_q}_{L^2_x(R\leq|x|)}\\ &\lesssim \|u(t)\|^{(p+1)(1-\theta_q)}_{H^1_x}\|u(t)\|^{(p+1)\theta_q}_{L^2_x(R\leq|x|)} \lesssim_{q,M(u),C_0} \|u(t)\|^{(p+1)\theta_q}_{L^2_x(R\leq|x|)}. \end{aligned}$$

We estimate $\mathcal{R}_3$.

$$\mathcal{R}_3 = -\int_{R\leq|x|\leq 3R}\left\{\frac{1}{R^2}\mathscr{X}^{(4)}\left(\frac{r}{R}\right)+\frac{4}{Rr}\mathscr{X}^{(3)}\left(\frac{r}{R}\right)\right\}|u(t,x)|^2dx \lesssim \frac{1}{R^2}\|u(t)\|^2_{L^2_x(R\leq|x|)}.$$

We estimate $\mathcal{R}_4$. By $V \geq 0$, $x \cdot \nabla V + 2V \geq 0$, Lemma 2.2, and 2.3, we have

$$\begin{aligned}
\mathcal{R}_4 &= -4 \int_{|x|\leq R} (x \cdot \nabla V + 2V)|u(t,x)|^2 dx - 8 \int_{R\leq |x|} V(x)|u(t,x)|^2 dx \\
&\qquad\qquad\qquad\qquad - 2 \int_{R\leq|x|\leq 3R} \frac{R}{r}\mathscr{X}'\left(\frac{r}{R}\right)(x\cdot\nabla V)|u(t,x)|^2 dx \\
&\leq -2 \int_{R\leq|x|\leq 3R} \frac{R}{r}\mathscr{X}'\left(\frac{r}{R}\right)(x\cdot\nabla V)|u(t,x)|^2 dx \\
&\lesssim \|x\cdot\nabla V\|_{L_x^{\frac{3}{2}}(R\leq|x|\leq 3R)} \|u(t)\|_{L_x^6(R\leq|x|\leq 3R)} \\
&\lesssim \|x\cdot\nabla V\|_{L_x^{\frac{3}{2}}(R\leq|x|\leq 3R)} \|\nabla u(t)\|_{L_x^2} \\
&\lesssim \|x\cdot\nabla V\|_{L_x^{\frac{3}{2}}(R\leq|x|\leq 3R)},
\end{aligned}$$

which completes the proof of this lemma. □

Lemma 4.20. *Let $d \geq 1$, $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$, and $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$. Assume that $u_0 \in H^1(\mathbb{R}^d)$ satisfies (1.20) and (1.22). Let u be a solution to (NLS_V) with (IC) given in Theorem 1.50 or Theorem 1.51. Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}} V \in L^{\frac{d}{2}}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}| = 1$ and $x \cdot \nabla V + 2V \geq 0$. Then, there exists $\delta > 0$ such that*

$$K_V(u(t)) \leq 2\|\nabla u(t)\|_{L_x^2}^2 - \frac{d(p-1)}{p+1}\|u(t)\|_{L_x^{p+1}}^{p+1} + 2\int_{\mathbb{R}^3} V(x)|u(t,x)|^2 dx < -\delta.$$

for any $t \in (T_{min}, T_{max})$.

Proof. The left inequality follows from

$$\begin{aligned}
-\int_{\mathbb{R}^3}(x\cdot\nabla V)|u(t,x)|^2 dx &= -\int_{\mathbb{R}^3}(x\cdot\nabla V + 2V)|u(t,x)|^2 dx + 2\int_{\mathbb{R}^3} V(x)|u(t,x)|^2 dx \\
&\leq 2\int_{\mathbb{R}^3} V(x)|u(t,x)|^2 dx.
\end{aligned}$$

We prove the second inequality. By the assumption (1.20),

$$\varepsilon_1 := \frac{1}{2}\left[\left\{\frac{M(Q_{1,0})}{M(u_0)}\right\}^{\frac{1-s_c}{s_c}} E_0(Q_{1,0}) - E_V(u_0)\right] > 0$$

and

$$E_V(u_0) < \frac{1}{2}E_V(u_0) + \frac{1}{2}\left\{\frac{M(Q_{1,0})}{M(u_0)}\right\}^{\frac{1-s_c}{s_c}} E_0(Q_{1,0}) = \left\{\frac{M(Q_{1,0})}{M(u_0)}\right\}^{\frac{1-s_c}{s_c}} E_0(Q_{1,0}) - \varepsilon_1. \quad (4.23)$$

Using the estimate (1.22), we have

$$\|(-\Delta_V)^{\frac{1}{2}}u(t)\|_{L_x^2}^2 > \left\{\frac{M(Q_{1,0})}{M(u_0)}\right\}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L_x^2}^2 \quad (4.24)$$

for any $t \in (T_{\min}, T_{\max})$. (4.23), (4.24), and (2.4) give

$$\begin{aligned}
&\|\nabla u(t)\|_{L_x^2}^2 - \frac{d(p-1)}{2(p+1)}\|u(t)\|_{L_x^{p+1}}^{p+1} + \int_{\mathbb{R}^3} V(x)|u(t,x)|^2 dx \\
&= \frac{d(p-1)}{2}E_V(u) - \frac{d(p-1)-4}{4}\|(-\Delta_V)^{\frac{1}{2}}u\|_{L_x^2}^2 \\
&< \frac{d(p-1)}{2}\left[\left\{\frac{M(Q_{1,0})}{M(u_0)}\right\}^{\frac{1-s_c}{s_c}} E_0(Q_{1,0}) - \varepsilon_1\right] - \frac{d(p-1)-4}{4}\left\{\frac{M(Q_{1,0})}{M(u_0)}\right\}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L^2}^2 \\
&= -\frac{d(p-1)}{2}\varepsilon_1 =: -\frac{1}{2}\delta.
\end{aligned}$$

□

Proof of blow-up or grow-up result in Main theorem 1.56 (2). We assume that

$$T_{\max} = \infty \quad \text{and} \quad C_0 := \sup_{t\in[0,\infty)} \|\nabla u(t)\|_{L^2_x} < \infty$$

for contradiction. By Lemma 4.20, there exists $\delta > 0$ such that

$$\|\nabla u(t)\|_{L^2_x}^2 - \frac{3(p-1)}{2(p+1)}\|u(t)\|_{L^{p+1}_x}^{p+1} + \int_{\mathbb{R}^3} V(x)|u(t,x)|^2 dx < -\delta$$

for any $t \in [0,\infty)$. We consider the function $I(t)$ as in Lemma 4.19. From Lemma 4.19 and 4.18, we have

$$\begin{aligned} I''(s) &\leq -8\delta + C\|u(s)\|_{L^2_x(R\leq|x|)}^{(p+1)\theta_q} + \frac{C}{R^2}\|u(s)\|_{L^2_x(R\leq|x|)}^2 + C\|x\cdot V\|_{L^{\frac{3}{2}}_x(R\leq|x|\leq 3R)} \\ &= -8\delta + C\eta^{\frac{(p+1)\theta_q}{2}} + o_R(1) \end{aligned} \tag{4.25}$$

for any $\eta > 0$, $R > 0$, and $s \in \left[0, \frac{\eta R}{4C_0 M(u_0)^{1/2}}\right]$. We take $\eta = \eta_0 > 0$ sufficiently small such as $C\eta_0^{\frac{(p+1)\theta_q}{2}} \leq 2\delta$. Then, (4.25) implies

$$I''(s) \leq -6\delta + o_R(1) \tag{4.26}$$

for any $R > 0$ and $s \in \left[0, \frac{\eta_0 R}{4C_0 M(u_0)^{1/2}}\right]$. We set

$$T = T(R) := \alpha_0 R := \frac{\eta_0 R}{4C_0 M(u_0)^{1/2}}.$$

Integrating (4.26) on $s \in [0,t]$ and integrating on $t \in [0,T]$, we have

$$I(T) \leq I(0) + I'(0)T + \frac{1}{2}\left(-6\delta + o_R(1)\right)T^2 = I(0) + I'(0)\alpha_0 R + \frac{1}{2}\left(-6\delta + o_R(1)\right)\alpha_0^2 R^2. \tag{4.27}$$

Here, we can prove

$$I(0) = o_R(1)R^2 \quad \text{and} \quad I'(0) = o_R(1)R. \tag{4.28}$$

Indeed,

$$\begin{aligned} I(0) &= \int_{|x|\leq\sqrt{R}} |x|^2|u_0(x)|^2 dx + \int_{\sqrt{R}\leq|x|\leq 3R} R^2 \mathscr{X}\left(\frac{r}{R}\right)|u_0(x)|^2 dx \\ &\leq RM(u_0) + cR^2\|u_0\|_{L^2_x(\sqrt{R}\leq|x|)} = o_R(1)R^2, \end{aligned}$$

and

$$\begin{aligned} I'(0) &= 4\mathrm{Im}\int_{|x|\leq\sqrt{R}} \overline{u_0(x)}x\cdot\nabla u_0(x)dx + 2\mathrm{Im}\int_{\sqrt{R}\leq|x|\leq 3R} \frac{R}{r}\mathscr{X}'\left(\frac{r}{R}\right)\overline{u_0(x)}x\cdot\nabla u_0(x)dx \\ &\leq 4\sqrt{R}\,\|u_0\|_{L^2_x}\|\nabla u_0\|_{L^2_x} + cR\|u_0\|_{L^2_x(\sqrt{R}\leq|x|)}\|\nabla u_0\|_{L^2_x} = o_R(1)R. \end{aligned}$$

Combining (4.27) and (4.28), we get

$$I(T) \leq (o_R(1) - 3\delta\alpha_0^2)R^2.$$

We take $R > 0$ sufficiently large such as $o_R(1) - 3\delta\alpha_0^2 < 0$. However, this contradict $I(T) \geq 0$. □

4.2.4. *Blow-up part.* In this subsubsection, we prove the blow-up part in Main theorem 1.56 (2).

Proof. We assume that $T_{\max} = \infty$.
Let $xu_0 \in L^2(\mathbb{R}^3)$.
Then, it follows from Proposition 4.8 and Lemma 4.20 that

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2_x}^2 = 4K_V(u(t)) < -4\delta < 0.$$

This is contradiction. Therefore, the solution u to (NLS$_V$) blows up.
Let V and u_0 be radially symmetric.
We define radial functions

$$F(r) = \begin{cases} \frac{1}{2}r^2 & (r \le 1), \\ \text{smooth} & (1 < r < 2), \\ \frac{3}{2}r & (2 \le r). \end{cases}$$

satisfying $1 - F'' \ge 0$ and $F_R(r) = R^2 F\left(\frac{r}{R}\right)$. We note that

$$\left| -\frac{2(p-1)}{p+1}\left\{-3 + F''\left(\frac{r}{R}\right) + \frac{2R}{r}F'\left(\frac{r}{R}\right)\right\}\right| \lesssim 4\left\{1 - F''\left(\frac{r}{R}\right)\right\} \tag{4.29}$$

for any $r \ge 0$. Using Proposition 4.9,

$$\begin{aligned}
&2\mathrm{Im}\frac{d}{dt}\int_{\mathbb{R}^3} \overline{u(t,x)}\nabla F_R(x)\cdot\nabla u(t,x)dx \\
&= 4\int_{\mathbb{R}^3} F''\left(\frac{r}{R}\right)|\nabla u|^2 dx - \frac{2(p-1)}{p+1}\int_{\mathbb{R}^3}\left\{F''\left(\frac{r}{R}\right) + \frac{2R}{r}F'\left(\frac{r}{R}\right)\right\}|u|^{p+1}dx \\
&\quad - \int_{\mathbb{R}^3}\left\{\frac{1}{R^2}F^{(4)}\left(\frac{r}{R}\right) + \frac{4}{Rr}F^{(3)}\left(\frac{r}{R}\right)\right\}|u|^2 dx - 2\int_{\mathbb{R}^3}\frac{R}{r}F'\left(\frac{r}{R}\right)(x\cdot\nabla V)|u|^2 dx \\
&= 2K_V(u(t)) - \int_{\mathbb{R}^3}\left[4\left\{1 - F''\left(\frac{r}{R}\right)\right\}|u'|^2 + \frac{2(p-1)}{p+1}\left\{-3 + F''\left(\frac{r}{R}\right) + \frac{2R}{r}F'\left(\frac{r}{R}\right)\right\}|u|^{p+1}\right. \\
&\quad \left. + \left\{\frac{1}{R^2}F^{(4)}\left(\frac{r}{R}\right) + \frac{4}{Rr}F^{(3)}\left(\frac{r}{R}\right)\right\}|u|^2\right]dx + 2\int_{\mathbb{R}^3}\left\{1 - \frac{R}{r}F'\left(\frac{r}{R}\right)\right\}(x\cdot\nabla V)|u|^2 dx \\
&= 2K_V(u(t)) + 2\int_{R\le|x|}\left\{1 - \frac{R}{r}F'\left(\frac{r}{R}\right)\right\}(x\cdot\nabla V)|u|^2 dx \\
&\quad - \int_{R\le|x|}\left[4\left\{1 - F''\left(\frac{r}{R}\right)\right\}|u'|^2 + \frac{2(p-1)}{p+1}\left\{-3 + F''\left(\frac{r}{R}\right) + \frac{2R}{r}F'\left(\frac{r}{R}\right)\right\}|u|^{p+1}\right]dx \\
&\quad - \int_{R\le|x|\le 2R}\left\{\frac{1}{R^2}F^{(4)}\left(\frac{r}{R}\right) + \frac{4}{Rr}F^{(3)}\left(\frac{r}{R}\right)\right\}|u|^2 dx \\
&=: 2K_V(u(t)) + I_1 - 4\int_{R\le|x|}\left\{1 - F''\left(\frac{r}{R}\right)\right\}|u'|^2 dx + I_2 + I_3.
\end{aligned} \tag{4.30}$$

