Pointwise convergence problems and some sharp inequalities arising in quantum mechanics

（量子力学に現れる各点収束問題と不等式の最良定数問題）

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Abstract

We are concerned three different themes strongly related to quantum mechanics by employing harmonic analytic approaches. The first chapter pertains to the pointwise convergence problem for the Schrödinger type operator, the so called Carleson's problem, initiated by the mathematical giant Lennart Carleson in 1980. He showed that some smoothness condition on the initial data is required for the solutions to the standard Schrödinger equation to converge to the initial data almost everywhere in $\mathbb{R}$. While the one spatial dimensional case was completely understood in a very early stage, the higher dimensional case turns out to be extremely difficult. In 2016, Jean Bourgain finally provided a plausible necessary condition, then soon later, Xiumin Du, Larry Guth, Xiaochun Li and Ruixian Zhang proved that Bourgain's regularity threshold is essentially sufficient as well. Their proof contains state-of-the-art technologies in harmonic analysis, which also reflects the well-known connections among Carleson's problem and other major open problems in harmonic analysis, such as Stein's restriction conjecture and the Kakeya conjecture. Many variations of Carleson's problem are also concerned, for instance, convergence along generalized paths and refinements by measuring the corresponding divergence sets in a more precise sense than Lebesgue measure. Chu-hee Cho, Sanghyuk Lee and Ana Vargas considered the following two distinct generalized paths in one spatial dimensional case; (1) paths along lines generated by a given fractal set, and (2) path along a tangential line onto the hyperplane $\mathbb{R}^d \times \{0\}$. We extend their results from the standard Schrödinger equation to the fractional Schrödinger equation, which has been also studied actively because of its useful applications. By our novel approach, we prove that the Minkowski dimension of the given fractal set influences to the smoothness condition on the initial data for pointwise convergence in the situation of (1), and for (2), the Hölder exponent of the curve and the order of the fractional Schrödinger operator influences the smoothness condition of the initial data. We also consider the refined problem of estimating the size of the associated divergence sets in case (2).

In the second part of the thesis, we consider the Strichartz estimate for the Klein–Gordon operator which can somehow be considered to be the hybrid of Schrödinger and wave operators. Strichartz estimates are one of the most important results in harmonic analysis since they have very useful applications in non-linear PDE theory and have connection with Stein's restriction conjecture. In 2007, Damiano Foschi obtained the sharp Strichartz estimate for wave equation in some special cases and a complete characterization of extremisers. Here, sharp estimate means the estimate with the optimal constant. The latest extension of this result is due to Neal Bez, Chirs Jeavons and Tohru Ozawa who further discussed this subject in the context of the so-called null-form estimates. René Quilodrán and soon later Jeavons, simultaneously, naturally extended

Foschi's argument from wave to the Klein–Gordon equation and obtained analogous results. Jeavons further proved an improved Strichartz estimate in five spatial dimensions. In this chapter, we take the philosophy of Bez–Jeavons–Ozawa and extend results due to Quilodrán and Jeavons to two different directions, which we call *the wave regime* and *the non-wave regime*. In the non-wave regime, we also obtained an improved Strichartz estimate in four spatial dimensions.

In the last chapter, Nelson's celebrated hypercontractivity inequality is concerned and a new perspective of supersolutions is provided. Jonathan Bennett and Bez have pursued a remarkable study of algebraic closure properties of supersolutions in their series of papers. For example, in 2009 they presented a new significantly simple proof of the sharp n-fold Young's convolution inequality and its inverse by combining the closure property and heat-flow monotonicity argument. The purpose of this chapter is to reprove the hypercontractivity inequality for the Ornstein–Uhlenbeck semigroup, another key object in quantum mechanics, by this technique and formally extend this result for far more abstract Markov semigroups which enjoy the diffusion property.

Acknowledgment

I wish to express my deep appreciation and sincere gratitude to my best supervisor Dr. Neal Bez for spending so much time that I hardly bring to mind the whole last five years since when I was an undergraduate student. The every opportunity he gave me has provided dynamic impacts on my research and unique experiences. This thesis would never completed without his continuous encouragement, considerable supports, and immense knowledge for sure. The way he professionally *speaks in mathematics* have improved me as a person in general more than anyone could or even would imagine. I am genuinely proud of myself to have been his student.

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Notation

Here, we note some notation used throughout the thesis. We denote $I = [-1, 1]$, $A \gtrsim B$ if $A \geq CB$, $A \lesssim B$ if $A \leq CB$ and $A \sim B$ if $C^{-1}B \leq A \leq CB$ for some constant $C > 0$. For $r > 0$ and $s > 0$, let $\phi_s(r) = \sqrt{s^2 + r^2}$ so that the Japanese brackets is denoted by $\langle r \rangle = \phi_1(r)$. The spatial Fourier transform $\widehat{\cdot}$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x,$$

while $\widetilde{\cdot}$ is specifically used for the space-time Fourier transform, i.e.

$$\widetilde{f}(\tau, \xi) = \int_{\mathbb{R}^{d+1}} e^{-i(x\cdot\xi + t\tau)} f(t, x)\,\mathrm{d}t\mathrm{d}x.$$

The operator $\sqrt{-\Delta_x}$ is sometimes written as D that is

$$\widehat{Df}(\xi) = |\xi|\widehat{f}(\xi)$$

for appropriate functions f on $\mathbb{R}^d$. Additionally, we define $D_\pm$ by

$$\widetilde{D_\pm f}(\tau, \xi) = ||\tau| \pm |\xi||\widetilde{f}(\tau, \xi)$$

for appropriate functions f on $\mathbb{R} \times \mathbb{R}^d$. The d'Alembertian operator $\partial_t^2 - \Delta_x$ will be denoted by $\Box$, so that $|\Box| = D_- D_+$.

For functional spaces, we may consider the homogeneous and inhomogeneous Sobolev spaces defined by

$$\dot{H}^s(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} \text{ satisfies } \|f\|_{\dot{H}^s} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^d)} < \infty\}$$

and

$$H^s(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} \text{ satisfies } \|f\|_{H^s} = \|(1-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^d)} < \infty\},$$

respectively. We denote the identity operator by id and the constant function equal to 1 by $\mathbb{1}$. Furthermore, $\mathbb{B}^d(x, r)$ denotes the ball of radius $r > 0$ the center of x, and it is abbreviated as $\mathbb{B}^d$ if $(x, r) = (0, 1)$. The real number $q' = \frac{q}{q-1}$ is given as Hölder conjugate of $q \in [1, \infty]$, and

$$\|F\|_{L^p_x L^q_t L^r_\theta} = \left(\int \left(\int \left(\int |F(x, t, \theta)|^r \, d\theta \right)^{\frac{q}{r}} dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}},$$

where the domains of integration will be clear from the context (also, $\|F\|_{L^p_x L^q_t}$ is similarly given).

For notation that has been used for more abstract setting in Chapter 3, the reader may refer to Section 3.2.1.

Contents

Chapter 1

A variety of pointwise convergence problems for Schrödinger-type equations

1.1 Introduction

Let $d \geq 1$. The Schrödinger equation is the "Newton's law" in quantum physics and is considered to be one of the greatest achievements in the modern science and named after its founder and Nobel laureate in physics in 1933, Erwin Schrödinger, and we state it as

$$\begin{cases} \partial_t u(x,t) = -i\Delta_x u(x,t) & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(x,0) = f(x) & x \in \mathbb{R}^d. \end{cases}$$

Since it was established, Schrödinger equations have attracted attention from an enormous number of researchers in a variety of fields in science, even today. As expressing its connection to wave, the solution can be formally written by using Fourier transform as

$$u(t,x) = e^{-it\Delta} f(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{f}(\xi)\, \mathrm{d}\xi.$$

In harmonic analysis, Lennart Carleson, a mathematical giant who is famous for the result in 1966 that the Fourier series of any L^2 periodic functions in one spatial dimension converges almost everywhere in 1966, raised a problem on the topic of statistical mechanics in 1980, and two years later Björn Dahlberg and Carlos Kenig rearranged the problem in the context of the Schrödinger equation.

Question 1.1.1. *Take the initial data from inhomogeneous Sobolev space H^s. Then, for which $s > 0$ are the solutions to the Schrödinger equation guaranteed to converge to the initial data almost everywhere in $\mathbb{R}$?*

By the fact that the parameter s of Sobolev space intuitively represents the smoothness of its members and the embedding property (i.e. $H^{s_2} \subset H^{s_1}$ if $s_1 \leq s_2$), Question 1.1.1 can be also understood as; *what is the smallest $s > 0$ such that*

$$\lim_{t\to 0} e^{-it\Delta} f(x) = f(x) \tag{1.1}$$

for almost all $x \in \mathbb{R}$?

Theorem (Carleson [43], Dahlberg–Kenig [52]). *The solution to the Schrödinger equation converges almost every $x \in \mathbb{R}$ to the initial data from $H^s(\mathbb{R})$ if and only if $s \geq \frac{1}{4}$.*

The sufficiency was proved by Carleson while Dahlberg and Kenig proved the necessity, and so nowadays Question 1.1.1 is often called Carleson's problem. The most natural generalization may then be identifying the required minimum s for (1.1) in higher spatial dimensions. It turns out that Question 1.1.1 in $\mathbb{R}^d$ instead of $\mathbb{R}$ is significantly more difficult and has some close connections with other problems, such as Stein's restriction conjecture. Per Sjölin in 1987 [134] and Luis Vega in 1988 [143], independently, showed that $s > \frac{1}{2}$ is sufficient for (1.1) in all spatial dimensions. Then, the first significant improvement in $\mathbb{R}^2$ was provided by Jean Bourgain in 1992 [33], which was later carefully studied and polished by Adela Moyua, Ana Vargas and Luis Vega [114, 115]. As a direct application of the improvement of bilinear restriction problem for the paraboloid in $\mathbb{R}^3$, Terence Tao and Vargas [140] (reader may also refer to their first part [139] as well as [141] by them with Vega, and [137] by Tao, where he also considered higher dimensions than two) showed (1.1) holds in $\mathbb{R}^2$ if $s > \frac{15}{32}$ (they also concerns in the same paper [140] the connection between their result of the bilinear restriction problem for the cone and the null form, the related problem to which will be discussed in Chapter 2). The result due to Sanghyuk Lee in 2006 [97] for (1.1) in $\mathbb{R}^2$ had been the best result for a long time. Here, he introduced a powerful trick so-called the time localization lemma which is widely and regularly used in harmonic analysis today.