From Lemma 4.20, there exists $\delta > 0$ such that

$$K_V(u(t)) < -\delta. \tag{4.31}$$

We can see that

$$I_3 \lesssim \frac{1}{R^2}M(u_0). \tag{4.32}$$

We estimate I_2. Applying (4.29), Lemma 2.4, and 2.1, we get

$$\begin{aligned}
I_2 &\lesssim \int_{R\le|x|}\left\{1 - F''\left(\frac{r}{R}\right)\right\}|u|^{p+1}dx \lesssim \int_R^\infty \left\{1 - F''\left(\frac{r}{R}\right)\right\}|u(r)|^{p+1}r^2 dr \\
&= \int_R^\infty \int_R^r \frac{d}{ds}\left\{1 - F''\left(\frac{s}{R}\right)\right\}ds|u(r)|^{p+1}r^2 dr \\
&= \int_R^\infty \int_s^\infty |u(r)|^{p+1}r^2 dr\frac{d}{ds}\left\{1 - F''\left(\frac{s}{R}\right)\right\}ds \\
&\lesssim \int_R^\infty \int_{s\le|x|}|u(x)|^{p+1}dx\frac{d}{ds}\left\{1 - F''\left(\frac{s}{R}\right)\right\}ds \\
&\lesssim \int_R^\infty \frac{1}{s^2}\|u(t)\|_{L_x^2(s\le|x|)}^{\frac{p+3}{2}}\|\nabla u(t)\|_{L_x^2(s\le|x|)}^{\frac{p-1}{2}}\frac{d}{ds}\left\{1 - F''\left(\frac{s}{R}\right)\right\}ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_R^\infty \left\{ \varepsilon^{-\frac{p-1}{4}} s^{-2} \|u(t)\|_{L_x^2(s\le|x|)}^{\frac{p+3}{2}} \right\} \left\{ \varepsilon^{\frac{p-1}{4}} \|\nabla u(t)\|_{L_x^2(s\le|x|)}^{\frac{p-1}{2}} \right\} \frac{d}{ds}\left\{1 - F''\left(\frac{s}{R}\right)\right\} ds \\
&\lesssim \int_R^\infty \left\{ \varepsilon^{-\frac{p-1}{5-p}} s^{-\frac{8}{5-p}} \|u(t)\|_{L_x^2(s\le|x|)}^{\frac{2(p+3)}{5-p}} + \varepsilon \|\nabla u(t)\|_{L_x^2(s\le|x|)}^2 \right\} \frac{d}{ds}\left\{1 - F''\left(\frac{s}{R}\right)\right\} ds \\
&\lesssim \varepsilon^{-\frac{p-1}{5-p}} R^{-\frac{8}{5-p}} \|u(t)\|_{L_x^2(R\le|x|)}^{\frac{2(p+3)}{5-p}} + \varepsilon \int_R^\infty \int_s^\infty |u'(r)|^2 r^2 dr \frac{d}{ds}\left\{1 - F''\left(\frac{s}{R}\right)\right\} ds \\
&= \varepsilon^{-\frac{p-1}{5-p}} R^{-\frac{8}{5-p}} \|u(t)\|_{L_x^2(R\le|x|)}^{\frac{2(p+3)}{5-p}} + \varepsilon \int_R^\infty \int_R^r \frac{d}{ds}\left\{1 - F''\left(\frac{s}{R}\right)\right\} ds |u'(r)|^2 r^2 dr \\
&= \varepsilon^{-\frac{p-1}{5-p}} R^{-\frac{8}{5-p}} \|u(t)\|_{L_x^2(R\le|x|)}^{\frac{2(p+3)}{5-p}} + \varepsilon \int_R^\infty \left\{1 - F''\left(\frac{r}{R}\right)\right\} |u'(r)|^2 r^2 dr. \qquad (4.33)
\end{aligned}
$$

By the same estimate as I_2, we have

$$
I_1 \lesssim \|x\cdot\nabla V\|_{L_x^{\frac{3}{2}}(R\le|x|)} \int_{R\le|x|} \left\{1 - F''\left(\frac{r}{R}\right)\right\} |u'(t,r)|^2 dx. \qquad (4.34)
$$

Combining (4.30), (4.31), (4.32), (4.33), and (4.34),

$$
\begin{aligned}
2\mathrm{Im}\frac{d}{dt}\int_{\mathbb{R}^3} \overline{u(t,x)}\nabla F_R(x)\cdot\nabla u(t,x)dx &\le -2\delta + \frac{c}{R^2}M(u_0) + c\varepsilon^{-\frac{p-1}{5-p}} R^{-\frac{8}{5-p}} M(u_0)^{\frac{(p+3)}{5-p}} \\
&\quad + \left(c\varepsilon + c\|x\cdot\nabla V\|_{L_x^{\frac{3}{2}}(R\le|x|)} - 4\right) \int_{R\le|x|} \left\{1 - F''\left(\frac{r}{R}\right)\right\} |u'(t,r)|^2 dx.
\end{aligned}
$$

Thus, if we take $\varepsilon > 0$ sufficiently small such as $c\varepsilon < 2$ and take $R > 0$ sufficiently large such as

$$
c\|x\cdot\nabla V\|_{L_x^{\frac{3}{2}}(R\le|x|)} < \frac{3p-7}{4} < 2 \quad \text{and} \quad \frac{c}{R^2}M(u_0) + c\varepsilon^{-\frac{p-1}{5-p}} R^{-\frac{8}{5-p}} M(u_0)^{\frac{(p+3)}{5-p}} < \delta, \qquad (4.35)
$$

then it follows that

$$
2\mathrm{Im}\frac{d}{dt}\int_{\mathbb{R}^3} \overline{u(t,x)}\nabla F_R(x)\cdot\nabla u(t,x)dx < -\delta < 0.
$$

Integrating this inequality on $[0,t)$,

$$
2\mathrm{Im}\int_{\mathbb{R}^3} \overline{u(t,x)}\nabla F_R(x)\cdot\nabla u(t,x)dx \le -\delta t + 2\mathrm{Im}\int_{\mathbb{R}^3} \overline{u_0(x)}\nabla F_R(x)\cdot\nabla u_0(x)dx. \qquad (4.36)
$$

If we take $T_0 > 0$ satisfying

$$
-\delta t + 2\mathrm{Im}\int_{\mathbb{R}^3} \overline{u_0(x)}\nabla F_R(x)\cdot\nabla u_0(x)dx < -\frac{1}{2}\delta t \qquad (4.37)
$$

for any $t > T_0$, then it follows from these inequalities that

$$
\frac{1}{2}\delta t \le \left| 2\mathrm{Im}\frac{d}{dt}\int_{\mathbb{R}^3} \overline{u(t,x)}\nabla F_R(x)\cdot\nabla u(t,x)dx \right| \le cRM(u_0)^{\frac{1}{2}}\|\nabla u(t)\|_{L_x^2}. \qquad (4.38)
$$

Therefore, we have $ct^2 \le \|\nabla u(t)\|_{L_x^2}^2$ for any $t \ge T_0$. From (4.30), Lemma 4.20, (4.34), Lemma 2.4, (4.32), (4.35), and Lemma 2.1,

$$
\begin{aligned}
&2\mathrm{Im}\frac{d}{dt}\int_{\mathbb{R}^3} \overline{u(t,x)}\nabla F_R(x)\cdot\nabla u(t,x)dx \\
&\qquad \le 4\int_{\mathbb{R}^3} \left\{ |\nabla u(t,x)|^2 + V(x)|u(t,x)|^2 - \frac{3(p-1)}{2(p+1)}|u(t,x)|^{p+1} \right\} dx \\
&\qquad\qquad + c\|x\cdot\nabla V\|_{L_x^{\frac{3}{2}}(R\le|x|)} \|\nabla u(t)\|_{L_x^2}^2 + \frac{c}{R^2}M(u_0) + \frac{c}{R^{p-1}}M(u_0)^{\frac{p+3}{4}}\|\nabla u(t)\|_{L_x^2}^{\frac{p-1}{2}} \\
&\qquad = 6(p-1)E_V(u_0) - (3p-7)\|\nabla u(t)\|_{L_x^2}^2 - (3p-7)\int_{\mathbb{R}^3} V(x)|u(t,x)|^2 dx \\
&\qquad\qquad + c\|x\cdot\nabla V\|_{L_x^{\frac{3}{2}}(R\le|x|)} \|\nabla u(t)\|_{L_x^2}^2 + \frac{c}{R^2}M(u_0) + \frac{c}{R^{p-1}}M(u_0)^{\frac{p+3}{4}}\|\nabla u(t)\|_{L_x^2}^{\frac{p-1}{2}}
\end{aligned}
$$

$$\leq 6(p-1)E_V(u_0) - \frac{3p-7}{2}\|\nabla u(t)\|_{L_x^2}^2 + \frac{c}{R^2}M(u_0) + \frac{c}{R^{\frac{4(p-1)}{5-p}}}M(u_0)^{\frac{p+3}{5-p}}.$$

Since $\|\nabla u(t)\|_{L_x^2}^2 \geq Ct^2$ and E_V, M are independent of t, there exists $T_1 \geq T_0$ such that

$$2\mathrm{Im}\frac{d}{dt}\int_{\mathbb{R}^3} \overline{u(t,x)}\nabla F_R(x)\cdot\nabla u(t,x)dx \leq -\frac{3p-7}{4}\|\nabla u(t)\|_{L_x^2}^2.$$

Integrating this inequality on $[T_1, t]$,

$$\mathrm{Im}\int_{\mathbb{R}^3} \overline{u(t,x)}\nabla F_R(x)\cdot\nabla u(t,x)dx - \mathrm{Im}\int_{\mathbb{R}^3} \overline{u(T_1,x)}\nabla F_R(x)\cdot\nabla u(T_1,x)dx$$
$$\leq -\frac{3p-7}{8}\int_{T_1}^t \|\nabla u(s)\|_{L^2}^2 ds.$$

From (4.36), (4.37), and (4.38), we get

$$\frac{3p-7}{8}\int_{T_1}^t \|\nabla u(s)\|_{L^2}^2 ds \leq -\mathrm{Im}\int_{\mathbb{R}^3} \overline{u(t,x)}\nabla F_R(x)\cdot\nabla u(t,x)dx \leq cRM(u_0)^{\frac{1}{2}}\|\nabla u(t)\|_{L_x^2}.$$

We set

$$S(t) := \int_{T_1}^t \|\nabla u(s)\|_{L_x^2}^2 ds \quad \text{and} \quad A := \frac{1}{M(u_0)}\left(\frac{3p-7}{8cR}\right)^2.$$

Then,

$$A \leq \frac{S'(t)}{S(t)^2}.$$

Integrating this inequality on $[T_1+1, t)$,

$$A(t-T_1-1) \leq \frac{1}{S(T_1+1)} - \frac{1}{S(t)} \leq \frac{1}{S(T_1+1)} < \infty.$$

However, this inequality is contradiction if we take a limit $t \to \infty$. □

4.2.5. *Corollary of Main theorem 1.56.* In this subsubsection, we prove Corollary 1.57 by using Lemma 4.18 and Lemma 4.19 with $p = \frac{7}{3}$. We also use the following lemma, which is a slight modification of Lemma 4.20.

Lemma 4.21. *Let $d = 3$ and $p = \frac{7}{3}$. Let V satisfy "$V \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap \mathcal{K}_0(\mathbb{R}^3)$ and $\|V_-\|_{\mathcal{K}} < 4\pi$" or $V \in L^\sigma(\mathbb{R}^3)$ for some $\frac{3}{2} < \sigma \leq \infty$. Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ for any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^3$ with $|\mathfrak{a}| = 1$ and $x\cdot\nabla V + 2V \geq 0$. Assume that $u_0 \in H^1(\mathbb{R}^3)$ satisfies $E_V(u_0) < 0$. Then, we have*

$$K_V(u(t)) \leq 2\|\nabla u(t)\|_{L_x^2}^2 - \frac{6}{5}\|u(t)\|_{L_x^{\frac{10}{3}}}^{\frac{10}{3}} + 2\int_{\mathbb{R}^3} V(x)|u(t,x)|^2 dx = 4E_V(u_0).$$

for any $t \in (T_{min}, T_{max})$, where u is the solution to (NLS$_V$) on (T_{min}, T_{max}).

Proof. The first inequality is proved by the same argument as the proof of Lemma 4.20. The second identity is proved by the definition of the energy E_V. □

Proof of Corollary 1.57. Corollary 1.57 is deduced by the same argument as the proof of Theorem 1.56 (2). In the argument, Lemma 4.18, 4.19, and 4.21 are used. □

4.3. **Proof of Main theorem 1.60.** In this subsection, we prove Main theorem 1.60.

4.3.1. *Non-radial case.* In this subsubsection, we prove the non-radial case in Main theorem 1.60.

Lemma 4.22. *Let $d=3$, $1<p\leq 5$, and $\omega>0$. Let $V\in L^{\frac{3}{2}}(\mathbb{R}^3)+L^{\infty}(\mathbb{R}^3)$ and $V\geq 0$. Then, it follows that*

$$n_{\omega,V}^{1,0}=\frac{p-1}{2(p+1)}\inf\{\|\phi\|_{L_x^{p+1}}^{p+1}:\phi\in H_x^1(\mathbb{R}^3)\setminus\{0\},\ \mathcal{N}_{\omega,V}(\phi)\leq 0\}$$
$$=\frac{p-1}{2(p+1)}\inf\{\|\phi\|_{H_{\omega,V}^1}^{2}:\phi\in H^1(\mathbb{R}^3)\setminus\{0\},\ \mathcal{N}_{\omega,V}(\phi)\leq 0\},$$

where

$$\|\phi\|_{H_{\omega,V}^1}^2:=\omega\|\phi\|_{L_x^2}^2+\|(-\Delta_V)^{\frac{1}{2}}\phi\|_{L_x^2}^2.$$

Proof. We only prove the first equality. The second equality holds by the same argument for the first equality. We set

$$\widetilde{n}_{\omega,V}^{1,0}:=\frac{p-1}{2(p+1)}\inf\{\|\phi\|_{L_x^{p+1}}^{p+1}:\phi\in H_x^1(\mathbb{R}^3)\setminus\{0\},\ \mathcal{N}_{\omega,V}(\phi)\leq 0\}.$$

When $\phi\in H_x^1(\mathbb{R}^3)\setminus\{0\}$ satisfies $\mathcal{N}_{\omega,V}(\phi)=0$, we have $S_{\omega,V}(\phi)=\frac{p-1}{2(p+1)}\|\phi\|_{L_x^{p+1}}^{p+1}$, so $n_{\omega,V}^{1,0}\geq\widetilde{n}_{\omega,V}^{1,0}$ holds. When $\phi\in H_x^1(\mathbb{R}^3)\setminus\{0\}$ satisfies $\mathcal{N}_{\omega,V}(f)\leq 0$, there exists $0<\lambda\leq 1$ such that $\mathcal{N}_{\omega,V}(\lambda\phi)=0$. For such $0<\lambda\leq 1$, it follows that

$$n_{\omega,V}^{1,0}\leq S_{\omega,V}(\lambda\phi)=\frac{p-1}{2(p+1)}\|\lambda\phi\|_{L_x^{p+1}}^{p+1}=\frac{p-1}{2(p+1)}\lambda^{p+1}\|\phi\|_{L_x^{p+1}}^{p+1}\leq\frac{p-1}{2(p+1)}\|\phi\|_{L_x^{p+1}}^{p+1}.\tag{4.39}$$

Therefore, we obtain $n_{\omega,V}^{1,0}\leq\widetilde{n}_{\omega,V}^{1,0}$. □

Proposition 4.23. *Let $d=3$, $1<p\leq 5$, and $\omega>0$. Let $V\in L^{\frac{3}{2}}(\mathbb{R}^3)+L^{\sigma}(\mathbb{R}^3)$ for some $\frac{3}{2}<\sigma<\infty$ and $V\geq 0$. Then, $n_{\omega,V}^{1,0}=n_{\omega,0}$ holds.*

Proof. This proof follows from the same argument with $(\alpha,\beta)=(1,0)$ as Proposition 4.36. □

Lemma 4.24. *Let $d=3$, $V\in\mathcal{K}_0(\mathbb{R}^3)$, and $\|V_-\|_{\mathcal{K}}<\pi$. Then, the integral kernel $K(x,y)$ of $(\omega-\Delta_V)^{-1}$ satisfies $K(x,y)>0$ for any $\omega>0$ and any $x,y\in\mathbb{R}^3$ with $x\neq y$.*

Proof. From the proof of [22, Proposition 5.1], it follows that

$$\widetilde{K}(t,x,y)=\frac{(2\pi t)^{-\frac{3}{2}}}{1-\|V_-\|_{\mathcal{K}}/\pi}e^{-\frac{|x-y|^2}{8t}},$$

where $\widetilde{K}(t,x,y)$ is the heat kernel of $e^{t\Delta_V}$. Combining this expression and the following formula:

$$(\omega-\Delta_V)^{-1}=\int_0^{\infty}e^{t\Delta_V}e^{-t\omega}dt,$$

we obtain the desired result. For the proof, see also [110]. □

Proposition 4.25. *Besides the assumptions of Proposition 4.23, we assume that $V\neq 0$ and "$V\in\mathcal{K}_0(\mathbb{R}^3)$ or $V>0$". Then, $n_{\omega,V}^{1,0}$ is not attained for any $\omega>0$.*

Proof. On the contrary, we assume that ϕ attains $n_{\omega,V}^{1,0}$. From $|\nabla|\phi||\leq|\nabla\phi|$ (see [52]) and Lemma 4.22, $|\phi|$ also attains $n_{\omega,V}^{1,0}$. We may assume $\phi\geq 0$. If $V\in\mathcal{K}_0(\mathbb{R}^3)$, then $\phi>0$ holds by Lemma 4.24. Therefore, we can take $y\in\mathbb{R}^3$ satisfying

$$\mathcal{N}_{\omega,V}(\phi(\cdot-y))<\mathcal{N}_{\omega,V}(\phi)=0.\tag{4.40}$$

Since $\mathcal{N}_{\omega,V}(\lambda\phi(\cdot-y))>0$ for small $\lambda\in(0,1)$ and $\mathcal{N}_{\omega,V}(\phi(\cdot-y))<0$, we can take $\lambda_0\in(0,1)$ such that $\mathcal{N}_{\omega,V}(\lambda_0\phi(\cdot-y))=0$. By the definition of $n_{\omega,V}^{1,0}$ and (4.40), we obtain

$$n_{\omega,V}^{1,0}\leq S_{\omega,V}(\lambda_0\phi(\cdot-y))<\frac{p-1}{2(p+1)}\|\phi(\cdot-y)\|_{H_{\omega,V}^1}^2<\frac{p-1}{2(p+1)}\|\phi\|_{H_{\omega,V}^1}^2=n_{\omega,V}^{1,0}.$$

This is contradiction. □

The following proposition is more general case of Proposition 1.63 (1).

Proposition 4.26. *Let $d = 3$ and $\frac{7}{3} < p < 5$, $V \in L^{\frac{3}{2}}(\mathbb{R}^3) + L^\sigma(\mathbb{R}^3)$ for some $\frac{3}{2} < \sigma < \infty$, and $V \geq 0$. Let $Q_{\omega,0}$ be the ground state to ($\mathrm{SP}_{\omega,0}$). The following two conditions (1) and (2) are equivalent.*

(1) $M(u_0)^{\frac{1-s_c}{s_c}} E_V(u_0) < M(Q_{1,0})^{\frac{1-s_c}{s_c}} E_0(Q_{1,0})$.
(2) *There exists* $\omega > 0$ *such that* $S_{\omega,V}(u_0) < n_{\omega,V}^{1,0} (= n_{\omega,0} = S_{\omega,0}(Q_{\omega,0}))$.

Proof. This proof follows from the same argument as Proposition 1.68 (1). □

4.3.2. *Radial case.* In this subsubsection, we prove the radial case in Main theorem 1.60.

Lemma 4.27 (Reflexivity). *Let $V \geq 0$ and $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$. Then, $(\dot{H}_V^1(\mathbb{R}^3), \langle\,\cdot\,,\,\cdot\,\rangle_{\dot{H}_V^1})$ is a real Hilbert space with an inner product*

$$\langle f, g\rangle_{\dot{H}_V^1} := \langle(-\Delta_V)^{\frac{1}{2}} f, (-\Delta_V)^{\frac{1}{2}} g\rangle_{L^2} = Re\int_{\mathbb{R}^3} \nabla f(x)\cdot\overline{\nabla g(x)} + V(x) f(x)\overline{g(x)}dx.$$

In particular, $(\dot{H}_V^1(\mathbb{R}^3), \langle\,\cdot\,,\,\cdot\,\rangle_{\dot{H}_V^1})$ is a reflexive space.

Proof. By the direct calculation, we can see linearity in the first exponent and conjugate symmetry of $\langle\,\cdot\,,\,\cdot\,\rangle_{\dot{H}_V^1}$. If $f = 0$, then $\|f\|_{\dot{H}_V^1} = 0$ holds clearly. Conversely, if $\|f\|_{\dot{H}_V^1} = 0$, then $f = 0$ from $0 \leq \|f\|_{\dot{H}^1} \leq \|f\|_{\dot{H}_V^1} = 0$. We prove that $(\dot{H}_V^1(\mathbb{R}^3), \|\cdot\|_{\dot{H}_V^1})$ is a Banach space. Let $\{f_n\} \subset \dot{H}_V^1(\mathbb{R}^3)$ be a Cauchy sequence. Then,

$$0 \leq \|f_m - f_n\|_{\dot{H}^1} \leq \|f_m - f_n\|_{\dot{H}_V^1} \longrightarrow 0 \quad \text{as} \quad n > m \to \infty$$

by $V \geq 0$. Thus, $\{f_n\}$ is a Cauchy sequence in $\dot{H}^1(\mathbb{R}^3)$. Since $\dot{H}^1(\mathbb{R}^3)$ is a Banach space, there exists a function $f_\infty \in \dot{H}^1(\mathbb{R}^3)$ such that $f_n \longrightarrow f_\infty$ in $\dot{H}^1(\mathbb{R}^3)$. By Sobolev's embedding, we have

$$0 \leq \|f_n - f_\infty\|_{\dot{H}_V^1}^2 \leq (1 + c\|V\|_{L^{\frac{3}{2}}})\|f_n - f_\infty\|_{\dot{H}^1}^2 \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

This implies that $(\dot{H}_V^1(\mathbb{R}^3), \|\cdot\|_{\dot{H}_V^1})$ is a Banach space. □

Lemma 4.28 (Compact embedding). *Let $d = 3$, $1 < p < 5$, $V \geq 0$, and $V \in L^{\frac{3}{2}}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. Then, the embedding $H_{V,rad}^1(\mathbb{R}^3) \subset\subset L^{p+1}(\mathbb{R}^3)$ is compact.*

Proof. This lemma follows from $\|f\|_{H^1} \leq \|f\|_{H_V^1}$ and Lemma 2.5. □

Lemma 4.29. *Let $d = 3$, $1 < p \leq 5$, and $\omega > 0$. Let $V \in L^{\frac{3}{2}}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ and $V \geq 0$. Then, it follows that*

$$\begin{aligned} r_{\omega,V}^{1,0} &= \frac{p-1}{2(p+1)} \inf\{\|\phi\|_{L^{p+1}}^{p+1} : \phi \in H_{rad}^1(\mathbb{R}^3) \setminus \{0\},\ \mathcal{N}_{\omega,V}(\phi) \leq 0\} \\ &= \frac{p-1}{2(p+1)} \inf\{\|\phi\|_{H_{\omega,V}^1}^2 : \phi \in H_{rad}^1(\mathbb{R}^3) \setminus \{0\},\ \mathcal{N}_{\omega,V}(\phi) \leq 0\}. \end{aligned}$$

Proof. This lemma follows from the same argument as Lemma 4.22. □

The following theorem implies attainability of $r_{\omega,V}^{1,0}$ in Main theorem 1.60 (radial case).

Theorem 4.30. *Let $d = 3$, $1 < p < 5$, and $\omega > 0$. Let $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $V \geq 0$. Then, there exists a function $Q_{\omega,V} \in H_{rad}^1(\mathbb{R}^3) \setminus \{0\}$ such that $Q_{\omega,V}$ attains $r_{\omega,V}^{1,0}$ and $\mathcal{N}_{\omega,V}(Q_{\omega,V}) = 0$.*

Proof. We take a minimizing sequence $\{\phi_n\} \subset H^1_{\mathrm{rad}} \setminus \{0\}$ of $r^{1,0}_{\omega,V}$, that is, ϕ_n satisfies

$$\frac{p-1}{2(p+1)}\|\phi_n\|^{p+1}_{L^{p+1}_x} = \frac{p-1}{2(p+1)}\|\phi_n\|^2_{H^1_{\omega,V}} = S_{\omega,V}(\phi_n) \searrow r^{1,0}_{\omega,V} \quad \text{and} \quad \mathcal{N}_{\omega,V}(\phi_n) = 0.$$

Then, $\{\phi_n\}$ is bounded in $L^2_x(\mathbb{R}^3)$ and $\dot{H}^1_V(\mathbb{R}^3)$. From Lemma 4.27 and Lemma 4.28, we can take a subsequence of $\{\phi_n\}$ (, which is denoted by the same symbol) satisfying $\phi_n \rightharpoonup Q_{\omega,V}$ in $L^2(\mathbb{R}^3)$ and $\dot{H}^1_V(\mathbb{R}^3)$ and $\phi_n \longrightarrow Q_{\omega,V}$ in $L^{p+1}_x(\mathbb{R}^3)$. Therefore, we get

$$\|Q_{\omega,V}\|_{L^2_x} \le \liminf_{n\to\infty} \|\phi_n\|_{L^2_x}, \quad \|(-\Delta_V)^{\frac{1}{2}} Q_{\omega,V}\|_{L^2_x} \le \liminf_{n\to\infty} \|(-\Delta_V)^{\frac{1}{2}}\phi_n\|_{L^2_x},$$
$$\frac{p-1}{2(p+1)}\|Q_{\omega,V}\|^{p+1}_{L^{p+1}_x} = \frac{p-1}{2(p+1)} \lim_{n\to\infty} \|\phi_n\|^{p+1}_{L^{p+1}_x} = r^{1,0}_{\omega,V} \ge n^{1,0}_{\omega,V} = n_{\omega,0} > 0.$$

The last inequality implies that $Q_{\omega,V} \neq 0$. In addition, these relations deduces that

$$\mathcal{N}_{\omega,V}(Q_{\omega,V}) \le \liminf_{n\to\infty} \mathcal{N}_{\omega,V}(\phi_n) = 0.$$

Therefore, there exists $\lambda \in (0, 1]$ such that $\mathcal{N}_{\omega,V}(\lambda Q_{\omega,V}) = 0$. For such λ, we have

$$r^{1,0}_{\omega,V} \le S_{\omega,V}(\lambda Q_{\omega,V}) = \frac{p-1}{2(p+1)}\|\lambda Q_{\omega,V}\|^{p+1}_{L^{p+1}_x} \le \frac{p-1}{2(p+1)}\|Q_{\omega,V}\|^{p+1}_{L^{p+1}_x} = r^{1,0}_{\omega,V}.$$

We can see that λ must be 1 and $\mathcal{N}(Q_{\omega,V}) = 0$ holds. □

Remark 4.31. Theorem 4.30 implies that $\mathcal{M}_{\omega,V,\mathrm{rad}}$ is not empty under the assumptions of the proposition.

Proposition 4.32. *Let $d = 3$, $1 < p \le 5$, and $V \in L^{\frac{3}{2}}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. Let V be radially symmetric. Then, $\mathcal{M}^{1,0}_{\omega,V,\mathrm{rad}} \subset \mathcal{G}_{\omega,V,\mathrm{rad}}$ holds.*

Proof. We take any $\phi \in \mathcal{M}^{1,0}_{\omega,V,\mathrm{rad}}$. Then,

$$\langle \mathcal{N}'_{\omega,V}(\phi), \phi\rangle = \partial_\lambda\, \mathcal{N}_{\omega,V}(e^\lambda \phi)\Big|_{\lambda=0} = -(p-1)\|\phi\|^{p+1}_{L^{p+1}_x} < 0. \tag{4.41}$$

Take any $w \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$. We define two functions f and g as follows:

$$f(s,t) := S_{\omega,V}(\phi + sw + t\phi), \quad g(s,t) := \mathcal{N}_{\omega,V}(\phi + sw + t\phi), \quad (s,t) \in \mathbb{R}^2.$$

Then, f and g satisfy $f, g \in C^1(\mathbb{R}^2)$,

$$g(0,0) = \mathcal{N}_{\omega,V}(\phi) = 0, \quad \text{and} \quad g_t(0,0) = \langle \mathcal{N}'_{\omega,V}(\phi), \phi\rangle \neq 0.$$

By the implicit function theorem, there exists a real-valued function $\gamma \in C^1(-\delta,\delta)$ such that $\gamma(0) = 0$,

$$g(s,\gamma(s)) = 0, \quad g_s(s,\gamma(s)) + g_t(s,\gamma(s))\gamma'(s) = 0, \quad s \in (-\delta,\delta).$$

Since $\mathcal{N}_{\omega,V}(\phi + sw + \gamma(s)\phi) = 0$ for $s \in (-\delta,\delta)$, $f(s,\gamma(s))$ has a local minimum at $s = 0$. Hence, we have

$$0 = \frac{d}{ds} f(0,\gamma(0)) = \langle S'_{\omega,V}(\phi), w\rangle - \langle S'_{\omega,V}(\phi), \phi\rangle \frac{\langle \mathcal{N}'_{\omega,V}(\phi), w\rangle}{\langle \mathcal{N}'_{\omega,V}(\phi), \phi\rangle}.$$