The condition $s \geq \frac{1}{4}$ had often been targeted as the necessary and sufficient condition for (1.1) even in every spatial dimension $\mathbb{R}^d$ until 2012 when Jean Bourgain in [35] constructed a counterexample showing that $s \geq \frac{1}{2} - \frac{1}{d}$ if $d \geq 4$ is necessary for (1.1). This result was slightly improved by Renato Lucà and Keith Rogers [105, 106] and Ciprian Demeter and Shaoming Guo [54]. In the same paper, he also gave an improved sufficient condition $s > \frac{1}{2} - \frac{1}{4d}$ in higher dimensions, which coincides with Carleson's result for $d = 1$ and Lee's result for $d = 2$. The proof is based on his earlier result with Larry Guth [37] and the techniques developed by Jonathan Bennett, Anthony Carbery and Terence Tao [23], the landmark paper which essentially completely solved the multilinear restriction problem (see also its remarkably shortened proof due to Guth [75]), whose endpoint case was later proved by Guth [74]. Finally, in 2016, Bourgain found the necessary condition

$$s \geq \frac{1}{2} - \frac{1}{2(d+1)}$$

for (1.1) in general $\mathbb{R}^d$ in his very short paper [36]. Lucà and Rogers also found a slightly simpler way; employing an ergodic argument instead of Gauss sums in [36], to reprove the Bourgain's necessary condition. Then, in the following year, Xiumin Du, Larry Guth and Xiaochun Li in [56] filled in the gap up to the critical point with the sufficient condition $s > \frac{1}{3}$ (which is $\frac{1}{2} - \frac{1}{2(d+1)}$ when $d = 2$) in $\mathbb{R}^2$ by combining sophisticated modern techniques, such as the wave packet decomposition, the polynomial partitioning method, induction on scale, the Bourgain–Demeter ℓ^2-decoupling theorem, and the refined Strichartz estimate. In 2018, Du and Ruixiang Zhang [57] proved Bourgain's necessary condition is essentially sufficient (except from the critical point, $s > \frac{1}{2} - \frac{1}{2(d+1)}$) for the remaining all the dimensional cases $d \geq 3$. Their proof is different to the earlier result for $d = 2$ by Du, Guth and Li, in particular, without use of the polynomial partitioning method and with critical use of the multilinear restriction estimates (see also the slightly earlier result by Du, Guth, Li and Zhang [58]). For more detailed history on classical Calreson's problem the reader may refer to the comprehensive introduction in the doctoral thesis of Andrew David Bailey [3].

As Carleson's problem grew in importance, many approaches to understand the problem and many variations of the problem have been considered. The following are the four major variations explored in this chapter.

1.1.1 To the fractional Schrödinger equation

It may not be so surprising that a simple generalization of Schrödinger equation turns out to be a fundamental equation in wider concept of quantum physics, more precisely, fractional quantum mechanics (fQM), for which one may be able to trace back to work of Nick Laskin [95, 96]. The interested reader may also refer to [45, 47, 50, 71, 72, 73, 83, 79, 86, 88, 120] among others. The fractional Schrödinger equation is stated as follows: Let $d \geq 1$ and $m > 0$, then

$$\begin{cases} \partial_t u(x,t) = i(-\Delta_x)^{\frac{m}{2}} u(x,t) & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(x,0) = f(x) & x \in \mathbb{R}^d. \end{cases}$$

As we have seen for the solution to the Schrödinger equation in the beginning of this introduction, the solution to the fractional Schrödinger equation has the naturally extended form as

$$u(x,t) = e^{it(-\Delta)^{\frac{m}{2}}} f(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^m)} \widehat{f}(\xi)\, d\xi.$$

Sjölin also considered the pointwise convergence problem for the fractional Schrödinger equation with $m > 1$ for one spatial dimension case in aforementioned work of him [134] and obtained the following.

Theorem (Sjölin [134]). *Let $d = 1$ and $m > 1$. The solution to the Schrödinger equation converges to the initial data, namely,*

$$\lim_{t\to 0} e^{it(-\Delta)^{\frac{m}{2}}} f(x) = f(x) \tag{1.2}$$

almost everywhere for all $f \in H^s(\mathbb{R})$ if and only if $s \geq \frac{1}{4}$.

The cases when $m = 1$ has also been considered yet there are relatively less paper about the problem since the nature is very different from the case when $m > 1$. The case when $m = 1$ is related to the wave equation and for $d = 1$ see the results by Michael Cowling [51] and Björn Walther [145] and for higher spatial dimensions, for instance, see work by Rogers and Paco Villarroya [121]. Walther has also worked on the case when $0 < m < 1$ [144].

In 2018, shortly after the incredible work of [56, 57] appeared in public, Chu-hee Cho and Hyerim Ko observed the methods in [56, 57] work well for the fractional Schrödinger equation since the hypersurface $\{(\xi, |\xi|^m) : \xi \in \mathbb{B}^d\} \subset \mathbb{R}^{d+1}$ has non-vanishing Gaussian curvature everywhere and concluded that $s > \frac{1}{2} - \frac{1}{2(d+1)}$ is sufficient for (1.2), thus extending Sjölin's result to $d \geq 2$. Unlike the case for the standard Schrödinger equation, the necessary part still remains open.

The key observation here is that the parameter $m > 1$ seems not to affect to the smooth regularity s in the context of the classical Carleson's problem. We will see that in some variations' of Carleson's problem, the sharp threshold depends on m.

1.1.2 Carleson's problem and the divergence sets

In 2011, Juan Barceló, Bennett, Carbery and Rogers proposed an interesting refinement of Carleson's problem [9]: Measuring more precisely the size of the sets on which convergence such as (1.1). Before [133], Peter Sjögren and Per Sjölin had also considered a similar problem in the context of the so-called C_s-capacity.

Definition. *For $s > 0$, define the divergence set*

$$\mathfrak{D}(f) = \{x \in \mathbb{R}^d : e^{it(-\Delta)^{m/2}} f(x) \not\to f(x) \text{ as } t \to 0\},$$

and

$$\alpha_d(s) := \sup_{f \in H^s(\mathbb{R}^d)} \dim_{\mathrm{H}} \mathfrak{D}(f),$$

where $\dim_{\mathrm{H}}$ *denotes the Hausdorff dimension.*

For instance, one of the results by Barceló, Bennett, Carbery and Rogers combined with result by Darko Žubrinić [147] completely refine classical results by Carleson [43], Dahlberg and Kenig [52] and Sjölin [134] in one spatial dimension as

$$\alpha_1(s) = 1 - 2s$$

for $m > 1$. As seen in Figure 1.1, there is the unexpected gap at the critical point $s = \frac{1}{4}$.

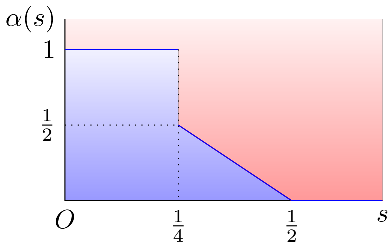


Figure 1.1: The result of the refined Carleson's problem in one spatial dimension

In higher spatial dimensional case, we describe the current known results. The broken lines in Figure 1.2 represent the previous results due to Lucà and Rogers [108] (before [56] was appeared) and Du, Guth, Li and Zhang [58] (before [57] was appeared).

$$\alpha_d(s) \leq \begin{cases} d + 1 - \frac{2(d+1)s}{d}, & \frac{d}{2(d+1)} < s < \frac{d}{4} \text{ (Du–Guth–Li [56], Du–Zhang [57])} \\ d - 2s, & \frac{d}{4} \leq s \leq \frac{d}{2} \text{ (Barceló–Bennett–Carbery–Rogers [9])} \end{cases}$$

$$\alpha_d(s) \geq \begin{cases} d, & s < \frac{d}{2(d+1)} \text{ (Bourgain [36])} \\ d + \frac{d}{d-1} - \frac{2(d+1)s}{d-1}, & \frac{d}{2(d+1)} \leq s < \frac{d+1}{8} \text{ (Lucà–Rogers [108])} \\ d + 1 - \frac{2(d+2)s}{d}, & \frac{d+1}{8} \leq s < \frac{d}{4} \text{ (Lucà–Rogers [108])} \\ d - 2s, & \frac{d}{4} \leq s \leq \frac{d}{2} \text{ (Žubrinić [147])} \end{cases}$$

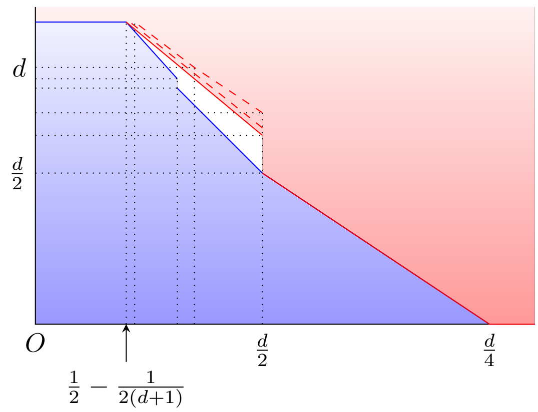


Figure 1.2: The results of the refined Carleson's problem in higher spatial dimension

1.1.3 Path along lines generated by a fractal set

Sjögren and Sjölin also considered the following generalized Carleson's problem in the sense of the manner of its convergence in [133]. Define for a compact set $\Theta \subset \mathbb{R}^d$,

$$\Gamma_x(\Theta) = \{(t, x + t\theta) : t \in [-1, 1] \text{ and } \theta \in \Theta.\}$$

While the usual limit for the classical Carleson's problem can be illustrated as the limit $(y, t) \to (x, 0)$, where $(y, t) \in \Gamma_x(\{0\})$ (see Figure 1.3), setting $\Theta = \mathbb{B}(0, 1)$, the unit ball of the center at the origin, they take the limit $(y, t) \to (x, t)$ where $(y, t) \in \Gamma_x([-1, 1])$ (see Figure 1.4).

As we have seen in the previous section, one of the early seminal generalizations of Carleson's problem is due to Sjögren and Sjölin in 1989 [133]. In the same paper they noted that the convergence $\lim_{t\to 0}$ in (1.1) can be considered as reaching $(t, y) \to (0, x)$ along the vertical line onto $(0, x)$; for each x

$$\Gamma_x(\{0\}) = \{(t, x + t\theta) : t \in [-1, 1] \text{ and } \theta \in \{0\}\}$$

and widened the region for the path of convergence from such standard vertical line to the conical region and obtain the following.