Since $\langle S'_{\omega,V}(\phi), z\rangle = 0$ and $\langle \mathcal{N}'_{\omega,V}(\phi), z\rangle = 0$ hold for any $z \in H^1_{\mathrm{rad}}(\mathbb{R}^3)^\perp$, there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that

$$S'_{\omega,V}(\phi) = \eta \mathcal{N}'_{\omega,V}(\phi). \tag{4.42}$$

Since

$$0 = \mathcal{N}_{\omega,V}(\phi) = \langle S'_{\omega,V}(\phi), \phi\rangle = \eta\langle \mathcal{N}'_{\omega,V}(\phi), \phi\rangle$$

and (4.41), we have $\eta = 0$. Combining $\eta = 0$ and (4.42), we have $S'_{\omega,V}(\phi) = 0$. We take any $\psi \in \mathcal{A}_{\omega,V,\mathrm{rad}}$. Then, $\mathcal{N}_{\omega,V}(\psi) = \langle S'_{\omega,V}(\psi), \psi\rangle = 0$. Since ϕ is in $\mathcal{M}^{1,0}_{\omega,V,\mathrm{rad}}$, we get $S_{\omega,V}(\phi) \le S_{\omega,V}(\psi)$. Therefore, we obtain $\phi \in \mathcal{G}_{\omega,V,\mathrm{rad}}$. □

Proposition 4.33. *Let $d = 3$, $1 < p \leq 5$, and $V \in L^{\frac{3}{2}}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. Let V be radially symmetric. If $\mathcal{M}^{1,0}_{\omega,V,\mathrm{rad}}$ is not empty, then $\mathcal{G}_{\omega,V,\mathrm{rad}} \subset \mathcal{M}^{1,0}_{\omega,V,\mathrm{rad}}$ holds.*

Proof. We take any $\phi \in \mathcal{G}_{\omega,V,\mathrm{rad}}$ and a function $\psi \in \mathcal{M}^{1,0}_{\omega,V,\mathrm{rad}} \subset \mathcal{G}_{\omega,V,\mathrm{rad}}$, where the last inclusion is used Proposition 4.32. Let $w \in H^1_{\mathrm{rad}}(\mathbb{R}^3) \setminus \{0\}$ satisfy $\mathcal{N}_{\omega,V}(w) = 0$. Then, we have

$$S_{\omega,V}(\phi) = S_{\omega,V}(\psi) \leq S_{\omega,V}(w).$$

Moreover, it follows that $\mathcal{N}_{\omega,V}(\phi) = \langle S'_{\omega,V}(\phi), \phi\rangle = 0$. Therefore, we obtain $\phi \in \mathcal{M}^{1,0}_{\omega,V,\mathrm{rad}}$. □

Corollary 4.34. *Let $d = 3$, $1 < p < 5$, and $\omega > 0$. Let $V \in L^{\frac{3}{2}}_{rad}(\mathbb{R}^3)$ and $V \geq 0$. Then, it follows that $\mathcal{G}_{\omega,V,\mathrm{rad}} = \mathcal{M}^{1,0}_{\omega,V,\mathrm{rad}}$.*

This corollary is deduced immediately by Theorem 4.30, Proposition 4.32, and 4.33.

4.4. **Proof of Theorem 1.62.**

In this subsection, we prove Theorem 1.62 and Proposition 1.63 (2).

Proof of Theorem 1.62. We note that $\mathcal{N}_{\omega,V}(u_0) = 0$ implies that $u_0(x) \equiv 0$ by the definition of $r^{1,0}_{\omega,V}$ and the assumption $S_{\omega,V}(u_0) < r^{1,0}_{\omega,V}$. Then, we consider only case $\mathcal{N}_{\omega,V}(u_0) > 0$. First, we prove that the solution u to (NLS$_V$) satisfies $\mathcal{N}_{\omega,V}(u(t)) > 0$ for any $t \in (T_{\min}, T_{\max})$. If there exists $t_0 \in (T_{\min}, T_{\max})$ such that $\mathcal{N}_{\omega,V}(u(t_0)) = 0$, then $S_{\omega,V}(u(t_0)) \geq r_{\omega,V}$ by the definition of $r_{\omega,V}$. On the other hands, we have $S_{\omega,V}(u(t_0)) = S_{\omega,V}(u_0) < r_{\omega,V}$ by the conservation laws. This is contradiction. Since $\mathcal{N}_{\omega,V}(u(t)) > 0$ for any $t \in (T_{\min}, T_{\max})$, we have

$$r^{1,0}_{\omega,V} > S_{\omega,V}(u_0) \geq \frac{p-1}{2(p+1)} \|u(t)\|^2_{H^1_{\omega,V}} \sim \|u(t)\|^2_{H^1_x}$$

for any $t \in (T_{\min}, T_{\max})$. This inequality implies that the solution u to (NLS$_V$) exists globally in time. □

Proof of Proposition 1.63 (2). We consider $\lambda Q_{\omega,V}$ for $\lambda > 0$ and the "radial" ground state $Q_{\omega,V}$ to (SP$_{\omega,V}$). We define a function f as $f(\lambda) = S_{\omega,V}(\lambda Q_{\omega,V})$. Solving $f'(\lambda_0) = 0$, we get $\lambda_0 = 1$ from $\mathcal{N}_{\omega,V}(Q_{\omega,V}) = 0$. Thus, the function f has a maximum value $f(\lambda_0) = S_{\omega,V}(Q_{\omega,V}) = r^{1,0}_{\omega,V}$ at $\lambda = \lambda_0$. Therefore, there exists $0 < \delta < 1$ such that $n^{1,0}_{\omega,V} \leq f(\lambda) < r^{1,0}_{\omega,V}$ for any $\lambda \in [\delta, 1)$. On the other hand, we have $\mathcal{N}_{\omega,V}(\lambda Q_{\omega,V}) \geq 0$ for any $\lambda \in (0, 1]$. Therefore, if we set $u_0 = \lambda Q_{\omega,V}$ for $\lambda \in [\delta, 1)$, then we obtain the desired result. □

4.5. **Proof of Main theorem 1.64.**

In this subsection, we prove Main theorem 1.64.

4.5.1. *Non-radial case.*

In this subsubsection, we prove the non-radial case in Main theorem 1.64.

We note that $K^{\alpha,\beta}_{\omega,V}$ can be written as follows:

$$\begin{aligned} K^{\alpha,\beta}_{\omega,V}(f) &= \frac{\omega(2\alpha - d\beta)}{2}\|f\|^2_{L^2_x} + \frac{2\alpha - (d-2)\beta}{2}\|\nabla f\|^2_{L^2_x} + \frac{2\alpha - d\beta}{2}\int_{\mathbb{R}^d} V(x)|f(x)|^2 dx \\ &\quad - \frac{\beta}{2}\int_{\mathbb{R}^d}(x\cdot\nabla V)|f(x)|^2dx - \frac{(p+1)\alpha - d\beta}{p+1}\|f\|^{p+1}_{L^{p+1}_x} \quad (4.43) \\ &= \frac{\omega\{2\alpha - (d-2)\beta\}}{2}\|f\|^2_{L^2_x} + \frac{2\alpha - (d-2)\beta}{2}\|(-\Delta_V)^{\frac{1}{2}} f\|^2_{L^2_x} \\ &\quad - \frac{\beta}{2}\int_{\mathbb{R}^d}(2\omega + 2V + x\cdot\nabla V)|f(x)|^2dx - \frac{(p+1)\alpha - d\beta}{p+1}\|f\|^{p+1}_{L^{p+1}_x} \quad (4.44) \end{aligned}$$

and (1.26) deduces the following relations:

$$\overline{\mu} := 2\alpha - (d-2)\beta \geq \underline{\mu} \geq 0, \quad \overline{\mu} > 0, \quad (p+1)\alpha - d\beta > (p-1)\alpha - 2\beta > 0.$$

We define the following functional:

$$T^{\alpha,\beta}_{\omega,V}(f) := S_{\omega,V}(f) - \frac{1}{2\alpha-(d-2)\beta}K^{\alpha,\beta}_{\omega,V}(f)$$
$$= \frac{\beta}{2\{2\alpha-(d-2)\beta\}}\int_{\mathbb{R}^d}(2\omega+2V+x\cdot\nabla V)|f(x)|^2dx + \frac{(p-1)\alpha-2\beta}{(p+1)\{2\alpha-(d-2)\beta\}}\|f\|^{p+1}_{L^{p+1}_x}.$$

Lemma 4.35. *Let $d \geq 1$, $1+\frac{4}{d} < p < \infty$ if $d = 1,2$, $1+\frac{4}{d} < p < 1+\frac{4}{d-2}$ if $d \geq 3$, and let (α,β) satisfy (1.26). Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\eta}(\mathbb{R}^d)+L^{\infty}(\mathbb{R}^d)$ for $\eta = 1$ if $d = 1$, some $\eta > 1$ if $d = 2$, $\eta = \frac{d}{2}$ if $d \geq 3$, and any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}| \leq 1$, $V \geq 0$, $x\cdot\nabla V \leq 0$, and $\omega_0 < \infty$. If $\omega > 0$ satisfies $\omega \geq \omega_0$, then*

$$n^{\alpha,\beta}_{\omega,V} = \inf\{T^{\alpha,\beta}_{\omega,V}(\phi) : \phi \in H^1(\mathbb{R}^d)\setminus\{0\},\ K^{\alpha,\beta}_{\omega,V}(\phi) \leq 0\}$$

holds.

In the next proposition, we prove that $n^{\alpha,\beta}_{\omega,V}$ is independent of (α,β) under the assumptions of Main theorem 1.64 (non-radial case).

Proof. This lemma follows from the same argument as Lemma 4.22. We note that corresponding inequality to (4.39) follows from $\omega \geq \omega_0$. □

Proposition 4.36. *Let $d \geq 1$, $1+\frac{4}{d} < p < \infty$ if $d = 1,2$, $1+\frac{4}{d} < p < 1+\frac{4}{d-2}$ if $d \geq 3$, $\omega > 0$, and let (α,β) satisfy (1.26). Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\eta}(\mathbb{R}^d)+L^{\sigma}(\mathbb{R}^d)$ for $\eta = 1$ if $d = 1$, some $\eta > 1$ if $d = 2$, $\eta = \frac{d}{2}$ if $d \geq 3$, some $\eta \leq \sigma < \infty$, and any $\mathfrak{a} \in (\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}| \leq 1$, $V \geq 0$, $x\cdot\nabla V \leq 0$, and $2V + x\cdot\nabla V \geq 0$. Then, the identity $n^{\alpha,\beta}_{\omega,V} = n_{\omega,0}$ holds.*

Proof. When $V = 0$, this proposition is trivial. We assume $V \neq 0$. First, we prove $n^{\alpha,\beta}_{\omega,V} \geq n_{\omega,0}$. We take any $\phi \in H^1(\mathbb{R}^d)\setminus\{0\}$ satisfying $K^{\alpha,\beta}_{\omega,V}(\phi) = 0$. Since $K^{\alpha,\beta}_{\omega,0}(\phi) \leq K^{\alpha,\beta}_{\omega,V}(\phi) = 0$ by $V \geq 0$ and $x\cdot\nabla V \leq 0$, we have

$$n_{\omega,0} \leq T^{\alpha,\beta}_{\omega,0}(\phi) \leq T^{\alpha,\beta}_{\omega,V}(\phi) = S_{\omega,V}(\phi)$$

by Lemma 4.35 and $2V + x\cdot\nabla V \geq 0$, which implies $n_{\omega,0} \leq n^{\alpha,\beta}_{\omega,V}$. Next, we prove $n_{\omega,0} \geq n^{\alpha,\beta}_{\omega,V}$. We note that the ground state $Q_{\omega,0}$ to ($\text{SP}_{\omega,0}$) attains $n_{\omega,0}$. For any $\{y_n\}$ satisfying $|y_n| \longrightarrow \infty$, it follows that

$$S_{\omega,V}(Q_{\omega,0}(\,\cdot\,-y_n)) \longrightarrow S_{\omega,0}(Q_{\omega,0}) = n_{\omega,0} \quad \text{as} \quad n \to \infty$$

by $V \in L^{\frac{d}{2}}(\mathbb{R}^d)+L^{\sigma}(\mathbb{R}^d)$ for some $\frac{d}{2} \leq \sigma < \infty$. For any $\{y_n\}$ satisfying $|y_n| \longrightarrow \infty$, we also have

$$K^{\alpha,\beta}_{\omega,V}(Q_{\omega,0}(\,\cdot\,-y_n)) > K^{\alpha,\beta}_{\omega,0}(Q_{\omega,0}(\,\cdot\,-y_n)) = K^{\alpha,\beta}_{\omega,0}(Q_{\omega,0}) = 0$$

by $V \geq 0$ and $x\cdot\nabla V \leq 0$. Since $K^{\alpha,\beta}_{\omega,V}(Q_{\omega,0}(\,\cdot\,-y_n)) > 0$ and $K^{\alpha,\beta}_{\omega,V}(\lambda Q_{\omega,0}(\,\cdot\,-y_n)) < 0$ for a sufficiently large $\lambda > 1$, we can take $\lambda_n > 1$ with $K^{\alpha,\beta}_{\omega,V}(\lambda_n Q_{\omega,0}(\,\cdot\,-y_n)) = 0$. For this sequence $\{\lambda_n\}$, we have $\lambda_n \longrightarrow 1$ as $n \to \infty$. Indeed, $K^{\alpha,\beta}_{\omega,V}(\lambda_n Q_{\omega,0}(\,\cdot\,-y_n)) = 0$ and $K^{\alpha,\beta}_{\omega,0}(Q_{\omega,0}) = 0$ deduces

$$0 = \frac{(p+1)\alpha-d\beta}{p+1}(1-\lambda_n^{p-1})\|Q_{\omega,0}\|^{p+1}_{L^{p+1}_x}$$
$$+ \frac{2\alpha-d\beta}{2}\int_{\mathbb{R}^d}V(x)Q_{\omega,0}(x-y_n)^2dx - \frac{\beta}{2}\int_{\mathbb{R}^d}(x\cdot\nabla V)Q_{\omega,0}(x-y_n)^2dx.$$

Thus, we have $\lambda_n \longrightarrow 1$ as $n \to \infty$. Hence, $S_{\omega,V}(\lambda_n Q_{\omega,0}(\,\cdot\,-y_n)) \longrightarrow S_{\omega,0}(Q_{\omega,0}) = n_{\omega,0}$ as $n \to \infty$ and $K^{\alpha,\beta}_{\omega,V}(\lambda_n Q_{\omega,V}(\,\cdot\,-y_n)) = 0$ for each $n \in \mathbb{N}$. This implies $n_{\omega,0} \geq n^{\alpha,\beta}_{\omega,V}$. □

In the next proposition, we complete the proof of non-radial case in Main theorem 1.64.

Proposition 4.37. *Besides the assumptions of Proposition 4.36, we assume that $x\cdot\nabla V < 0$. Then, $n^{\alpha,\beta}_{\omega,V}$ is not attained.*

Proof. We note that $V \geq 0$ and $x \cdot \nabla V < 0$ imply $V > 0$. We assume for contradiction that there exists $\phi \in H^1(\mathbb{R}^d)$ such that ϕ attains $n_{\omega,V}^{\alpha,\beta}$. We take $y \in \mathbb{R}^d$ satisfying

$$K_{\omega,V}^{\alpha,\beta}(\phi(\cdot - y)) < K_{\omega,V}^{\alpha,\beta}(\phi) = 0 \tag{4.45}$$

and

$$\int_{\mathbb{R}^d} (2V + x \cdot \nabla V)|\phi(x-y)|^2 dx \leq \int_{\mathbb{R}^d} (2V + x \cdot \nabla V)|\phi(x)|^2 dx. \tag{4.46}$$

We note that if we take $y \in \mathbb{R}^d$ with sufficiently large $|y|$, then this inequality holds by $x^{\mathfrak{a}} \partial^{\mathfrak{a}} V \in L^{\frac{d}{2}}(\mathbb{R}^d) + L^{\sigma}(\mathbb{R}^d)$ for some $\frac{d}{2} \leq \sigma < \infty$ and any $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^d$ with $|\mathfrak{a}| \leq 1$, $V > 0$, and $x \cdot \nabla V < 0$. Since $K_{\omega,V}^{\alpha,\beta}(\lambda\phi(\cdot - y)) > 0$ for some $0 < \lambda < 1$, combining this inequality and (4.45), there exists $0 < \lambda_0 < 1$ such that $K_{\omega,V}^{\alpha,\beta}(\lambda_0\phi(\cdot - y)) = 0$. Therefore, the definition of $n_{\omega,V}^{\alpha,\beta}$ and (4.46) imply

$$n_{\omega,V}^{\alpha,\beta} \leq S_{\omega,V}(\lambda_0\phi(\cdot - y)) = T_{\omega,V}^{\alpha,\beta}(\lambda_0\phi(\cdot - y)) < T_{\omega,V}^{\alpha,\beta}(\phi(\cdot - y)) \leq T_{\omega,V}^{\alpha,\beta}(\phi) = S_{\omega,V}(\phi) = n_{\omega,V}^{\alpha,\beta}.$$

This is contradiction. □

Proof of Proposition 1.68 (1). By the equation ($\mathrm{SP}_{\omega,0}$), we have $Q_{\omega,0} = \omega^{\frac{1}{p-1}} Q_{1,0}(\omega^{\frac{1}{2}} \cdot)$. Then,

$$S_{\omega,0}(Q_{\omega,0}) = \omega^{\frac{d+2-(d-2)p}{2(p-1)}} S_{1,0}(Q_{1,0})$$

holds. Thus, the condition $S_{\omega,V}(u_0) < n_{\omega,V}^{\alpha,\beta}$ in Proposition 1.68 (1) is equivalent to

$$S_{\omega,V}(u_0) < \omega^{\frac{d+2-(d-2)p}{2(p-1)}} S_{1,0}(Q_{1,0})$$

by using Proposition 4.36. Here, we define a function $f(\omega) := \omega^{\frac{d+2-(d-2)p}{2(p-1)}} S_{1,0}(Q_{1,0}) - S_{\omega,V}(u_0)$ on $\omega \in (0, \infty)$. Solving $f'(\omega_0) = 0$, we get

$$\omega_0 = \left\{ \frac{p-1}{d+2-(d-2)p} \cdot \frac{M(u_0)}{S_{1,0}(Q_{1,0})} \right\}^{\frac{2(p-1)}{d+4-dp}}.$$

The function f has a maximum value at $\omega = \omega_0$ by $p > 1 + \frac{4}{d}$. Therefore, if there exists $\omega > 0$ such that $S_{\omega,V}(u_0) < S_{\omega,0}(Q_{\omega,0})$, then $f(\omega_0) > 0$ holds. Since

$$f(\omega_0) = \left\{ \frac{d+2-(d-2)p}{p-1} \right\}^{\frac{2(p-1)}{dp-(d+4)}} \frac{dp-(d+4)}{2\{d+2-(d-2)p\}} \frac{S_{1,0}(Q_{1,0})^{\frac{2(p-1)}{dp-(d+4)}}}{M(u_0)^{\frac{d+2-(d-2)p}{dp-(d+4)}}} - E_V(u_0).$$

$f(\omega_0) > 0$ implies

$$\left\{ \frac{d+2-(d-2)p}{p-1} \right\}^{\frac{1}{s_c}} \frac{dp-(d+4)}{2\{d+2-(d-2)p\}} S_{1,0}(Q_{1,0})^{\frac{1}{s_c}} > M(u_0)^{\frac{1-s_c}{s_c}} E_V(u_0).$$

Here, calculating $S_{1,0}(Q_{1,0})$ by using Proposition 2.6, we have

$$S_{1,0}(Q_{1,0}) = \frac{p-1}{d+2-(d-2)p} \left[\frac{2\{d+2-(d-2)p\}}{dp-(d+4)} \right]^{s_c} M(Q_{1,0})^{1-s_c} E_0(Q_{1,0})^{s_c}.$$

Therefore, we obtain

$$M(Q_{1,0})^{\frac{1-s_c}{s_c}} E_0(Q_{1,0}) > M(u_0)^{\frac{1-s_c}{s_c}} E_V(u_0).$$

□

4.5.2. *Radial case.* In this subsubsection, we prove the radial case in Main theorem 1.64.

Lemma 4.38 (Positivity of $T^{\alpha,\beta}_{\omega,V}$). *Let $d \geq 1$, $1+\frac{4}{d} < p < \infty$ if $d = 1, 2$, $1+\frac{4}{d} < p < 1+\frac{4}{d-2}$ if $d \geq 3$, and let (α, β) satisfy (1.26). We assume that $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^\eta(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for $\eta = 1$ if $d = 1$, some $\eta > 1$ if $d = 2$, $\eta = \frac{d}{2}$ if $d \geq 3$, and any $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^d$ with $|\mathfrak{a}| \leq 1$, $V \geq 0$, and $\omega_0 < \infty$. If $\omega > 0$ satisfies $\omega \geq \omega_0$ and $f \in H^1(\mathbb{R}^d) \setminus \{0\}$ satisfies $K^{\alpha,\beta}_{\omega,V}(f) = 0$, then we have $T^{\alpha,\beta}_{\omega,V}(f) > 0$.*

Proof. This proposition follows from (1.26) and $2\omega + 2V + x \cdot \nabla V \geq 0$ a.e. $x \in \mathbb{R}^d$. □

Lemma 4.39 (Equivalence of H^1-norm and $S_{\omega,V}$). *Let $d \geq 1$, $1+\frac{4}{d} < p < \infty$ if $d = 1, 2$, $1+\frac{4}{d} < p < 1+\frac{4}{d-2}$ if $d \geq 3$, and let (α, β) satisfy (1.26). Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\frac{d}{2}}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for $\eta = 1$ if $d = 1$, some $\eta > 1$ if $d = 2$, $\eta = \frac{d}{2}$ if $d \geq 3$, and any $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^d$ with $|\mathfrak{a}| \leq 1$, $V \geq 0$, and $\omega_0 < \infty$. We assume that $\omega > 0$ satisfies $\omega \geq \omega_0$ and $f \in H^1(\mathbb{R}^d)$ satisfies $K^{\alpha,\beta}_{\omega,V}(f) \geq 0$. Then,*

$$\{(p-1)\alpha - 2\beta\}S_{\omega,V}(f) \leq \frac{(p-1)\alpha - 2\beta}{2}\|f\|^2_{H^1_{\omega,V}} \leq \{(p+1)\alpha - d\beta\}S_{\omega,V}(f)$$

holds.

Proof. The first inequality holds clearly. We prove the second inequality.

$$\begin{aligned}\frac{(p-1)\alpha - 2\beta}{2}J_{\omega,V}(f) &\leq \frac{(p-1)\alpha - 2\beta}{2}J_{\omega,V}(f) + K^{\alpha,\beta}_{\omega,V}(f)\\ &= \{(p+1)\alpha - d\beta\}S_{\omega,V}(f) - \frac{\beta}{2}\int_{\mathbb{R}^d}(2\omega + 2V + x\cdot\nabla V)|f(x)|^2dx\\ &\leq \{(p+1)\alpha - d\beta\}S_{\omega,V}(f).\end{aligned}$$

□

Lemma 4.40 (Positivity of $K^{\alpha,\beta}_{\omega,V}$). *Let $d \geq 1$, $1+\frac{4}{d} < p < \infty$ if $d = 1, 2$, $1+\frac{4}{d} < p < 1+\frac{4}{d-2}$ if $d \geq 3$, and (α, β) satisfy (1.26). Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^\eta(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for $\eta = 1$ if $d = 1$, some $\eta > 1$ if $d = 2$, $\eta = \frac{d}{2}$ if $d \geq 3$, and any $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^d$ with $|\mathfrak{a}| \leq 1$, $V \geq 0$, and $x \cdot \nabla V \leq 0$. Suppose that $\{f_n\}$ is a bounded sequence in $H^1(\mathbb{R}^d) \setminus \{0\}$ and satisfies $\|\nabla f_n\|_{L^2} \longrightarrow 0$ as $n \to \infty$. Then, there exists $n_0 \in \mathbb{N}$ such that $K^{\alpha,\beta}_{\omega,V}(f_n) > 0$ for any $n \geq n_0$.*

Proof. We take a positive constant C such as $\sup_{n\in\mathbb{N}}\|f_n\|_{L^2_x} \leq C$. Applying Proposition 1.16, we have

$$\begin{aligned}K^{\alpha,\beta}_{\omega,V}(f_n) \geq{}& \frac{\omega(2\alpha - d\beta)}{2}\|f_n\|^2_{L^2_x} + \frac{2\alpha - (d-2)\beta}{2}\|\nabla f_n\|^2_{L^2_x} + \frac{2\alpha - d\beta}{2}\int_{\mathbb{R}^d}V(x)|f_n(x)|^2dx\\ &- \frac{\beta}{2}\int_{\mathbb{R}^d}(x\cdot\nabla V)|f_n(x)|^2dx - \frac{(p+1)\alpha - d\beta}{p+1}C_{\mathrm{GN}}\|\nabla f_n\|^{\frac{d(p-1)}{2}}_{L^2_x}\|f_n\|^{\frac{d+2-(d-2)p}{2}}_{L^2_x}\\ \geq{}& \left\{\frac{2\alpha - (d-2)\beta}{2} - \frac{(p+1)\alpha - d\beta}{p+1}C_{\mathrm{GN}}\,C^{\frac{d+2-(d-2)p}{2}}\|\nabla f_n\|^{\frac{dp-(d+4)}{2}}_{L^2_x}\right\}\|\nabla f_n\|^2_{L^2_x}.\end{aligned}$$

When $\|\nabla f_n\|_{L^2_x} \neq 0$ is sufficiently small, we obtain $K^{\alpha,\beta}_{\omega,V}(f_n) > 0$. □

Lemma 4.41 (Positivity of $r^{\alpha,\beta}_{\omega,V}$). *Let $d \geq 1$, $1+\frac{4}{d} < p < \infty$ if $d = 1, 2$, $1+\frac{4}{d} < p < 1+\frac{4}{d-2}$ if $d \geq 3$, and let (α, β) satisfy (1.26). Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^\eta(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for $\eta = 1$ if $d = 1$, some $\eta > 1$ if $d = 2$, $\eta = \frac{d}{2}$ if $d = 3$, and any $\mathfrak{a} \in (\mathbb{N} \cup \{0\})^d$ with $|\mathfrak{a}| \leq 1$, $V \geq 0$, $x \cdot \nabla V \leq 0$, and $\omega_0 < \infty$. If $\omega \geq \omega_0$, then we have $r^{\alpha,\beta}_{\omega,V} > 0$.*

Proof. We take any $\phi \in H^1_{\mathrm{rad}}(\mathbb{R}^d) \setminus \{0\}$ satisfying $K^{\alpha,\beta}_{\omega,V}(\phi) = 0$. Then,

$$\frac{2\alpha - (d-2)\beta}{2}\|\nabla\phi\|^2_{L^2_x} \leq \frac{\omega(2\alpha - d\beta)}{2}\|\phi\|^2_{L^2_x} + \frac{2\alpha - (d-2)\beta}{2}\|\nabla\phi\|^2_{L^2_x}$$

$$+\frac{2\alpha-d\beta}{2}\int_{\mathbb{R}^d}V(x)|\phi(x)|^2dx-\frac{\beta}{2}\int_{\mathbb{R}^d}(x\cdot\nabla V)|\phi(x)|^2dx$$
$$=\frac{(p+1)\alpha-d\beta}{p+1}\|\phi\|_{L_x^{p+1}}^{p+1}$$
$$\leq\frac{(p+1)\alpha-d\beta}{p+1}C_{\mathrm{GN}}\|\phi\|_{L_x^2}^{\frac{d+2-(d-2)p}{2}}\|\nabla\phi\|_{L_x^2}^{\frac{d(p-1)}{2}}$$

by $K_{\omega,V}^{\alpha,\beta}(\phi)=0$ and Proposition 1.16. Using this inequality and Lemma 4.39, we have

$$1\lesssim\|\phi\|_{L_x^2}^{\frac{d+2-(d-2)p}{2}}\|\nabla\phi\|_{L_x^2}^{\frac{dp-(d+4)}{2}}\leq\|\phi\|_{H_x^1}^{p-1}\sim\|\phi\|_{H_{\omega,V}^1}^{p-1}\sim S_{\omega,V}(\phi)^{\frac{p-1}{2}}.$$

□

Lemma 4.42. *Let $d\geq 1$, $1+\frac{4}{d}<p<\infty$ if $d=1,2$, $1+\frac{4}{d}<p<1+\frac{4}{d-2}$ if $d\geq 3$, and let (α,β) satisfy (1.26). Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V\in L^{\eta}(\mathbb{R}^d)+L^{\infty}(\mathbb{R}^d)$ for $\eta=1$ if $d=1$, some $\eta>1$ if $d=2$, $\eta=\frac{d}{2}$ if $d\geq 3$, and any $\mathfrak{a}\in(\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}|\leq 1$, $V\geq 0$, $x\cdot\nabla V\leq 0$, and $\omega_0<\infty$. If $\omega>0$ satisfies $\omega\geq\omega_0$ and $\phi\in H_{rad}^1(\mathbb{R}^d)\setminus\{0\}$ satisfies $K_{\omega,V}^{\alpha,\beta}(\phi)\leq 0$, then there exists $0<\lambda\leq 1$ such that*

$$K_{\omega,V}^{\alpha,\beta}(\lambda\phi)=0\quad\textit{and}\quad r_{\omega,V}^{\alpha,\beta}\leq S_{\omega,V}(\lambda\phi)=T_{\omega,V}^{\alpha,\beta}(\lambda\phi)\leq T_{\omega,V}^{\alpha,\beta}(\phi).$$

In particular,

$$r_{\omega,V}^{\alpha,\beta}=\inf\{T_{\omega,V}^{\alpha,\beta}(\phi):\phi\in H_{rad}^1(\mathbb{R}^d)\setminus\{0\},\ K_{\omega,V}^{\alpha,\beta}(\phi)\leq 0\}$$

holds.