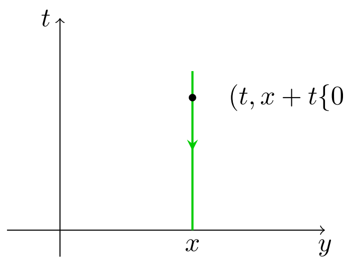


Figure 1.3: Convergence along the vertical line

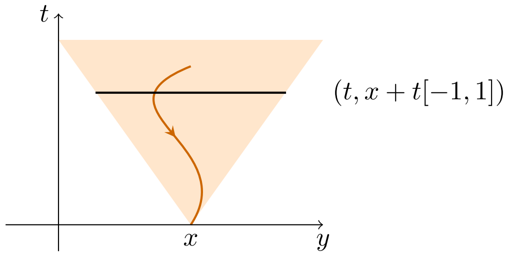


Figure 1.4: Convergence along the conical region

The conical region admits a larger variety of paths that converge to $(x, 0) \in \mathbb{R}^d \times \mathbb{R}$, so one may expect somewhat stronger smoothness condition on s than the classical setting (with convergence along the vertical line); Sjögren and Sjölin proved that the guess is actually true.

Theorem (Sjögren–Sjölin [133])**.** *Let $d \geq 1$. Then,*

$$\lim_{\substack{(t,y)\to(0,x)\\(t,y)\in\Gamma_x([-1,1])}} e^{-it\Delta} f(y) = f(x)$$

for almost everywhere in $\mathbb{R}^d$ for any $f \in H^s(\mathbb{R}^d)$ if and only if $s \geq \frac{d}{2}$.

Their credit is rather constructing the counterexample which shows its necessity since its sufficiency can be trivially proved by Hölder's inequality and the fact that

$$\int_0^\infty (1 + r^2)^{-s} r^{d-1} \, \mathrm{d}r < \infty$$

if $s > \frac{d}{2}$. When $d = 1$, the classical Carleson's problem (considering (1.1)) and the result of Sjögren–Sjölin has been unified by Chu-hee Cho, Sanghyuk Lee and Ana Vargas [48]. Let Θ be a compact set in $\mathbb{R}$. Now Θ could be other than $\{0\}$ or $\mathbb{B}^1 = [-1, 1]$, for instance, a fractal set such as the middle third Cantor set. To measure the size of Θ more precisely than Lebesgue sense, we introduce the following Minkowski dimension.

Definition (Minkowski dimension)**.** *For a compact set $\Theta \subset \mathbb{R}^d$, denote by $\beta(\Theta)$ the Minkowski dimension of Θ given by*

$$\beta(\Theta) := \inf\{\varsigma \in [0, d] : \limsup_{\delta\to 0} N(\Theta : \delta)\delta^\varsigma < \infty\},$$

where $N(\Theta : \delta) = \min\{k \in \mathbb{N} : \Theta \subset \bigcup_{j=1}^k \Omega_j, \ |\Omega_j| < \delta\}$.

Then, Cho, Lee and Vargas considered paths within $\Gamma_x(\Theta)$ (see Figure 1.5) to unify the above results in the sense of Minkowski dimension. Their interesting result is the following.

Theorem (Cho–Lee–Vargas [48])**.** *Let $d = 1$ and Θ be a compact subset of $\mathbb{R}$. Then,*

$$\lim_{\substack{(t,y)\to(0,x)\\(t,y)\in\Gamma_x(\Theta)}} e^{-it\Delta} f(y) = f(x)$$

for almost everywhere in $\mathbb{R}$ for any $f \in H^s(\mathbb{R})$ if $s > \frac{1}{2} - \frac{1-\beta(\Theta)}{4}$. Here, for each $x \in \mathbb{R}$

$$\Gamma_x(\Theta) = \{(t, x + t\theta) : t \in [-1, 1] \text{ and } \theta \in \Theta\}.$$

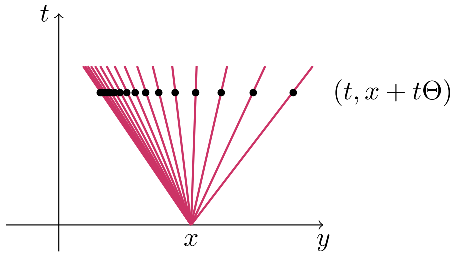


Figure 1.5: Convergence along a fractal set Θ

The necessity of $s > \frac{1}{2} - \frac{1-\beta(\Theta)}{4}$ is still open.

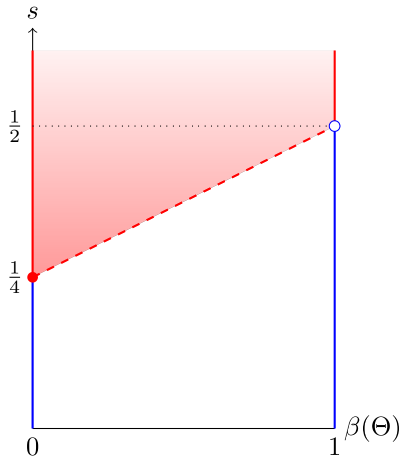


Figure 1.6: The illustration of the result due to Cho–Lee–Vargas

1.1.4 Path along a tangential curve

In the study of pointwise convergence problem for the Schrödinger equation with harmonic oscillator potential, Lee and Rogers [100] showed that any $\gamma \in C^1(\mathbb{R}^d \times [-1,1] \to \mathbb{R}^d)$, such as $\gamma(x,t) = x - (t^\kappa, 0, \cdots, 0)$ with $\kappa \geq 1$, is essentially equivalent to the vertical line in the context of pointwise convergence problem of (1.15).

Theorem (Lee–Rogers [100]). *Suppose that*

$$\lim_{t\to 0} e^{-it\Delta} f(x) = f(x)$$

for all $f \in H^s(\mathbb{R}^d)$ *whenever* $s > s_0$, *then*

$$\lim_{\substack{(y,t)\to(0,x)\\ (y,t)\in\Gamma_x(\gamma,C^1)}} e^{-it\Delta} f(y) = f(x)$$

for all $f \in H^s(\mathbb{R}^d)$ *whenever* $s > s_0$*. Here, for each* $x \in \mathbb{R}^d$

$$\Gamma_x(\gamma, C^1) := \left\{ (t, \gamma(t,x)) : \substack{t\in[-1,1], \\ \gamma(t,y) \,\in\, C^1(\mathbb{R}^d, \mathbb{R}^d \times [0,1]) \textit{ satisfies } \gamma(0,x) = x} \right\}.$$

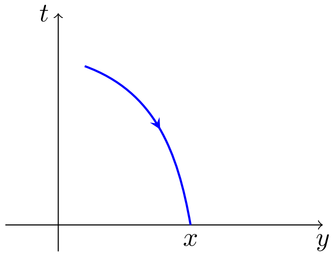


Figure 1.7: Convergence along a non-tangential curve

The typical curve $\gamma(t,x)$ satisfying the conditions of the above theorem is, for instance, $\gamma(t,x) = x + (t^\kappa, 0, \ldots, 0)$ with $\kappa \geq 1$ and, as the theorem above states, fits in the frame work of "non-tangential case" which has been discussed in the previous section. Then, the natural question is; what happens to s if we take the convergence path that approaches to $(0,x)$ tangentially against the hyperplane $\mathbb{R}^d \times \{0\}$, such as along the curve given by $\gamma(t,x) = x + (t^\kappa, 0, \ldots, 0)$ with $0 < \kappa \leq 1$? In the paper [48] where Cho, Lee and Vargas considered the path of lines generated by a fractal set, they also study this problem associated with a tangential curve and gave an answer to the question in one dimensional case. The nature turns to be remarkably different from the non-tangential case. To state their result, let us introduce some classes of curves.

Definition. *We say the curve* γ *satisfy Hölder condition of order* $\kappa \in (0,1]$ *in* t *if*

$$|\gamma(x,t) - \gamma(x,t')| \leq C_1 |t - t'|^\kappa, \quad x \in \mathbb{R}^d, \quad t, t' \in [-1,1] \tag{1.3}$$

and is bilipschitz in x *if*

$$\frac{1}{C_2}|x - x'| \leq |\gamma(x,t) - \gamma(x',t)| \leq C_2|x - x'|, \quad t \in [-1,1], \quad x, x' \in \mathbb{R}^d \tag{1.4}$$

for some $C_1, C_2 > 0$.

Theorem (Cho–Lee–Vargas [48])**.** *Let* $d = 1$*. Then,*

$$\lim_{\substack{(t,y)\to(0,x) \\ (t,y)\in\Gamma_x(\gamma,\kappa)}} e^{-it\Delta} f(y) = f(x)$$

for almost everywhere in $\mathbb{R}$ *for any* $f \in H^s(\mathbb{R})$ *if* $s > \frac{1}{2} - \min\{\frac{1}{4}, \kappa\}$*. Here,*

$$\Gamma_x(\gamma, \kappa) = \{(t, \gamma(x,t)) : t \in [-1,1],\ \gamma \textit{ satisfies } \gamma(x,0) = x,\ (1.3) \textit{ and } (1.4)\}.$$

Note that $\Gamma_x(\gamma, \kappa)$ contains $\gamma(x,t) = x - (t^\kappa, 0, \cdots, 0)$ with $0 \leq \kappa \leq 1$. The result above has been refined in the context of measuring its divergence set in Section 1.1.2.

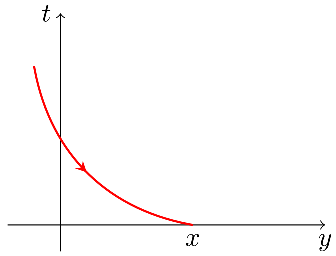


Figure 1.8: Convergence along a tangential curve

Definition. *Suppose γ satisfies $\gamma(x,0)=x$, (1.3) and (1.4). For $s>0$ define the divergence set*

$$\mathfrak{D}(f,\gamma)=\{x\in\mathbb{R}^d : e^{it(-\Delta)^{m/2}}f(\gamma(x,t))\not\to f(x) \text{ as } t\to 0\},$$

and

$$\alpha_d^\gamma(s) := \sup_{f\in H^s(\mathbb{R}^d)} \dim_{\mathrm{H}}\mathfrak{D}(f,\gamma).$$

Theorem (Cho–Lee [46])**.** *Let $d=1$, $0<\kappa\le 1$, and γ on $\mathbb{R}\times[-1,1]$ satisfies $\gamma(x,0)=x$, (1.3) and (1.4). Then, for $s>\frac{1}{4}$,*

$$\alpha_2^\gamma(s)\le \max\left\{1-2s, \frac{1-2s}{2\kappa}\right\}.$$

1.1.5 Reduction

In the study of Carleson's problem, or pointwise convergence problem in general, it is usually considered as the problem asking whether or not the associated maximal inequality via the reduction argument below hold. The fundamental idea can be seen in most text books which discuss the Lebesgue differentiation theorem as a corollary of the weak-type boundedness of the Hardy–Littlewood maximal operator. To describe such a reduction in a wide sense as required in the present thesis, first let us define a family of measures.

Definition (α-dimensional measure)**.** *Let $0<\alpha\le d$. A positive Borel measure μ is said to be α-dimensional if there exists a constant c such that*

$$\mu(\mathbb{B}^d(x,r))\le cr^\alpha, \tag{1.5}$$

where $\mathbb{B}^d(x,r)$ is the ball centered at $x\in\mathbb{R}^d$ with radius $r>0$.

For simplicity, we shall denote $S_t=S_{t,m}=e^{it(-\Delta)^{\frac{m}{2}}}$.

Proposition 1.1.2. *Let $d\ge 1$, $m>1$, $q\in[1,\infty)$, γ be a function on $\mathbb{R}^d\times[-1,1]$, and μ be an α-dimensional measure. Suppose that there exists some constant $C>0$ such that*

$$\left\|\sup_{t\in[-1,1]}|S_tf(\gamma(\cdot,t))|\right\|_{L^q(\mathbb{B}^d)}\le C\|f\|_{H^s}$$

for all $f \in H^s$ whenever $s > s_0$, then

$$\lim_{t\to 0} S_t f(\gamma(x,t)) = f(x)$$

for μ-almost everywhere for all $f \in H^s$ whenever $s > s_0$.

Proof of Proposition 1.1.2. Fix an arbitrary $f \in H^s(\mathbb{R})$. Then, it is enough to show that $\mu(\mathfrak{D}(\gamma, f)) = 0$. Now, choose a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R})$ which converges in H^s-norm to $f \in H^s(\mathbb{R})$. Then, we divide the divergence set into localized pieces as follows and show that all terms turn out to be 0.

$$\mu(\mathfrak{D}(\gamma, f)) \le \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{\infty} \mu(\{x \in I + j \,:\, \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \ell^{-1}\}).$$

Now, for each $n \ge 1$, $j = 0$ and $\lambda \ge 1$ observe that

$$\begin{aligned}
&\mu(\{x \in I \,:\, \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \lambda^{-1}\}) \\
&\le \mu(\{x \in I \,:\, \limsup_{t\to 0} |S_t f(\gamma(x,t)) - S_t f_n(\gamma(x,t))| > (3\lambda)^{-1}\}) \\
&\quad + \mu(\{x \in I \,:\, \limsup_{t\to 0} |S_t f_n(\gamma(x,t)) - f_n(x)| > (3\lambda)^{-1}\}) \\
&\quad\quad + \mu(\{x \in I \,:\, |f_n(x) - f(x)| > (3\lambda)^{-1}\}) \\
&\le \mu(\{x \in I \,:\, \sup_{t\in I} |S_t\big(f(\gamma(x,t)) - f_n(\gamma(x,t))\big)| > (3\lambda)^{-1}\}) \\
&\quad + 0 + \mu(\{x \in I \,:\, |f_n(x) - f(x)| > (3\lambda)^{-1}\}).
\end{aligned}$$

By invoking Chebyshev's inequality and Theorem 1.3.3 we obtain

$$\mu(\{x \in I \,:\, \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \lambda^{-1}\}) \lesssim \lambda^2 \|f - f_n\|_{H^s(\mathbb{R})}^2, \tag{1.6}$$

which tends to 0 as $n \to \infty$. For other j, make translation $x \mapsto x + j$ and we define a measure μ_j by $\mu_j(x) = \mu(x + j)$ and a curve γ_j by $\gamma_j(x,t) = \gamma(x + j, t)$, both of which satisfy the required conditions for Theorem 1.3.3 so that (1.6) holds with I replaced by $I + j$. Therefore, for all $j \in \mathbb{Z}$ and $\ell \ge 1$,

$$\mu(\{x \in I + j \,:\, \lim_{t\to 0} |S_t f(\gamma(x,t)) - f(x)| > \ell^{-1}\}) = 0$$

holds as desired. □

1.2 First new result

The first result is associated to the path along lines generated by a fractal set. This result has been published in the paper [132] by the author.

Theorem 1.2.1 (S.)**.** *Let $d = 1$, $a > 1$ and Θ be a compact subset of $\mathbb{R}$. Then,*

$$\lim_{\substack{(t,y)\to(0,x) \\ (t,y)\in\Gamma_x(\Theta)}} e^{it(-\Delta)^a} f(y) = f(x)$$

for almost everywhere in $\mathbb{R}$ for any $f \in H^s(\mathbb{R})$ if $s > \frac{1}{2} - \frac{1-\beta(\Theta)}{4}$.

We will consider a generalized setting. Throughout the Section 1.2, let the evolution operator S_t on appropriate input functions by

$$S_t f(x) = S_t^{\Phi} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x\xi + t\Phi(\xi))} \widehat{f}(\xi)\, d\xi.$$

Here, $\Phi : \mathbb{R} \to \mathbb{R}$ is a C^2 function which satisfies for some $C_1 > 0$,

$$|\xi| \left| \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \Phi(\xi) \right| \geq C_1 \tag{1.7}$$

for all $|\xi| \geq 1$. Moreover, for some $C_2 > 0$,

$$|\xi| \left| \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \Phi(\xi) \right| \geq C_2 \left| \frac{\mathrm{d}}{\mathrm{d}\xi} \Phi(\xi) \right| \tag{1.8}$$

for all $|\xi| \geq 1$. It is trivial to verify that $\Phi(\xi) = |\xi|^a$ satisfies these conditions when $a > 1$. By the reduction argument in Section 1.1.5, Theorem 1.2.1 follows from the subsequent maximal estimate.

Theorem 1.2.2. *Let $\Theta \subset \mathbb{R}$ be compact and suppose $\Phi \in C^2(\mathbb{R})$ satisfies* (1.7) *and* (1.8)*. For any $q \in [1, 4]$ and $s > \frac{1}{4} + \frac{\beta(\Theta)}{4}$, there exists a constant $C_{q,s}$ such that*

$$\left\| \sup_{(t,\theta) \in [-1,1] \times \Theta} |S_t f(\cdot + t\theta)| \right\|_{L^q(-1,1)} \leq C_{q,s} \|f\|_{H^s(\mathbb{R})}$$

whenever $f \in H^s(\mathbb{R})$.

Theorem 1.2.2 improves the result in [48] in two respects; the class of evolution operators has been widened from the case $\Phi(\xi) = |\xi|^2$ to those satisfying (1.7) and (1.8), and our maximal estimates are valid for $q \in [1, 4]$ (the estimate in [48] was proved in only the cases $q \in [1, 2]$). While the proof in [48] may be modified in a straightforward way to go beyond the classical case $\Phi(\xi) = |\xi|^2$ to a certain extent, it seems to us to be difficult to handle case $\Phi(\xi) = |\xi|^a$ with a close to 1. Indeed, the argument in [48] rests on a certain widely used time localization argument which becomes increasingly weak as a approaches 1. To overcome this significant obstacle, we remove the use of the time localization lemma; this simplification to the proof has allowed us to handle the case $\Phi(\xi) = |\xi|^a$ for any $a > 1$. Further explanation of this point will follow our proof of Theorem 1.2.2.

1.2.1 Lemmas

The following lemmas will be crucial for the oscillatory integral estimates in the proof of Theorem 1.2.1. Applying these lemmas appropriately essentially allows us to avoid the time localization lemma, which is used in [48].

Lemma 1.2.3 (van der Corput's lemma)**.** *Let $-\infty < a < b < \infty$, ϕ be a sufficiently smooth real-valued function and ψ be a bounded smooth complex-valued function. Suppose we have $|\phi^{(k)}(x)| \geq 1$ for all $x \in [a, b]$. If $k = 1$ and ϕ' is monotonic on (a, b), or simply $k \geq 2$, then there exists a constant C_k such that*

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x)\, dx \right| \leq C_k \lambda^{-\frac{1}{k}} \left(\int_a^b \left| \frac{\mathrm{d}}{\mathrm{d}\xi} \psi(x) \right| dx + \|\psi\|_{L^\infty} \right)$$

for all $\lambda > 0$.

For a proof of van der Corput's lemma, we refer the reader to [135].

Lemma 1.2.4. *Let $1 \leq q \leq 4$. There exists a constant C_q such that*

$$\left| \iiiint g(x,t)h(x',t')|x-x'|^{-\frac{1}{2}}\, dxdtdx'dt' \right| \leq C_q \|g\|_{L_x^{q'}L_t^1} \|h\|_{L_x^{q'}L_t^1},$$

where the integrals are taken over $(x,t),(x',t') \in I \times I$.

Proof. Denoting $G(x) = \|g(x,\cdot)\|_{L^1}$ and $H(x') = \|h(x',\cdot)\|_{L^1}$,

$$\left| \iiiint g(x,t)h(x',t')|x-x'|^{-\frac{1}{2}}\, dxdx'dtdt' \right| \leq \int_{-1}^{1}\int_{-1}^{1} G(x)H(x')|x-x'|^{-\frac{1}{2}}\, dxdx'.$$

By the Hardy–Littlewood–Sobolev inequality,

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1} G(x)H(x')|x-x'|^{-\frac{1}{2}}\, dxdx' &\lesssim \|G\|_{L^{\frac{4}{3}}(I)} \|H\|_{L^{\frac{4}{3}}(I)} \\ &\lesssim \|g\|_{L_x^{q'}L_t^1} \|h\|_{L_x^{q'}L_t^1},\end{aligned}$$

where the last inequality is obtained by Hölder's inequality since $\frac{4}{3} \leq q'$ from our assumption. □

1.2.2 Proof of Theorem 1.2.1

We fix $q \in [2,4]$. The case $q \in [1,2)$ follows immediately by Hölder's inequality.

The proof begins with a reduction to the case where f is frequency-localised to a large annulus and θ belongs to an interval of an appropriately small length. This reduction to the forthcoming Proposition 1.2.5 essentially follows the argument in [48]; our main novelty is the proof of Propositon 1.2.5.