Proof. This lemma follows from the same argument as Lemma 4.22. □

The following theorem is attainability of $r_{\omega,V}^{\alpha,\beta}$ in Main theorem 1.64 (radial case). The proof is similar to the argument in Theorem 4.30. However, the argument does not need a reflexivity of $\dot{H}_V^1$.

Theorem 4.43. *Let $d\geq 2$, $1+\frac{4}{d}<p<\infty$ if $d=2$, $1+\frac{4}{d}<p<1+\frac{4}{d-2}$ if $d\geq 3$, and let (α,β) satisfies (1.26). Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V\in L^{\eta}(\mathbb{R}^d)+L^{\infty}(\mathbb{R}^d)$ for some $\eta>1$ if $d=2$, $\eta=\frac{d}{2}$ if $d\geq 3$, and any $\mathfrak{a}\in(\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}|\leq 1$, $V\geq 0$, $x\cdot\nabla V\leq 0$, and $\omega_0<\infty$. If $\omega>0$ satisfies $\omega\geq\omega_0$, then $r_{\omega,V}^{\alpha,\beta}$ is attained.*

Proof. We take a minimizing sequence $\{\phi_n\}\subset H_{\mathrm{rad}}^1(\mathbb{R}^d)\setminus\{0\}$, that is, ϕ_n satisfies

$$K_{\omega,V}^{\alpha,\beta}(\phi_n)=0\ \text{ for any }\ n\in\mathbb{N}\tag{4.47}$$

and

$$S_{\omega,V}(\phi_n)=T_{\omega,V}^{\alpha,\beta}(\phi_n)\searrow r_{\omega,V}^{\alpha,\beta}\quad\text{as}\quad n\to\infty.$$

We see that $\{\phi_n\}$ is a bounded sequence in $L_x^2(\mathbb{R}^d)$ and $\dot{H}_V^1(\mathbb{R}^d)$ by Lemma 4.39. From $V\geq 0$, $\{\phi_n\}$ is also a bounded sequence in $\dot{H}_x^1(\mathbb{R}^d)$. From Lemma 2.5, we can take a subsequence of $\{\phi_n\}$ (, which is denoted by the same symbol) satisfying $\phi_n\rightharpoonup Q_{\omega,V}$ in $H_x^1(\mathbb{R}^d)$ and $\phi_n\longrightarrow Q_{\omega,V}$ in $L_x^{p+1}(\mathbb{R}^d)$. Therefore, we get

$$\|Q_{\omega,V}\|_{L_x^2}\leq\liminf_{n\to\infty}\|\phi_n\|_{L_x^2},\tag{4.48}$$

$$\|(-\Delta_V)^{\frac{1}{2}}Q_{\omega,V}\|_{L_x^2}\leq\liminf_{n\to\infty}\|(-\Delta_V)^{\frac{1}{2}}\phi_n\|_{L_x^2},\tag{4.49}$$

$$\|Q_{\omega,V}\|_{L_x^{p+1}}=\lim_{n\to\infty}\|\phi_n\|_{L_x^{p+1}}.\tag{4.50}$$

To prove (4.49), we used the following inequality

$$\int_{\mathbb{R}^d}V(x)|Q_{\omega,V}(x)|^2dx\leq\liminf_{n\to\infty}\int_{\mathbb{R}^d}V(x)|\phi_n(x)|^2dx,\tag{4.51}$$

which holds by the following argument. Using the Hölder's inequality,

$$\left|\int_{\mathbb{R}^d} V(x)\phi_n(x)\overline{Q_{\omega,V}(x)}dx\right| \leq \left(\int_{\mathbb{R}^d} V(x)|\phi_n(x)|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d} V(x)|Q_{\omega,V}(x)|^2dx\right)^{\frac{1}{2}}$$

and taking liminf of the both sides, we get the desired inequality (4.49). The relations (4.48), (4.49), and (4.50) deduce

$$S_{\omega,V}(Q_{\omega,V}) \leq \liminf_{n\to\infty} S_{\omega,V}(\phi_n) = r^{\alpha,\beta}_{\omega,V}.$$

We prove that $Q_{\omega,V}$ is not trivial. We assume $Q_{\omega,V}=0$ for contradiction. Then, we have

$$\begin{aligned} 0 &= \frac{(p+1)\alpha - d\beta}{p+1}\lim_{n\to\infty}\|\phi_n\|^{p+1}_{L^{p+1}_x} \\ &= \lim_{n\to\infty}\left\{\frac{\omega(2\alpha-d\beta)}{2}\|\phi_n\|^2_{L^2_x} + \frac{2\alpha-(d-2)\beta}{2}\|\nabla\phi_n\|^2_{L^2}\right. \\ &\qquad \left. + \frac{2\alpha - d\beta}{2}\int_{\mathbb{R}^d} V(x)|\phi_n(x)|^2dx - \frac{\beta}{2}\int_{\mathbb{R}^d}(x\cdot\nabla V)|\phi_n(x)|^2dx\right\} \\ &\geq \frac{2\alpha-(d-2)\beta}{2}\liminf_{n\to\infty}\|\nabla\phi_n\|^2_{L^2_x} \geq 0 \end{aligned}$$

by (4.50) and (4.47), that is, $\liminf_{n\to\infty}\|\nabla\phi_n\|_{L^2_x}=0$. From Lemma 4.40, we get $K^{\alpha,\beta}_{\omega,V}(\phi_n)>0$ for sufficiently large n. This contradict (4.47). We prove that $Q_{\omega,V}$ attains $r^{\alpha,\beta}_{\omega,V}$. Combining (4.48), (4.49), and (4.50), we have

$$\begin{aligned} K^{\alpha,\beta}_{\omega,V}(Q_{\omega,V}) &\leq \liminf_{n\to\infty} K^{\alpha,\beta}_{\omega,V}(\phi_n) = 0, \\ T^{\alpha,\beta}_{\omega,V}(Q_{\omega,V}) &\leq \liminf_{n\to\infty} T^{\alpha,\beta}_{\omega,V}(\phi_n) = r^{\alpha,\beta}_{\omega,V}. \end{aligned} \tag{4.52}$$

We note that the inequalities

$$\|(-x\cdot\nabla V)^{\frac{1}{2}}Q_{\omega,V}\|_{L^2_x} \leq \liminf_{n\to\infty}\|(-x\cdot\nabla V)^{\frac{1}{2}}\phi_n\|_{L^2_x},$$

$$\|(2\omega+2V+x\cdot\nabla V)^{\frac{1}{2}}Q_{\omega,V}\|_{L^2_x} \leq \liminf_{n\to\infty}\|(2\omega+2V+x\cdot\nabla V)^{\frac{1}{2}}\phi_n\|_{L^2_x}$$

follows from the same argument as (4.51). From (4.52), there exists $\lambda\in(0,1]$ such that

$$K^{\alpha,\beta}_{\omega,V}(\lambda Q_{\omega,V}) = 0,$$

and hence, we have

$$r^{\alpha,\beta}_{\omega,V} \leq S_{\omega,V}(\lambda Q_{\omega,V}) = T^{\alpha,\beta}_{\omega,V}(\lambda Q_{\omega,V}) \leq T^{\alpha,\beta}_{\omega,V}(Q_{\omega,V}) \leq r^{\alpha,\beta}_{\omega,V}.$$

Therefore, λ must be 1 and $Q_{\omega,V}$ attains $r^{\alpha,\beta}_{\omega,V}$. □

Lemma 4.44. *Let $d\geq 1$, $1+\frac{4}{d}<p<\infty$ if $d=1,2$, $1+\frac{4}{d}<p<1+\frac{4}{d-2}$ if $d\geq 3$, and let (α,β) satisfy (1.26). Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V\in L^\eta(\mathbb{R}^d)+L^\infty(\mathbb{R}^d)$ for $\eta\geq 1$ if $d=1$, some $\eta>1$ if $d=2$, $\eta=\frac{d}{2}$ if $d\geq 3$, and any $\mathfrak{a}\in(\mathbb{N}\cup\{0\})^d$ with $|\mathfrak{a}|\leq 2$ and $3x\cdot\nabla V + x\nabla^2 V x^T\leq 0$. Then,*

$$(\mathcal{D}^{\alpha,\beta}-\overline{\mu})(\mathcal{D}^{\alpha,\beta}-\underline{\mu})S_{\omega,V}(f) \leq -\frac{(p-1)\alpha\{(p-1)\alpha-2\beta\}}{p+1}\|f\|^{p+1}_{L^{p+1}_x}$$

holds for any $f\in H^1(\mathbb{R}^d)$. In particular, if f satisfies $K^{\alpha,\beta}_{\omega,V}(f)=0$, then it follows that

$$\mathcal{D}^{\alpha,\beta}K^{\alpha,\beta}_{\omega,V}(f) \leq -\overline{\mu}\underline{\mu}T^{\alpha,\beta}_{\omega,V}(f) - \frac{(p-1)\alpha\{(p-1)\alpha-2\beta\}}{p+1}\|f\|^{p+1}_{L^{p+1}_x}.$$

Proof. By the simple calculation, we have

$$\mathcal{D}^{\alpha,\beta}\|f\|^2_{L^2_x} = \underline{\mu}\|f\|^2_{L^2_x}, \quad \mathcal{D}^{\alpha,\beta}\|\nabla f\|^2_{L^2_x} = \overline{\mu}\|\nabla f\|^2_{L^2_x}, \quad \mathcal{D}^{\alpha,\beta}\|f\|^{p+1}_{L^{p+1}_x} = \{(p+1)\alpha-d\beta\}\|f\|^{p+1}_{L^{p+1}_x},$$

$$\mathcal{D}^{\alpha,\beta}\int_{\mathbb{R}^d}V(x)|f(x)|^2dx = \underline{\mu}\int_{\mathbb{R}^d}V(x)|f(x)|^2dx - \beta\int_{\mathbb{R}^d}(x\cdot\nabla V)|f(x)|^2dx,$$

and

$$\mathcal{D}^{\alpha,\beta}\int_{\mathbb{R}^d}(x\cdot\nabla V)|f(x)|^2dx = \{2\alpha-(d+1)\beta\}\int_{\mathbb{R}^d}(x\cdot\nabla V)|f(x)|^2dx - \beta\int_{\mathbb{R}^d}(x\nabla^2Vx^T)|f(x)|^2dx.$$

These identities and $3x\cdot\nabla V + x\nabla^2Vx^T \leq 0$ imply the desired results. □

Proposition 4.45. *Besides the assumptions of Lemma 4.44, we assume that V is radially symmetric. Then, $\mathcal{M}^{\alpha,\beta}_{\omega,V,rad} \subset \mathcal{G}_{\omega,V,rad}$ holds.*

Proof. We take any $\phi \in \mathcal{M}^{\alpha,\beta}_{\omega,V,\mathrm{rad}}$. From Lemma 4.44, we have

$$\begin{aligned}\langle (K^{\alpha,\beta}_{\omega,V})'(\phi), \mathcal{D}^{\alpha,\beta}\phi\rangle &= \mathcal{D}^{\alpha,\beta}K^{\alpha,\beta}_{\omega,V}(\phi)\\ &\leq -\overline{\mu}\underline{\mu}T^{\alpha,\beta}_{\omega,V}(\phi) - \frac{(p-1)\alpha\{(p-1)\alpha-2\beta\}}{p+1}\|\phi\|^{p+1}_{L^{p+1}_x} < 0.\end{aligned} \tag{4.53}$$

Thus, there exists the Lagrange multiplier $\eta\in\mathbb{R}$ such that

$$S'_{\omega,V}(\phi) = \eta(K^{\alpha,\beta}_{\omega,V})'(\phi). \tag{4.54}$$

This identity deduces

$$0 = K^{\alpha,\beta}_{\omega,V}(\phi) = \mathcal{D}^{\alpha,\beta}S_{\omega,V}(\phi) = \langle S'_{\omega,V}(\phi), \mathcal{D}^{\alpha,\beta}\phi\rangle = \eta\langle (K^{\alpha,\beta}_{\omega,V})'(\phi), \mathcal{D}^{\alpha,\beta}\phi\rangle.$$

This implies $\eta = 0$ by (4.53). Therefore, we obtain $S'_{\omega,V}(\phi) = 0$ by (4.54). We take any $\psi \in \mathcal{A}_{\omega,V,\mathrm{rad}}$. Then, we have $K^{\alpha,\beta}_{\omega,V}(\psi) = \langle S'_{\omega,V}(\psi), \mathcal{D}^{\alpha,\beta}(\psi)\rangle = 0$. Therefore, we obtain $S_{\omega,V}(\phi) \leq S_{\omega,V}(\psi)$, that is, $\phi\in\mathcal{G}_{\omega,V,\mathrm{rad}}$. □

Proposition 4.46. *We assume the same conditions as Proposition 4.45. If $\mathcal{M}^{\alpha,\beta}_{\omega,V,rad}$ is not empty, then $\mathcal{G}_{\omega,V,rad} \subset \mathcal{M}^{\alpha,\beta}_{\omega,V,rad}$ holds.*

Proof. We take any $\phi\in\mathcal{G}_{\omega,V,\mathrm{rad}}$. We take a $\psi \in \mathcal{M}^{\alpha,\beta}_{\omega,V,\mathrm{rad}} \subset \mathcal{G}_{\omega,V,\mathrm{rad}}$, where the last inclusion holds by Proposition 4.45. Let $w \in H^1_{\mathrm{rad}}(\mathbb{R}^d)\setminus\{0\}$ satisfy $K^{\alpha,\beta}_{\omega,V}(w) = 0$. Then, it follows that $S_{\omega,V}(\phi) = S_{\omega,V}(\psi) \leq S_{\omega,V}(w)$. In addition, we have $K^{\alpha,\beta}_{\omega,V}(\phi) = \langle S'_{\omega,V}(\phi), \mathcal{D}^{\alpha,\beta}\phi\rangle = 0$. Therefore, we obtain $\phi\in\mathcal{M}^{\alpha,\beta}_{\omega,V,\mathrm{rad}}$, that is, $\mathcal{G}_{\omega,V,\mathrm{rad}} \subset \mathcal{M}^{\alpha,\beta}_{\omega,V,\mathrm{rad}}$ holds. □

Corollary 4.47. *Let $d\geq 2$, $1+\frac{4}{d} < p < \infty$ if $d = 2$, $1+\frac{4}{d} < p < 1+\frac{4}{d-2}$ if $d\geq 3$, and let (α,β) satisfy (1.26). Let $x^{\mathfrak{a}}\partial^{\mathfrak{a}}V \in L^{\eta}(\mathbb{R}^d)+L^{\infty}(\mathbb{R}^d)$ for some $\eta > 1$ if $d = 2$, $\eta = \frac{d}{2}$ if $d\geq 3$, and any $\mathfrak{a}\in(\mathbb{N}\cup\{0\})^d$ with $|\alpha|\leq 2$, $V\geq 0$, $x\cdot\nabla V\leq 0$, $3x\cdot\nabla V + x\nabla^2Vx^T\leq 0$, and $\omega_0 < \infty$. If $\omega > 0$ satisfies $\omega\geq\omega_0$, then $\mathcal{M}^{\alpha,\beta}_{\omega,V,rad} = \mathcal{G}_{\omega,V,rad}$ holds.*

This corollary holds by Theorem 4.43, Proposition 4.45, and 4.46.