Suppose $\psi_0 \in C_0^\infty(I)$ and $\psi \in C_0^\infty((-2,-\frac{1}{2}) \cup (\frac{1}{2},2))$ give rise to a standard dyadic partition of unity

$$\psi_0(\xi) + \sum_{k\geq 1} \psi_k \equiv 1,$$

where $\psi_k = \psi(\frac{\cdot}{2^{k-1}})$. For each $0 \leq k \in \mathbb{Z}$, the frequency localization operator P_k is defined by

$$\widehat{P_k f}(\xi) = \psi_k(\xi)\widehat{f}(\xi).$$

Then, by letting $M_\Theta f = \sup\{|S_t f(\cdot + t\theta)| : 0 < t < 1, \theta \in \Theta\}$

$$\|M_\Theta f\|_{L^q(I)} \lesssim \|M_\Theta P_0 f\|_{L^q(I)} + \sum_{k\geq 1} \|M_\Theta P_k f\|_{L^q(I)}. \tag{1.9}$$

The first term is relatively easy to estimate. In fact,

$$\begin{aligned}\|M_\Theta P_0 f\|_{L^q(I)} &\lesssim \int_{\mathbb{R}} \psi_0(\xi)|\widehat{f}(\xi)|\, d\xi \\ &\lesssim \|f\|_{L^2} \\ &\lesssim \|f\|_{H^s}\end{aligned}$$

for $s \geq 0$, and thereby this term can be easily handled.

For the remaining terms, first note that for each $k \geq 1$, there exists a finite collection of intervals $\{\Omega_{k,j}\}_{j=1}^{N_k}$ which satisfies

$$\Theta \subset \bigcup_{j=1}^{N_k} \Omega_{k,j},$$

where $|\Omega_{k,j}| \leq 2^{-\frac{qk}{4}}$ for each j and $N_k = N(\Theta, 2^{-\frac{qk}{4}})$ is the smallest number of $2^{-\frac{qk}{4}}$-intervals which cover Θ. (The reason for the choice of scale $2^{-\frac{qk}{4}}$ will become clear as we proceed.) For $x \in I$,

$$M_\Theta P_k f(x)^q \leq \sum_{j=1}^{N_k} \sup_{\substack{t \in I \\ \theta \in \Omega_{k,j}}} |S_t P_k f(x + t\theta)|^q,$$

therefore

$$\sum_{k \geq 1} \|M_\Theta P_k f\|_{L^q(I)} \leq \sum_{k \geq 1} \left(\sum_{j=1}^{N_k} \left\|M_{\Omega_{k,j}} P_k f\right\|_{L^q(I)}^q \right)^{\frac{1}{q}}.$$

Now, we shall introduce the following crucial proposition.

Proposition 1.2.5. *Let $2 \leq q \leq 4$, $k \geq 1$ and Ω be an interval with $|\Omega| \leq 2^{-\frac{qk}{4}}$. Then, there exists a constant C_q such that*

$$\|M_\Omega P_k f\|_{L^q(I)} \leq C_q 2^{\frac{k}{4}} \|f\|_{L^2} \tag{1.10}$$

holds for all $f \in L^2(\mathbb{R})$.

Proof of Proposition 1.2.5. Set $\lambda = 2^k$ and

$$Tf(x,t,\theta) := \chi(x,t,\theta) \int_{\mathbb{R}} e^{i((x+t\theta)\xi + t\Phi(\xi))} f(\xi) \psi(\tfrac{\xi}{\lambda})\, d\xi,$$

where $\chi = \chi_{I \times I \times \Omega}$. Then (1.10) follows from

$$\|Tf\|_{L^q_x L^\infty_t L^\infty_\theta} \lesssim \lambda^{\frac{1}{4}} \|f\|_{L^2} \quad (\lambda \gtrsim 1) \tag{1.11}$$

since

$$\begin{aligned} \|M_\Omega P_k f\|_{L^q(I)} &\sim \|T\widehat{f}\|_{L^q_x L^\infty_t L^\infty_\theta} \\ &\lesssim \lambda^{\frac{1}{4}} \|\widehat{f}\|_{L^2} \\ &\lesssim \lambda^{\frac{1}{4}} \|f\|_{L^2} \end{aligned}$$

by Plancherel's theorem. Let us consider the dual form of (1.11), which is

$$\|T^* F\|_{L^2} \lesssim \lambda^{\frac{1}{4}} \|F\|_{L^{q'}_x L^1_t L^1_\theta} \tag{1.12}$$

where

$$T^* F(\xi) = \psi(\tfrac{\xi}{\lambda}) \iiint \chi(x',t',\theta') e^{-i((x'+t'\theta')\xi + t'\Phi(\xi))} F(x',t',\theta')\, dx' dt' d\theta'.$$

Then,

$$
\begin{aligned}
&\|T^*F\|_{L^2}^2\\
&= \lambda \int \psi^2(\xi) \iiint \iiint \chi(x,t,\theta)\chi(x',t',\theta')\\
&\quad \times e^{i(\lambda(x-x'+t\theta-t'\theta')\xi+(t-t')\Phi(\lambda\xi))}\bar{F}(x,t,\theta)F(x',t',\theta')\, dxdtd\theta dx'dt'd\theta' d\xi\\
&= \int_W \int_{W'} \chi(w)\chi(w')\bar{F}(w)F(w')K_\lambda(w,w')\, dwdw'\\
&= \sum_{\ell=1}^{3} \iint_{V_\ell} \chi(w)\chi(w')\bar{F}(w)F(w')K_\lambda(w,w')\, dwdw'\\
&=: \mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_1.
\end{aligned}
$$

Here, we denote $w = (x,t,\theta) \in W$ and $w' = (x',t',\theta') \in W$, where $W := I \times I \times \Omega$. Also,

$$
\begin{aligned}
K_\lambda(w,w') &= \int_{\mathbb{R}} e^{i\phi(\xi,w,w')}\psi^2(\tfrac{\xi}{\lambda})\, d\xi\\
&= \lambda \int_{\mathbb{R}} e^{i\phi(\lambda\xi,w,w')}\psi^2(\xi)\, d\xi,
\end{aligned}
$$

$$
\phi(\xi,w,w') = (x-x'+t\theta-t'\theta')\xi + (t-t')\Phi(\xi),
$$

and

$$
\begin{cases}
V_1 &= \{(w,w') \in W \times W : |x-x'| < 4\lambda^{-\frac{q}{4}}\},\\
V_2 &= \{(w,w') \in W \times W : |x-x'| \geq 4\lambda^{-\frac{q}{4}} \text{ and } |x-x'| < 4|t-t'|\ \},\\
V_3 &= \{(w,w') \in W \times W : |x-x'| \geq 4\lambda^{-\frac{q}{4}} \text{ and } |x-x'| \geq 4|t-t'|\ \}.
\end{cases}
$$

Thus, (1.12) follows from

$$
\mathcal{I}_\ell \lesssim \lambda^{\frac{1}{2}} \|F\|_{L_x^{q'} L_t^1 L_\theta^1}^2
$$

for each $\ell = 1,2,3$.

The term $\mathcal{I}_1$

Let us start with an estimate of $\mathcal{I}_1$.

$$
\mathcal{I}_1 \lesssim \lambda^{\frac{1}{2}} \|F\|_{L_x^{q'} L_t^1 L_\theta^1}^2.
$$

Trivially,

$$
|K_\lambda(w,w')| \lesssim \lambda
$$

so by the dual form of Young's convolution inequality

$$
\begin{aligned}
&\lambda \int_{-1}^{1}\int_{-1}^{1} \|F(x,\cdot,\cdot)\|_{L_t^1 L_\theta^1}\|F(x',\cdot,\cdot)\|_{L_t^1 L_\theta^1}\chi_{[-4\lambda^{-\frac{q}{4}},4\lambda^{-\frac{q}{4}}]}(x-x')\, dxdx'\\
&\lesssim \lambda \|F\|_{L_x^{q'} L_t^1 L_\theta^1}^2 \|\chi_{[-4\lambda^{-\frac{q}{4}},4\lambda^{-\frac{q}{4}}]}\|_{L^{\frac{q}{2}}}\\
&\sim \lambda^{\frac{1}{2}} \|F\|_{L_x^{q'} L_t^1 L_\theta^1}^2.
\end{aligned}
$$

The term $\mathcal{I}_2$

Since

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\lambda\xi)\right| = \lambda^2|t-t'|\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Phi(\lambda\xi)\right| \gtrsim \lambda|x-x'|$$

holds from (1.7), we are allowed to apply Lemma 1.2.3 to get

$$|K_\lambda(w,w')| \lesssim \lambda(\lambda|x-x'|)^{-\frac{1}{2}}.$$

By using Lemma 1.2.4, it follows that

$$\begin{aligned}\mathcal{I}_2 &\le \lambda^{\frac{1}{2}}\iint_{V_1}\chi(w')|F(w')|\chi(w)|\bar{F}(w)||x-x'|^{-\frac{1}{2}}\,dwdw'\\ &\lesssim \lambda^{\frac{1}{2}}\|F\|^2_{L_x^{q'}L_t^1L_\theta^1}.\end{aligned}$$

The term $\mathcal{I}_3$

It remains to show $\mathcal{I}_3$. In this case, we firstly observe the following key relationship:

$$|x-x'+t\theta-t'\theta'| \sim |x-x'|. \tag{1.13}$$

Indeed,

$$\begin{aligned}|x-x'+t\theta-t'\theta'| &\ge |x-x'|-|t-t'|-|\theta-\theta'|\\ &\ge \frac{3}{4}|x-x'|-\lambda^{-\frac{q}{4}}\\ &\ge \frac{1}{2}|x-x'|.\end{aligned}$$

Similarly, the other way holds, too.

Now, let us observe that for all $(w,w')\in V_2$, we have

$$|K_\lambda(w,w')| \lesssim \lambda(\lambda|x-x'|)^{-\frac{1}{2}}. \tag{1.14}$$

Before proving (1.14), we note that

$$\mathcal{I}_3 \lesssim \lambda^{\frac{1}{2}}\|F\|^2_{L_x^{q'}L_t^1L_\theta^1}$$

immediately follows by using Lemma 1.2.4 as before.

To see (1.14), let us split K_λ into $\mathcal{K}_1$ and $\mathcal{K}_2$ as follows

$$\begin{aligned}K_\lambda(w,w') &= \lambda\int_{U_1}e^{i\phi(\lambda\xi,w,w')}\psi^2(\xi)\,d\xi + \lambda\int_{U_2}e^{i\phi(\lambda\xi,w,w')}\psi^2(\xi)\,d\xi\\ &=: \mathcal{K}_1+\mathcal{K}_2,\end{aligned}$$

where

$$U_1 = \{\xi\in\operatorname{supp}\psi : |x-x'+t\theta-t'\theta'| \ge 2|t-t'||\Phi'(\lambda\xi)|\}$$

and

$$U_2 = \{\xi\in\operatorname{supp}\psi : |x-x'+t\theta-t'\theta'| < 2|t-t'||\Phi'(\lambda\xi)|\}.$$

For $\mathcal{K}_1$, we have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\lambda\xi)\right| &\geq \lambda|x-x'+t\theta-t'\theta'|-\lambda|t-t'|\left|\frac{\mathrm{d}}{\mathrm{d}\xi}\Phi(\lambda\xi)\right| \\ &\geq \frac{\lambda}{2}|x-x'+t\theta-t'\theta'| \\ &\geq \frac{\lambda}{4}|x-x'| \\ &> \lambda^{1-\frac{q}{4}} \\ &\geq 1,\end{aligned}$$

where we have used the fact that $q \leq 4$. From (1.7) and the intermediate value theorem, $\Phi''(\xi)$ is single-signed on $(-\infty,-1]$ and $[1,\infty)$, which guarantees that $\Phi'(\xi)$ is monotone on these intervals. Hence, U_1 consists of at most two intervals. Invoking Lemma 1.2.3,

$$\mathcal{K}_1 \lesssim \lambda(\lambda|x-x'|)^{-1} \lesssim \lambda(\lambda|x-x'|)^{-\frac{1}{2}}.$$

On the other hand, for $\mathcal{K}_2$, it follows from (1.8) that

$$\begin{aligned}\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\lambda\xi)\right| &= \lambda^2|t-t'|\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Phi(\lambda\xi)\right| \\ &\gtrsim \lambda|t-t'|\left|\frac{\mathrm{d}}{\mathrm{d}\xi}\Phi(\lambda\xi)\right| \\ &\gtrsim \lambda|x-x'+t\theta-t\theta'| \\ &\gtrsim \lambda|x-x'|.\end{aligned}$$

Then, by using Lemma 1.2.3, we obtain

$$\mathcal{K}_2 \lesssim \lambda(\lambda|x-x'|)^{-\frac{1}{2}}.$$

Therefore, (1.14) holds.

Here, we have used the fact that $q \geq 2$. Therefore, we conclude that

$$\mathcal{I}_1 \lesssim \lambda^{\frac{1}{2}}\|F\|^2_{L_x^{q'}L_t^1L_\theta^1}$$

as claimed. □

By the definition of the upper Minkowski dimension, for small $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ depending on ε such that

$$N(\Theta, 2^{-\frac{qk}{4}}) \leq C_\varepsilon 2^{\frac{qk}{4}(\beta(\Theta)+\varepsilon)}.$$

Thus, if we also let $\widehat{\tilde{P}_k f} = \tilde{\psi}_k\widehat{f}$, where $\tilde{\psi} \in C_0^\infty((-4,-\frac{1}{4})\cup(\frac{1}{4},4))$ with $\tilde{\psi} \equiv 1$ on

$(-2,-\frac{1}{2})\cup(\frac{1}{2},2)$, then

$$\begin{aligned}\sum_{k\geq 1}\left(\sum_{j=1}^{N_k}\|M_{\Omega_{k,j}}P_kf\|_{L^q(I)}^q\right)^{\frac{1}{q}} &= \sum_{k\geq 1}\left(\sum_{j=1}^{N_k}\|M_{\Omega_{k,j}}P_k\tilde{P}_kf\|_{L^q(I)}^q\right)^{\frac{1}{q}}\\ &\lesssim \sum_{k\geq 1}\left(\sum_{j=1}^{N_k}2^{\frac{qk}{4}}\|\tilde{P}_kf\|_{L^2}^q\right)^{\frac{1}{q}}\\ &\lesssim \sum_{k\geq 1}2^{k(\frac{1}{4}+\frac{\beta(\Theta)}{4}+\frac{\varepsilon}{4})}\|\tilde{P}_kf\|_{L^2}\\ &\sim \sum_{k\geq 1}2^{-\frac{3}{4}k\varepsilon}\left(\int_{\operatorname{supp}\tilde{\psi}_k}2^{2k(\frac{1}{4}+\frac{\beta(\Theta)}{4}+\varepsilon)}|\widehat{f}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}\\ &\lesssim \|f\|_{H^{\frac{1}{4}+\frac{\beta(\Theta)}{4}+\varepsilon}}.\end{aligned}$$

Therefore, for arbitrary $\varepsilon>0$,

$$\|M_\Theta f\|_{L^q(I)}\lesssim\|f\|_{H^{\frac{1}{4}+\frac{\beta(\Theta)}{4}+\varepsilon}}$$

holds, which ends the proof. □

Remarks. The crucial component in the above proof of Theorem 1.2.1 is Proposition 1.2.5. The corresponding result in [48] (Lemma 3.1), stated for $q=2$ and $\Phi(\xi)=|\xi|^2$, is established through the following steps: TT^* argument, the time localization lemma, Schur's lemma and then an oscillatory integral argument. Following this approach in the case $\Phi(\xi)=|\xi|^a$, one may extend by simple modification to the range $a\geq\frac{3}{2}$. However, the time localization lemma reduces to the case of time intervals of length λ^{1-a}, and for a close to 1 this causes certain technical difficulties in the estimation of the oscillatory integrals which arise; in particular, the relationship (1.13) breaks down if we follow their argument as it stands. In order to overcome the significant technical difficulty, we removed the use of the time localization lemma and replaced this with appropriate decompositions of the domain $W\times W$.

1.3 Second new result

The second result is associated to the path along a tangential curve. This result has shown in the paper [49] by author collaborated with Chu-hee Cho.

Theorem 1.3.1 (Cho–S.)**.** *Let $d=1$ and $m>1$. Then,*

$$\lim_{\substack{(t,y)\to(0,x)\\(t,y)\in\Gamma_x(\gamma,\kappa)}} e^{it(-\Delta)^{\frac{m}{2}}}f(y)=f(x) \tag{1.15}$$

for almost everywhere in $\mathbb{R}$ for any $f\in H^s(\mathbb{R})$ if $s>\frac{1}{2}-\min\{\frac{1}{4},\frac{m\kappa}{2}\}$.

Theorem 1.3.1 can be obtained as a corollary of the following Theorem 1.3.2 in the case when $\alpha=1$.

Theorem 1.3.2. *Let $m > 1$, $0 < \kappa \leq 1$, μ be an α-dimensional measure and $\gamma \in \Gamma(\kappa)$. If $s > \max\left\{\frac{1}{4}, \frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\right\}$, then*

$$\lim_{t \to 0} S_t(f(\gamma(x,t))) = f(x), \quad \mu\text{-a.e. } x$$

for all $f \in H^s(\mathbb{R})$.

By a standard argument, this is reduced to the following local maximal estimate.

Theorem 1.3.3. *Let $m > 1$, $0 < \kappa \leq 1$, μ be an α-dimensional measure and $\gamma \in \Gamma(\kappa)$. If $s > \frac{1}{2} - \min\left\{\frac{1}{4}, \frac{\alpha}{2}, \frac{m\alpha\kappa}{2}\right\}$, then there exists a constant C such that*

$$\left(\int_{-1}^{1} \sup_{t \in [-1,1]} |S_t f(\gamma(\cdot,t))|^2 \, \mathrm{d}\mu(x)\right)^{\frac{1}{2}} \leq C\|f\|_{H^s} \tag{1.16}$$

for all $f \in H^s(\mathbb{R})$.

It is straightforward to obtain some results of the above type for $m \neq 2$ (more specifically $m \geq 2$) by appropriately modifying the argument in [48] or [55], however, as the second author observed in his study of a related problem in a different setting [132], there are some barriers with such an approach to treat m near 1. In particular, building on the ideas in [132] in which we completely avoid time localization techniques, we are able to handle the full range of $m > 1$ and give us the sharp sufficient conditions. Here, saying sharp is meant by that: Suppose $s < \max\left\{\frac{1}{4}, \frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\right\}$, then there exists $\gamma \in \Gamma(\kappa)$, α-dimensional μ and $f \in H^s(\mathbb{R})$ such that (1.3.2) fails. Since the counterexamples can be provided by adjusting the corresponding well-known constructions (for instance, [48] and [134]) without any major difficulty, we rather focus on the sufficient conditions. As corollaries of Theorem 1.3.2, we have the following.

Corollary 1.3.4. *Let $m > 1$, $0 < \kappa \leq 1$, $\gamma \in \Gamma(\kappa)$. If $s > \frac{1}{4}$, then*

$$\dim_c(\mathfrak{D}(\gamma, f)) \leq \max\left\{1 - 2s, \frac{1-2s}{m\kappa}\right\}.$$

The special case when μ is the (1-dimensional) Lebesgue measure extends the result in [48] from $m = 2$ to $m > 1$ as follows. Here, note that the required regularity on s for (1.15) depends not only on κ but m as well.

Corollary 1.3.5. *Let $m > 1$, $0 < \kappa \leq 1$ and $\gamma \in \Gamma(\kappa)$. If $s > \frac{1}{2} - \min\{\frac{1}{4}, \frac{m\kappa}{2}\}$, then*

$$\left(\int_{-1}^{1} \sup_{t \in [-1,1]} |S_t f(\gamma(\cdot,t))|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \leq C\|f\|_{H^s} \tag{1.17}$$

holds for all $f \in H^s(\mathbb{R})$.

Combining Corollary 1.3.5 with the result from [132], the results in [48] have been completely extended from the standard Schrödinger equation to the fractional Schrödinger equation with$m > 1$.

Remark. Although Theorem 1.3.3 is stated for the fractional Schrödinger evolution operator, by simply following our proof in the same conclusion is valid for a wider class of evolution operators such as

$$S_t^{\Phi} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x\xi + t\Phi(\xi))} \widehat{f}(\xi) \, \mathrm{d}\xi,$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a C^2-function for which there exist constants $C_3, C_4 > 0$ such that

$$|\xi|^{2-m} \left| \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \Phi(\xi) \right| \geq C_3 \quad \text{and} \quad |\xi| \left| \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \Phi(\xi) \right| \geq C_4 \left| \frac{\mathrm{d}}{\mathrm{d}\xi} \Phi(\xi) \right|$$

for all $|\xi| \geq 1$. This class trivially contains $|\xi|^m$ whenever $m > 1$.

1.3.1 Lemmas

In this section, as we have informed, we introduce useful lemmas which we use multiple times in the rest of the paper.

Lemma 1.3.6 (Frostman's lemma)**.** *Let $d \geq 1$ and X be a Borel set in $\mathbb{R}^d$. Then, the Hausdorff measure of order α of X is positive if and only if there exists α-dimensional measure such that $\operatorname{supp} \mu \subset X$ and $0 < \mu(\mathbb{R}^d) < \infty$. Further, $\mu(X) > 0$.*

For a proof of Lemma 1.3.6, we refer the reader to [110].