4.6. **Proof of Theorem 1.67.** In this subsection, we prove Theorem 1.67 and Proposition 1.68 (2). The non-radial case in Theorem 1.67 is deduced by the same argument as the radial case in Theorem 1.67 and Proposition 1.68 (1). Hence, we only prove the radial case.

Proof of Theorem 1.67. We note that $K^{\alpha,\beta}_{\omega,V}(u_0) = 0$ implies that $u_0(x)\equiv 0$ by the definition of $r^{\alpha,\beta}_{\omega,V}$ and the assumption $S_{\omega,V}(u_0) < r^{\alpha,\beta}_{\omega,V}$. Then, we consider only case $K^{\alpha,\beta}_{\omega,V}(u_0) > 0$. First, we prove that $K^{\alpha,\beta}_{\omega,V}(u(t)) > 0$. If the conclusion does not hold, then there exists $t_0\in(T_{\min}, T_{\max})$ such that $K^{\alpha,\beta}_{\omega,V}(u(t_0)) = 0$. For such t_0, we have $r^{\alpha,\beta}_{\omega,V}\leq S_{\omega,V}(u(t_0))$ by the definition of $r^{\alpha,\beta}_{\omega,V}$. On the other hand, the conservation laws implies $S_{\omega,V}(u(t_0)) = S_{\omega,V}(u_0) < r^{\alpha,\beta}_{\omega,V}$. This is contradiction. From Lemma 4.39, we have

$$r^{\alpha,\beta}_{\omega,V} > S_{\omega,V}(u_0) = S_{\omega,V}(u(t)) \sim \|u(t)\|^2_{H^1_x}.$$

Therefore, the solution u to (NLS_V) exists globally in time. □

Proof of Proposition 1.68 (2). The proof is the same as the argument of Proposition 1.63, so we omit it. ☐

4.7. **Proof of Main theorem 1.74.** In this subsection, we prove Theorem 1.73, Main theorem 1.74, and Theorem 1.76.

We note that (1.26) deduces the following relations:

$$\overline{\mu} := 2\alpha - (d-2)\beta \geq 2\alpha - (d-\mu)\beta \geq \underline{\mu}, \quad 2\alpha - (d-\mu)\beta > 0,$$
$$(p+1)\alpha - d\beta > (p-1)\alpha - 2\beta > 0.$$

Proof of Proposition 1.72. The proof follows from the same argument as Proposition 1.54. ☐

We prove that $r_{\omega,\gamma}^{\alpha,\beta}$ is independent of (α,β) for $d=1$. In Main theorem 1.64, it has been already shown that $r_{\omega,\gamma}^{\alpha,\beta}$ is independent of (α,β) for $d \geq 2$.

Proposition 4.48. *Let $d=1$, $1<p<\infty$, $\gamma>0$, and $0<\mu<1$. Let (α,β) satisfies (1.26). Then, we have*

$$r_{\omega,\gamma}^{\alpha,\beta} = \inf_{c\in\mathcal{C}} \max_{\tau\in[0,1]} S_{\omega,\gamma}(c(\tau)),$$

where

$$\mathcal{C} := \{c \in C([0,1]; H^1_{rad}(\mathbb{R})) : c(0)=0,\ S_{\omega,\gamma}(c(1)) < 0\}.$$

In particular, $r_{\omega,\gamma}^{\alpha,\beta}$ is independent of (α,β).

The proof is based on [71, Lemma 2.3].

Proof. We set

$$\mathscr{R} := \inf_{c\in\mathcal{C}} \max_{\tau\in[0,1]} S_{\omega,\gamma}(c(\tau)).$$

To prove $\mathscr{R} \leq r_{\omega,\gamma}^{\alpha,\beta}$, we prove that there exists $\{c_n\} \subset \mathcal{C}$ such that

$$\max_{\tau\in[0,1]} S_{\omega,\gamma}(c_n(\tau)) \longrightarrow r_{\omega,\gamma}^{\alpha,\beta}$$

as $n\to\infty$. We take a minimizing sequence $\{\phi_n\}$ to $r_{\omega,\gamma}^{\alpha,\beta}$, that is,

$$S_{\omega,\gamma}(\phi_n) \longrightarrow r_{\omega,\gamma}^{\alpha,\beta} \quad \text{as} \quad n\to\infty \quad \text{and} \quad K_{\omega,\gamma}^{\alpha,\beta}(\phi_n) = 0 \quad \text{for each} \quad n\in\mathbb{N}.$$

We set $\widetilde{c}_n(\tau) := e^{\alpha\tau}\phi_n(e^{\beta\tau}\,\cdot\,)$ for $\tau\in\mathbb{R}$. Then,

$$\begin{aligned} S_{\omega,\gamma}(\widetilde{c}_n(\tau)) = {} & \frac{\omega}{2} e^{(2\alpha-\beta)\tau}\|\phi_n\|_{L^2_x}^2 + \frac{1}{2} e^{(2\alpha+\beta)\tau}\|\nabla\phi_n\|_{L^2_x}^2 \\ & + \frac{1}{2} e^{\{2\alpha-(1-\mu)\beta\}\tau}\int_{\mathbb{R}} \frac{\gamma}{|x|^\mu}|\phi_n(x)|^2 dx - \frac{1}{p+1} e^{\{(p+1)\alpha-\beta\}\tau}\|\phi_n\|_{L^{p+1}_x}^{p+1}, \end{aligned}$$

so $S_{\omega,\gamma}(\widetilde{c}_n(\tau)) < 0$ for sufficiently large $\tau > 0$. Moreover, we have $\max_{\tau\in\mathbb{R}} S_{\omega,\gamma}(\widetilde{c}_n(\tau)) = S_{\omega,\gamma}(\widetilde{c}_n(0)) = S_{\omega,\gamma}(\phi_n) \longrightarrow r_{\omega,\gamma}^{\alpha,\beta}$ as $n\to\infty$ by $K_{\omega,\gamma}^{\alpha,\beta}(\phi_n) = 0$. We define a function c_n for $\tau\in[0,1]$, $L>0$, and $M>0$ as follows:

$$c_n(\tau) := \begin{cases} \widetilde{c}_n(2L\tau - L), & (\frac{1}{4} \leq \tau \leq 1), \\ (4\tau)^M \widetilde{c}_n(-\frac{L}{2}), & (0 \leq \tau < \frac{1}{4}). \end{cases}$$

If $L>0$ and $M=M(n)$ are sufficiently large, then $c_n\in\mathcal{C}$, $S_{\omega,\gamma}(c_n(1)) < 0$, and

$$\max_{\tau\in[0,1]} S_{\omega,\gamma}(c_n(\tau)) = S_{\omega,\gamma}(\phi_n) \longrightarrow r_{\omega,\gamma}^{\alpha,\beta} \quad \text{as} \quad n\to\infty.$$

Therefore, we obtain $\mathscr{R} \leq r_{\omega,\gamma}^{\alpha,\beta}$. We prove $\mathscr{R} \geq r_{\omega,\gamma}^{\alpha,\beta}$. We take any $c\in\mathcal{C}$, that is, $c(0)=0$ and $S_{\omega,\gamma}(c(1)) < 0$. Since $K_{\omega,\gamma}^{\alpha,\beta}(c(0)) = 0$ and $K_{\omega,\gamma}^{\alpha,\beta} \in C(H^1(\mathbb{R});\mathbb{R})$, it follows from Lemma 4.40 that $K_{\omega,\gamma}^{\alpha,\beta}(c(\tau)) > 0$ for sufficiently small $\tau\in(0,1)$. On the other hand, we have

$$K_{\omega,\gamma}^{\alpha,\beta}(c(1)) \leq \{(p+1)\alpha - \beta\} S_{\omega,\gamma}(c(1)) < 0.$$

By the intermediate value theorem, there exists $\tau_c \in (0,1)$ such that $K^{\alpha,\beta}_{\omega,\gamma}(c(\tau_c)) = 0$. Therefore, we obtain

$$\mathscr{R} \leq S_{\omega,\gamma}(c(\tau_c)) \leq \max_{\tau\in[0,1]} S_{\omega,\gamma}(c(\tau))$$

for any $c \in \mathcal{C}$. Taking infimum of this inequality over $c \in \mathcal{C}$, the desired result is gotten. □

4.7.1. *Global well-posedness in Theorem 1.73 and 1.76.* In this subsubsection, we prove global well-posedness in Theorem 1.73 and 1.76.

Lemma 4.49 (Coercivity I)**.** *Let $d \geq 1$, $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, $\gamma > 0$, $0 < \mu < \min\{2, d\}$, and $u_0 \in PW_{+,5} \cup PW_{-,5}$. Let $Q_{1,0}$ be the ground state to* (SP$_{\omega,0}$)*. We assume that u_0 satisfies* (4.1)*.*

- *(Case $PW_{+,5}$) If $u_0 \in PW_{+,5}$, then a solution u to* (NLS$_\gamma$) *with* (IC) *satisfies the following: there exists $\delta' > 0$ such that*

$$\|u(t)\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|(-\Delta_\gamma)^{\frac{1}{2}} u(t)\|_{L^2_x} < (1-\delta')^{\frac{1}{s_c}} \|Q_{1,0}\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L^2_x}$$

  for any $t \in (T_{min}, T_{max})$.
- *(Case $PW_{-,5}$) If $u_0 \in PW_{-,5}$, then a solution u to* (NLS$_\gamma$) *with* (IC) *satisfies the following: there exists $\delta' > 0$ such that*

$$\|u(t)\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|(-\Delta_\gamma)^{\frac{1}{2}} u(t)\|_{L^2_x} > (1+\delta')^{\frac{1}{s_c}} \|Q_{1,0}\|_{L^2_x}^{\frac{1-s_c}{s_c}} \|\nabla Q_{1,0}\|_{L^2_x}$$

  for any $t \in (T_{min}, T_{max})$.

Proof. This lemma follows from the same argument with Proposition 4.11 (1) and 4.17. □

The case $PW_{+,5}$ result in Lemma 4.49 deduced global well-posedness in Theorem 1.73 with $j = 5$.

Proof of global well-posedness in Theorem 1.73 with $j = 5$. The desired result follows from the fact that H^1-norm of the solutions is uniformly bounded with respect to time t. □

Lemma 4.50 (Coercivity II)**.** *Let $d \geq 1$, $1 + \frac{4}{d} < p < \infty$ if $d = 1, 2$, $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ if $d \geq 3$, $\gamma > 0$, and $0 < \mu < \min\{2, d\}$.*

- *($PW_{+,4}$ (resp. $PW_{+,6}$) case) If $u_0 \in PW_{+,4}$ (resp. $PW_{+,6}$), then a solution u to* (NLS$_\gamma$) *with* (IC) *satisfies $u(t) \in PW_{+,4}$ (resp. $PW_{+,6}$) for each $t \in (T_{min}, T_{max})$.*
- *($PW_{-,4}$ (resp. $PW_{-,6}$) case) If $u_0 \in PW_{-,4}$ (resp. $PW_{-,6}$), then a solution u to* (NLS$_\gamma$) *with* (IC) *satisfies $u(t) \in PW_{-,4}$ (resp. $PW_{-,6}$) for each $t \in (T_{min}, T_{max})$ and*

$$K_\gamma(u(t)) < 4(S_{\omega,\gamma}(u_0) - n_{\omega,\gamma} \ (\text{resp. } r_{\omega,\gamma})) < 0.$$

Proof. The case $PW_{+,6}$ (resp. $PW_{-,6}$) follows from the same argument as the case $PW_{+,4}$ (resp. $PW_{-,4}$), so we treat only $PW_{+,4}$ and $PW_{-,4}$. When $K_\gamma(u_0) = 0$, we have $u_0 \equiv 0$ by the definition of $n_{\omega,\gamma}$. Thus, Lemma 4.50 holds. Suppose that $K_\gamma(u_0) \neq 0$. If there exists $t_0 \in (T_{\min}, T_{\max})$ such that $K_\gamma(u(t_0)) = 0$, then

$$S_{\omega,\gamma}(u(t_0)) = S_{\omega,\gamma}(u_0) < n_{\omega,\gamma} \leq S_{\omega,\gamma}(u(t_0)).$$

This is contradiction. Therefore, $K_\gamma(u(t)) \neq 0$ for each $t \in (T_{\min}, T_{\max})$. In particular, the sign of $K_\gamma(u(t))$ corresponds with that of $K_\gamma(u_0)$ by the continuity of the solution. Let $K_\gamma(u_0) < 0$. We define a function

$$J_{\omega,\gamma}(\lambda) = S_{\omega,\gamma}(e^{d\lambda} u(t, e^{2\lambda} \cdot)).$$

We note that

$$J_{\omega,\gamma}(0) = S_{\omega,\gamma}(u(t)), \quad \frac{d}{d\lambda} J_{\omega,\gamma}(0) = K_\gamma(u(t)), \quad \frac{d^2}{d\lambda^2} J_{\omega,\gamma}(\lambda) < 4\frac{d}{d\lambda} J_{\omega,\gamma}(\lambda).$$

The equation $\frac{d}{d\lambda}J_{\omega,\gamma}(\lambda)=0$ for λ has only one negative solution. We set that the solution $\lambda=\lambda_0<0$. Integrating $\frac{d^2}{d\lambda^2}J_{\omega,\gamma}(\lambda)<4\frac{d}{d\lambda}J_{\omega,\gamma}(\lambda)$ on $[\lambda_0,0]$, we have

$$K_\gamma(u(t))-0<4(S_{\omega,\gamma}(u(t))-J_{\omega,\gamma}(\lambda_0))\leq 4(S_{\omega,\gamma}(u(t))-n_{\omega,\gamma})<0.$$

Therefore, we obtain

$$K_\gamma(u(t))<4(S_{\omega,\gamma}(u_0)-n_{\omega,\gamma})<0$$

for any $t\in(T_{\min},T_{\max})$. □

As a corollary, global well-posedness in Theorem 1.73 with $j=4$ and Theorem 1.76 holds.