Lemma 1.3.7. *Let $0 < \alpha \leq 1$ and μ be an α-dimensional measure. There exists a constant C such that for any interval $[a, b]$ $(-\infty < a, b < \infty)$*

$$\begin{aligned} &\left| \iint \iint g(x,t) h(x',t') \chi_{[a,b]}(x - x') \, \mathrm{d}\mu(x) \mathrm{d}t \mathrm{d}\mu(x') \mathrm{d}t' \right| \\ &\leq C(b-a)^{\alpha} \|g\|_{L^2_x(\mathrm{d}\mu) L^1_t} \|h\|_{L^2_x(\mathrm{d}\mu) L^1_t}. \end{aligned} \tag{1.18}$$

Moreover, for $0 < \rho < \alpha$ there exists a constant C such that

$$\begin{aligned} &\left| \iint \iint g(x,t) h(x',t') |x - x'|^{-\rho} \, \mathrm{d}\mu(x) \mathrm{d}t \mathrm{d}\mu(x') \mathrm{d}t' \right| \\ &\leq C \|g\|_{L^2_x(\mathrm{d}\mu) L^1_t} \|h\|_{L^2_x(\mathrm{d}\mu) L^1_t}. \end{aligned} \tag{1.19}$$

Here, the both integrals are taken over (x, t), $(x', t') \in I \times I$.

Proof of Lemma 1.3.7. Denoting $G(x) = \|g(x, \cdot)\|_{L^1}$ and $H(x') = \|h(x', \cdot)\|_{L^1}$,

$$\begin{aligned} &\left| \iint \iint g(x,t) h(x',t') \chi_{[a,b]}(x - x') \, \mathrm{d}\mu(x) \mathrm{d}t \mathrm{d}\mu(x') \mathrm{d}t' \right| \\ &\leq \int_{-1}^{1} \int_{-1}^{1} G(x) H(x') \chi_{[a,b]}(x - x') \mathrm{d}\mu(x) \mathrm{d}\mu(x'). \end{aligned}$$

By invoking the Cauchy–Schwarz inequality on $L^2(I \times I, \mathrm{d}\mu \mathrm{d}\mu)$ and (1.5),

$$\begin{aligned} &\int_{-1}^{1} \int_{-1}^{1} G(x) H(x') \chi_{[a,b]}(x - x') \mathrm{d}\mu(x) \mathrm{d}\mu(x') \\ &\lesssim \left(\iint G(x)^2 \chi_{[a,b]}(x - x') \, \mathrm{d}\mu(x) \mathrm{d}\mu(x') \right)^{\frac{1}{2}} \left(\iint H(x')^2 \chi_{[a,b]}(x - x') \, \mathrm{d}\mu(x) \mathrm{d}\mu(x') \right)^{\frac{1}{2}} \\ &\lesssim (b-a)^{\alpha} \|G\|_{L^2_x(\mathrm{d}\mu)} \|H\|_{L^2_x(\mathrm{d}\mu)}. \end{aligned}$$

Now, (1.19) follows from (1.18), immediately. In fact, by applying a dyadic decomposition,

$$\begin{aligned}
&\left| \iint \iint g(x,t)h(x',t')|x-x'|^{-\rho}\,\mathrm{d}\mu(x)\mathrm{d}t\mathrm{d}\mu(x')\mathrm{d}t' \right| \\
&\lesssim \sum_{j=0}^{\infty} 2^{\rho j} \iint G(x)H(x')\chi_{[2^{-j},2^{-j+1}]}(x-x')\,\mathrm{d}\mu(x)\mathrm{d}\mu(x') \\
&\lesssim \sum_{j=0}^{\infty} 2^{(\rho-\alpha)j}\|G\|_{L^2_x(\mathrm{d}\mu)}\|H\|_{L^2_x(\mathrm{d}\mu)} \\
&\lesssim \|G\|_{L^2_x(\mathrm{d}\mu)}\|H\|_{L^2_x(\mathrm{d}\mu)}
\end{aligned}$$

whenever $\rho - \alpha < 0$. □

1.3.2 Proof of (Theorem 1.3.2 $\Longrightarrow$ Corollary 1.3.4)

Let $s > \frac{1}{4}$ and $f \in H^s(\mathbb{R})$. If we suppose $\dim_c(\mathfrak{D}(f,\gamma)) > \max\{1-2s, \frac{1-2s}{m\kappa}\} \geq 0$, then one would find $0 < \alpha < 1$ satisfying $\dim_c(\mathfrak{D}(f,\gamma)) > \alpha > \max\{1-2s, \frac{1-2s}{m\kappa}\} \geq 0$. Here, note that the second inequality is equivalent to $s > \max\{\frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\}$. Hence, by Lemma 1.3.6 there would exist an α-dimensional measure μ such that $\mu(\mathfrak{D}(f,\gamma)) > 0$, which contradicts Theorem 1.3.2, and we must have $\dim_c(\mathfrak{D}(f,\gamma)) \leq \max\{1-2s, \frac{1-2s}{m\kappa}\}$. □

1.3.3 Proof of Theorem 1.3.1

Let

$$s_* = \min\left\{\frac{1}{4}, \frac{\alpha}{2}, \frac{m\alpha\kappa}{2}\right\}.$$

By following the standard steps via Littlewood–Paley decomposition, it is enough to show the following proposition. (For the details, for instance, see [132].)

Proposition 1.3.8. *Let $\varepsilon > 0$. Then, there exists a constant C_ε such that*

$$\left\| \sup_{t\in I} |S_t f(\gamma(\cdot,t))| \right\|_{L^2(I,\mathrm{d}\mu)} \leq C_\varepsilon \lambda^{\frac{1}{2}-s_*+\varepsilon}\|f\|_{L^2} \tag{1.20}$$

holds for all $\lambda \geq 1$ and $f \in L^2(\mathbb{R})$ whose Fourier support is contained in $\{\xi \in \mathbb{R} : \frac{\lambda}{2} \leq |\xi| \leq 2\lambda\}$.

Proof of Proposition 1.3.8. Let

$$Tf(x,t) = \chi(x,t)\int_{\mathbb{R}} e^{i(\gamma(x,t)\xi + t|\xi|^m)} f(\xi)\psi(\tfrac{\xi}{\lambda})\,\mathrm{d}\xi,$$

where $\chi = \chi_{I\times I}$ and $\psi \in C_0^\infty((-2,-\frac{1}{2}) \cup (\frac{1}{2},2))$. Then, by Plancherel's theorem, (1.20) follows from

$$\|Tf\|^2_{L^2_x(\mathrm{d}\mu)L^\infty_t} \lesssim \lambda^{1-2s_*+\varepsilon}\|f\|^2_{L^2}. \tag{1.21}$$

By duality, (1.21) is equivalent to

$$\|T^*F\|^2_{L^2} \lesssim \lambda^{1-2s_*+\varepsilon}\|F\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}, \tag{1.22}$$

where

$$T^*F(\xi) = \psi(\tfrac{\xi}{\lambda}) \iint \chi(x',t') e^{-i(\gamma(x',t')\xi + t'|\xi|^m)} F(x',t')\, \mathrm{d}\mu(x')\mathrm{d}t'.$$

Then,

$$\begin{aligned}
\|T^*F\|_{L^2}^2 &= \int \psi(\tfrac{\xi}{\lambda})^2 \iint \iint \chi(x,t)\chi(x',t') \\
&\quad \times e^{i((\gamma(x,t)-\gamma(x',t'))\xi + (t-t')|\xi|^m)} \overline{F}(x,t) F(x',t')\, \mathrm{d}\mu(x)\mathrm{d}t\mathrm{d}\mu(x')\mathrm{d}t'\mathrm{d}\xi \\
&= \int_W \int_{W'} \chi(w)\chi(w')\overline{F}(w)F(w')K_\lambda(w,w')\, \mathrm{d}_\mu w \mathrm{d}_\mu w' \\
&= \sum_{\ell=1}^{3} \iint_{V_\ell} \chi(w)\chi(w')\overline{F}(w)F(w')K_\lambda(w,w')\, \mathrm{d}_\mu w \mathrm{d}_\mu w' \\
&= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}$$

Here, we denote $W = I \times I$, $w = (x,t) \in W$, $w' \in (x',t') \in W$ and $\mathrm{d}_\mu w = \mathrm{d}\mu(x)\mathrm{d}t$. Also,

$$\begin{aligned}
K_\lambda(w,w') &= \int_{\mathbb{R}} e^{i\phi(\xi,w,w')} \psi(\tfrac{\xi}{\lambda})^2\, \mathrm{d}\xi \\
&= \lambda \int_{\mathbb{R}} e^{i\phi(\lambda\xi,w,w')} \psi(\xi)^2\, \mathrm{d}\xi, \\
\phi(\xi,w,w') &= (\gamma(x,t)-\gamma(x',t'))\xi + (t-t')|\xi|^m
\end{aligned}$$

and

$$\begin{cases}
V_1 = \{(w,w') \in W \times W : |x-x'| \le 2\lambda^{-\frac{2s_*}{\alpha}}\}, \\
V_2 = \{(w,w') \in W \times W : |x-x'| > 2\lambda^{-\frac{2s_*}{\alpha}} \text{ and } \frac{1}{C_2}|x-x'| \le 2C_1|t-t'|^\kappa\}, \\
V_3 = \{(w,w') \in W \times W : |x-x'| > 2\lambda^{-\frac{2s_*}{\alpha}} \text{ and } \frac{1}{C_2}|x-x'| > 2C_1|t-t'|^\kappa\}.
\end{cases}$$

Then, (1.22) follows from

$$\mathcal{I}_\ell \lesssim \lambda^{1-2s_*+\varepsilon} \|F\|_{L_x^2(\mathrm{d}\mu)L_t^1}^2$$

for each $\ell = 1,2,3$.