Corollary 4.51 (Global well-posedness). *Let $d\geq 1$, $1+\frac{4}{d}<p<\infty$ if $d=1,2$, $1+\frac{4}{d}<p<1+\frac{4}{d-2}$ if $d\geq 3$, $\gamma>0$, and $0<\mu<\min\{2,d\}$. If $u_0\in PW_{+,j}$ ($j=4,6$), then a solution u to (NLS$_\gamma$) with (IC) exists globally in time.*

Proof. From $u_0\in PW_{+,4}$ (resp. $PW_{+,6}$) and Lemma 4.50, we have $u(t)\in PW_{+,4}$ (resp. $PW_{+,6}$) for each $t\in(T_{\min},T_{\max})$. $K_\gamma(u(t))\geq 0$ deduces

$$2\|(-\Delta_\gamma)^{\frac{1}{2}}u(t)\|_{L_x^2}^2\geq\frac{d(p-1)}{p+1}\|u(t)\|_{L_x^{p+1}}^{p+1}.$$

Therefore, we obtain

$$n_{\omega,\gamma}\ (\text{resp. } r_{\omega,\gamma})>S_{\omega,\gamma}(u_0)\geq\frac{\omega}{2}\|u(t)\|_{L_x^2}^2+\frac{d(p-1)-4}{2d(p-1)}\|(-\Delta_\gamma)^{\frac{1}{2}}u(t)\|_{L_x^2}^2\gtrsim\|u(t)\|_{H_x^1}^2,$$

which implies the desired result. □

4.7.2. *Blow-up or grow-up result in Theorem 1.73 and 1.76.* In this subsubsection, we prove blow-up or grow-up results in Theorem 1.73 and 1.76.

Lemma 4.52. *Let $d\geq 1$, $1<p<\infty$ if $d=1,2$, $1<p<1+\frac{4}{d-2}$ if $d\geq 3$, $\gamma>0$, and $0<\mu<\min\{2,d\}$. Let $u\in C([0,\infty);H^1(\mathbb{R}^d))$ be a time global solution to (NLS$_\gamma$). We define a function*

$$I(t):=\int_{\mathbb{R}^d}\mathscr{X}_R(x)|u(t,x)|^2dx,$$

where $\mathscr{X}_R$ is defined as (2.1). Then, for $q\in(p+1,\infty)$ if $d=1,2$ and $q\in(p+1,\frac{2d}{d-2})$ if $d\geq 3$, there exist constants $C=C(q,M(u),C_0)>0$ and $\theta_q>0$ such that the estimate

$$I''(t)\leq 4K_\gamma(u(t))+C\,\|u(t)\|_{L_x^2(R\leq|x|)}^{(p+1)\theta_q}+\frac{C}{R^2}$$

holds for any $R>0$ and $t\in[0,\infty)$, where $\theta_q:=\frac{2\{q-(p+1)\}}{(p+1)(q-2)}\in(0,\frac{2}{p+1})$ and C_0 is given in Lemma 4.18.

Proof. This proof follows from the similar argument to Lemma 4.19. Using Proposition 4.9, we have

$$I''(t)=4K_\gamma(u(t))+\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3+\mathcal{R}_4,$$

where $\mathcal{R}_k=\mathcal{R}_k(t)$ ($k=1,2,3,4$) are defined as

$$\mathcal{R}_1:=4\int_{\mathbb{R}^d}\left\{\frac{1}{r^2}\mathscr{X}''\left(\frac{r}{R}\right)-\frac{R}{r^3}\mathscr{X}'\left(\frac{r}{R}\right)\right\}|x\cdot\nabla u|^2dx+4\int_{\mathbb{R}^d}\left\{\frac{R}{r}\mathscr{X}'\left(\frac{r}{R}\right)-2\right\}|\nabla u(t,x)|^2dx, \tag{4.55}$$

$$\mathcal{R}_2:=-\frac{2(p-1)}{p+1}\int_{\mathbb{R}^d}\left\{\mathscr{X}''\left(\frac{r}{R}\right)+\frac{(d-1)R}{r}\mathscr{X}'\left(\frac{r}{R}\right)-2d\right\}|u(t,x)|^{p+1}dx, \tag{4.56}$$

$$\mathcal{R}_3:=-\int_{\mathbb{R}^d}\left\{\frac{1}{R^2}\mathscr{X}^{(4)}\left(\frac{r}{R}\right)+\frac{2(d-1)}{Rr}\mathscr{X}^{(3)}\left(\frac{r}{R}\right)+\frac{(d-1)(d-3)}{r^2}\mathscr{X}''\left(\frac{r}{R}\right)\right.$$

$$+\frac{(d-1)(3-d)R}{r^3}\mathscr{X}'\left(\frac{r}{R}\right)\bigg\}|u(t,x)|^2dx, \quad (4.57)$$

$$\mathcal{R}_4 := 2\mu\int_{R\le|x|}\left\{\frac{R}{r}\mathscr{X}'\left(\frac{r}{R}\right)-2\right\}\frac{\gamma}{|x|^\mu}|u(t,x)|^2dx. \quad (4.58)$$

We estimate $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ by the same argument as Lemma 4.19 and can get

$$\mathcal{R}_1\le 0, \quad \mathcal{R}_2\le C\|u(t)\|_{L_x^2(R\le|x|)}^{(p+1)\theta_q}, \quad \mathcal{R}_3\le\frac{C}{R^2}.$$

$\mathcal{R}_4$ is estimated as $\mathcal{R}_4\le 0$, which completes the proof of the lemma. □

Proof of blow-up or grow-up in Theorem 1.73 and 1.76. This proof follows from the same argument with blow-up or grow-up in Main theorem 1.56. In the argument, we use Lemma 4.52. □

Using Theorem 1.71 and 1.73, we prove Main theorem 1.74.

Proof of Main theorem 1.74. We note that

$$PW_{+,j}\cup PW_{-,j}=\{u_0\in H^1(\mathbb{R}^d):(1.20)\}$$

for any $j=3,4,5$. If $u_0\in PW_{+,j}$ ($j=3,4,5$), then a solution u to (NLS$_\gamma$) is uniformly bounded in $H^1(\mathbb{R}^d)$. On the other hand, if $u_0\in PW_{-,j}$ ($j=3,4,5$), then a solution u to (NLS$_\gamma$) is unbounded in $H^1(\mathbb{R}^d)$. □

To complete this subsubsection, we prove Corollary 1.75 by Main theorem 1.74.

Proof of Corollary 1.75. Let $E_\gamma(u_0)\le 0$ and $u_0\ne 0$. (1.20) holds clearly. $E_\gamma(u_0)\le 0$ implies

$$\frac{1}{2}\|(-\Delta_\gamma)^{\frac{1}{2}}u_0\|_{L_x^2}^2\le\frac{1}{p+1}\|u_0\|_{L_x^{p+1}}^{p+1},$$

so we have

$$K_\gamma(u_0)<2\|(-\Delta_\gamma)^{\frac{1}{2}}u_0\|_{L_x^2}^2-\frac{d(p-1)}{p+1}\|u_0\|_{L_x^{p+1}}^{p+1}\le\frac{4-d(p-1)}{2}\|(-\Delta_\gamma)^{\frac{1}{2}}u_0\|_{L_x^2}^2<0.$$

□

4.7.3. *Blow-up result in Theorem 1.73 and 1.76.* In this subsubsection, we prove the blow-up results in Theorem 1.73 and 1.76. This proof is based on [48] and [100] (see also [67]). First, we prove the following lemma to get the blow-up results.

Lemma 4.53. *Let $d\ge 1$, $1+\frac{4}{d}<p<\infty$ if $d=1,2$, $1+\frac{4}{d}<p<1+\frac{4}{d-2}$ if $d\ge 3$, $\gamma>0$, and $0<\mu<\min\{2,d\}$. Let $\omega>0$. Then, we have*

$$n_{\omega,\gamma}=\inf\{U_{\omega,\gamma}(f):f\in H^1(\mathbb{R}^d)\setminus\{0\},\ K_\gamma(f)\le 0\},$$
$$r_{\omega,\gamma}=\inf\{U_{\omega,\gamma}(f):f\in H^1_{rad}(\mathbb{R}^d)\setminus\{0\},\ K_\gamma(f)\le 0\},$$

where $U_{\omega,\gamma}$ is defined as $U_{\omega,\gamma}(f):=S_{\omega,\gamma}(f)-\frac{1}{d(p-1)}K_\gamma(f)$.

Proof. This proof follows from the same argument with Lemma 4.22. □

Proof of blow-up in Theorem 1.73 and 1.76.
Case $u_0\in|x|^{-1}L^2(\mathbb{R}^d)$:
When $u_0\in|x|^{-1}L^2(\mathbb{R}^d)$, there exists a positive constant $\delta>0$ such that

$$\frac{d^2}{dt^2}\|xu(t)\|_{L_x^2}^2=4K_\gamma(u(t))<-\delta$$

for any $t\in(T_{\min},T_{\max})$ from Proposition 4.8, Lemma 4.50, and 4.20. This inequality implies the desired result.
Case $u_0\in H^1_{\mathrm{rad}}(\mathbb{R}^d)$:
Let $u_0\in PW_{-,j}$ ($j=4,5$). We consider the functional I in Lemma 4.52.

$$I''(t)=4K_\gamma(u)+\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3+\mathcal{R}_4,$$

where $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$ are defined as (4.55), (4.56), (4.57), and (4.58) respectively. We have already gotten $R_1 \leq 0$, $R_3 \leq \frac{C}{R^2}$, and $R_4 \leq 0$ in Lemma 4.52. We estimate $\mathcal{R}_2$.

$$\mathcal{R}_2 \leq c\,\|u(t)\|_{L_x^{p+1}(R\leq|x|)}^{p+1} \leq \frac{c}{R^{\frac{(d-1)(p-1)}{2}}\varepsilon} M(u)^{\frac{p+3}{4}} \cdot \varepsilon \|\nabla u(t)\|_{L_x^2(R\leq|x|)}^{\frac{p-1}{2}}$$
$$\leq \begin{cases} \dfrac{c}{R^2}\|\nabla u(t)\|_{L_x^2}^2, & (d=2,\ p=5), \\ \dfrac{c}{R^{\frac{2(d-1)(p-1)}{5-p}}\varepsilon^{\frac{4}{5-p}}} M(u)^{\frac{p+3}{5-p}} + 2\{d(p-1)-4\}\varepsilon\|\nabla u(t)\|_{L_x^2}^2, & (\text{otherwise}) \end{cases}$$

by Lemma 2.4 and 2.1. Let $0 < \varepsilon < \frac{2d(p-1)-4\mu}{2d(p-1)-8}$. We take a positive constant $\delta > 0$ such as $S_{\omega,\gamma}(u) < (1-\delta)n_{\omega,\gamma}$. Since $n_{\omega,\gamma} \leq U_{\omega,\gamma}(u(t))$ by Lemma 4.53, we have

$$\begin{aligned} I''(t) &\leq 4K_\gamma(u) + \frac{c}{R^{\frac{2(d-1)(p-1)}{5-p}}\varepsilon^{\frac{4}{5-p}}} M(u)^{\frac{p+3}{5-p}} + 2\{d(p-1)-4\}\varepsilon\|\nabla u(t)\|_{L_x^2}^2 + \frac{C}{R^2} \\ &< 4d(p-1)S_{\omega,\gamma}(u) - 2\omega d(p-1)M(u) - 2(1-\varepsilon)\{d(p-1)-4\}\|\nabla u(t)\|_{L_x^2}^2 + \frac{C}{R^2} \\ &\qquad - \{2d(p-1)(1-\varepsilon) + 4(2\varepsilon-\mu)\}\int_{\mathbb{R}^d} \frac{\gamma}{|x|^\mu}|u(t,x)|^2 dx + \frac{c}{R^{\frac{2(d-1)(p-1)}{5-p}}\varepsilon^{\frac{4}{5-p}}} M(u)^{\frac{p+3}{5-p}} \\ &< 4d(p-1)S_{\omega,\gamma}(u) - 4d(p-1)(1-\varepsilon)U_{\omega,\gamma}(u) + \frac{C}{R^2} + \frac{c}{R^{\frac{2(d-1)(p-1)}{5-p}}\varepsilon^{\frac{4}{5-p}}} M(u)^{\frac{p+3}{5-p}} \\ &< 4d(p-1)(1-\delta)n_{\omega,\gamma} - 4d(p-1)(1-\varepsilon)n_{\omega,\gamma} + \frac{C}{R^2} + \frac{c}{R^{\frac{2(d-1)(p-1)}{5-p}}\varepsilon^{\frac{4}{5-p}}} M(u)^{\frac{p+3}{5-p}} \\ &= 4d(p-1)(\varepsilon-\delta)n_{\omega,\gamma} + \frac{C}{R^2} + \frac{c}{R^{\frac{2(d-1)(p-1)}{5-p}}\varepsilon^{\frac{4}{5-p}}} M(u)^{\frac{p+3}{5-p}} \end{aligned}$$

for $d \geq 2$ and $p < 5$. Taking $\frac{c}{R^2} \leq 2\{d(p-1)-4\}\varepsilon$, we have

$$I''(t) < 4d(p-1)(\varepsilon-\delta)n_{\omega,\gamma} + \frac{C}{R^2}$$

for $d = 2$ and $p = 5$ by the same manner. Thus, if we take sufficiently small $0 < \varepsilon < \min\{\delta, \frac{2d(p-1)-4\mu}{2d(p-1)-8}\}$ and sufficiently large $R > 0$, then we obtain $I''(t) < 0$. This implies the solution u to (NLS$_\gamma$) blows up. The proof of the case $u_0 \in PW_{-,6}$ is proved by replacing $n_{\omega,\gamma}$ with $r_{\omega,\gamma}$. □

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