The term $\mathcal{I}_1$

By using the trivial estimate

$$|K_\lambda(w,w')| \lesssim \lambda \tag{1.23}$$

and Lemma 1.3.7, we obtain

$$\mathcal{I}_1 \lesssim \lambda^{1-2s_*} \|F\|_{L_x^2(\mathrm{d}\mu)L_t^1}^2.$$

The term $\mathcal{I}_2$

In this case, observe that

$$\begin{aligned}
\left| \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \phi(\lambda\xi,w,w') \right| &\gtrsim \lambda^m |t-t'||\xi|^{m-2} \\
&\gtrsim \lambda^m |x-x'|^{\frac{1}{\kappa}} \\
&\gtrsim \lambda^m \lambda^{-\frac{2s_*}{\alpha\kappa}} \\
&\ge 1
\end{aligned}$$

since $\frac{2s_*}{\alpha\kappa} = \min\{\frac{1}{2\alpha\kappa}, \frac{1}{\kappa}, m\} \leq m$. Then, by Lemma 1.2.3 for arbitrary small $\varepsilon > 0$

$$\begin{aligned}|K_\lambda(w,w')| &\lesssim \lambda(\lambda^m|x-x'|^{\frac{1}{\kappa}})^{-\frac{1}{2}} \\ &\lesssim \lambda(\lambda^m|x-x'|^{\frac{1}{\kappa}})^{-\frac{2s_*}{m}} \\ &\sim \lambda^{1-2s_*}|x-x'|^{-\frac{2s_*}{m\kappa}} \\ &\lesssim \lambda^{1-2s_*+\varepsilon}|x-x'|^{-\frac{2s_*}{m\kappa}+\varepsilon}\end{aligned}$$

since $\frac{2s_*}{m} = \min\{\frac{1}{2m}, \frac{\alpha}{m}, \alpha\kappa\} < \frac{1}{2}$ and our separation assumption again. Therefore, applying Lemma 1.3.7 with $\rho = \frac{2s_*}{m\kappa} - \varepsilon = \min\{\frac{1}{2m\kappa}, \frac{\alpha}{m\kappa}, \alpha\} - \varepsilon < \alpha$, it follows that

$$\mathcal{I}_2 \lesssim \lambda^{1-2s_*+\varepsilon}\|F\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}.$$

The term $\mathcal{I}_3$

It remains to consider $\mathcal{I}_3$. First note that we have

$$|\gamma(w) - \gamma(w')| \geq \frac{1}{2C_2}|x-x'| \tag{1.24}$$

for $(w,w') \in V_2$ by using (1.3) and (1.4). Next, we split K_λ into $\mathcal{K}_1$ and $\mathcal{K}_2$ as follows.

$$\begin{aligned}K_\lambda(w,w') &= \lambda\int_{U_1} e^{i\phi(\lambda\xi,w,w')}\psi(\xi)^2\,\mathrm{d}\xi + \lambda\int_{U_2} e^{i\phi(\lambda\xi,w,w')}\psi(\xi)^2\,\mathrm{d}\xi \\ &=: \mathcal{K}_1 + \mathcal{K}_2,\end{aligned}$$

where

$$\begin{cases}U_1 = \{\xi \in \operatorname{supp}\psi : \frac{1}{C_2}|x-x'| > 4m\lambda^{m-1}|t-t'||\xi|^{m-1}\}, \\ U_2 = \{\xi \in \operatorname{supp}\psi : \frac{1}{C_2}|x-x'| \leq 4m\lambda^{m-1}|t-t'||\xi|^{m-1}\}.\end{cases}$$

For $\mathcal{K}_1$, we use (1.24) in order to estimate the phase

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\lambda\xi,w,w')\right| &\geq \lambda|\gamma(w)-\gamma(w')| - m\lambda^m|t-t'||\xi|^{m-1} \\ &\geq \frac{1}{2C_2}\lambda|x-x'| - m\lambda^m|t-t'||\xi|^{m-1} \\ &> \frac{1}{4C_2}\lambda|x-x'| \\ &\gtrsim \lambda^{1-\frac{2s_*}{\alpha}} \\ &\geq 1\end{aligned}$$

since $\frac{2s_*}{\alpha} = \min\{\frac{1}{2\alpha}, 1, m\kappa\} \leq 1$. Here, note that the interval U_1 consists of at most two intervals since $\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\lambda\xi,w,w')$ is monotone on each interval $(-\infty,-1]$ and $[1,\infty)$. Thus, Lemma 1.2.3 gives that

$$\mathcal{K}_1 \lesssim \lambda(\lambda|x-x'|)^{-1} \lesssim \lambda(\lambda|x-x'|)^{-\min\{\frac{1}{2},\alpha\}}.$$

On the other hand, for $\mathcal{K}_2$,

$$\begin{aligned}\left|\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\lambda\xi,w,w')\right| &\gtrsim \lambda^m|t-t'||\xi|^{m-2} \\ &\gtrsim \lambda|x-x'|\end{aligned}$$

so that we are allowed to apply Lemma 1.2.3 to obtain

$$\begin{aligned}\mathcal{K}_2 &\lesssim \lambda(\lambda|x-x'|)^{-\frac{1}{2}}\\ &\lesssim \lambda(\lambda|x-x'|)^{-\min\{\frac{1}{2},\alpha\}}.\end{aligned}$$

Hence, for $(w,w')\in V_2$ we have the following kernel estimate

$$\begin{aligned}|K_\lambda(w,w')| &\lesssim \lambda^{1-\min\{\frac{1}{2},\alpha\}}|x-x'|^{-\min\{\frac{1}{2},\alpha\}}\\ &\lesssim \lambda^{1-\min\{\frac{1}{2},\alpha\}+\varepsilon}|x-x'|^{-\min\{\frac{1}{2},\alpha\}+\varepsilon}.\end{aligned}$$

Here, we used the separation assumption, $|x-x'|\gtrsim \lambda^{-\frac{2s_*}{\alpha}}$. By Lemma 1.3.7 with $\rho=\min\{\frac{1}{2},\alpha\}-\varepsilon<\alpha$ we conclude that

$$\begin{aligned}\mathcal{I}_3 &\lesssim \lambda^{1-\min\{\frac{1}{2},\alpha\}+\varepsilon}\|F\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}\\ &\lesssim \lambda^{1-2s_*+\varepsilon}\|F\|^2_{L^2_x(\mathrm{d}\mu)L^1_t}.\end{aligned}$$

□

1.3.4 The necessary conditions regarding Theorem 1.3.3

In this section, we present $s\ge \max\left\{\frac{1}{4},\frac{1-\alpha}{2},\frac{1-m\alpha\kappa}{2}\right\}$ is necessary for Theorem 1.3.3, otherwise there exist $\gamma\in\Gamma(\kappa)$ and α-dimensional measure μ such that (1.16) fails. Throughout the section, we shall let $\lambda\ge 1$, $\gamma(x,t)=x-t^\kappa$, $\mu(x)=|x|^{-1+\alpha}\,\mathrm{d}x$ and ψ_0 be a smooth radial bump function whose support is in a small neighborhood of the origin. Also, we fix $m>1$ and $0<\kappa\le 1$, and we assume that the maximal estimate (1.16) holds.

The necessity of $s\ge\frac{1-\alpha}{2}$

In this case, we will follow the idea in [48]. Let

$$\widehat{f_1}(\xi)=\psi_0(\lambda^{-\frac{1}{m}}\xi).$$

With this initial data,

$$\begin{aligned}|S_tf_1(\gamma(x,t))| &\sim \left|\int e^{i((x-t^\kappa)\xi+t|\xi|^m)}\widehat{f_1}(\xi)\,\mathrm{d}\xi\right|\\ &= \lambda^{\frac{1}{m}}\left|\int e^{i\phi_1(\eta,x,t)}\psi_0(\eta)\,\mathrm{d}\eta\right|,\end{aligned}$$

where

$$\phi_1(\eta,x,t)=\lambda^{\frac{1}{m}}(x-t^\kappa)\eta+\lambda t|\eta|^m.$$

For $x\in(0,\frac{1}{100}\lambda^{-\frac{1}{m}})$ and $|t|<\frac{1}{100}\lambda^{-1}$, we have

$$|\phi_1(\eta,x,t)|\le\frac{1}{2}$$

so that

$$\begin{aligned}|S_tf_1(\gamma(x,t))| &\gtrsim \lambda^{\frac{1}{m}}\left|\int(\cos\phi_1(\eta,x,t))\psi_0(\eta)\,\mathrm{d}\eta\right|\\ &\gtrsim \lambda^{\frac{1}{m}}\chi_{(0,\frac{1}{100}\lambda^{-\frac{1}{m}})\times(0,\frac{1}{100}\lambda^{-1})}(x,t).\end{aligned}$$

Hence,

$$\left\|\sup_{t\in I}|S_t f_1(\gamma(\cdot,t))|\right\|_{L^2(I,\mathrm{d}\mu)} \geq \left\|\sup_{t\in(0,\frac{1}{100}\lambda^{-1})}|S_t f(\gamma(\cdot,t))|\right\|_{L^2((0,\frac{1}{100}\lambda^{-\frac{1}{m}}),\mathrm{d}\mu)}$$
$$\gtrsim \lambda^{\frac{1}{m}}\lambda^{-\frac{\alpha}{2m}}.$$

On the other hand,

$$\|f_1\|_{H^s} \sim \left(\int(1+|\xi|^2)^s|\psi_0(\lambda^{-\frac{1}{m}}\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}$$
$$\lesssim \lambda^{\frac{s}{m}}\lambda^{\frac{1}{2m}}.$$

Therefore, combining the above calculations, we obtain

$$\lambda^{\frac{1}{m}}\lambda^{-\frac{\alpha}{2m}} \lesssim \lambda^{\frac{s}{m}}\lambda^{\frac{1}{2m}}.$$

As letting $\lambda\to\infty$, it is necessary that

$$\frac{1}{m}-\frac{\alpha}{2m} \leq \frac{s}{m}+\frac{1}{2m},$$

which is

$$s \geq \frac{1-\alpha}{2}.$$

The necessity of $s \geq \frac{1-m\alpha\kappa}{2}$

Here, choose the same initial data f_1 as above. For $x\in(0,\frac{1}{100}\lambda^{-\kappa})$ and $t=t(x)=x^{\frac{1}{\kappa}}$, one can estimate

$$|\phi_1(\eta,x,t)| \leq \frac{1}{2}.$$

Then, following a similar argument as above, we have

$$\lambda^{\frac{1}{m}}\lambda^{-\frac{\alpha\kappa}{2}} \lesssim \lambda^{\frac{s}{m}}\lambda^{\frac{1}{2m}}.$$

As letting $\lambda\to\infty$, it is necessary that

$$\frac{1}{m}-\frac{\alpha\kappa}{2} \leq \frac{s}{m}+\frac{1}{2m},$$

which clearly gives

$$s \geq \frac{1-m\alpha\kappa}{2}.$$

The necessity of $s \geq \frac{1}{4}$

In this case, we will refer to the idea in [134] (see page 712). Let

$$\widehat{f_2}(\xi) = \lambda^{-1}\psi_0(\lambda^{-1}\xi+\lambda).$$

Then, by the change of variables $-\eta = \lambda^{-1}\xi + \lambda$,

$$\begin{aligned}|S_t f_2(\gamma(x,t))| &\sim \left| \int e^{i((x-t^\kappa)\xi + t|\xi|^m)} \lambda^{-1}\psi_0(\lambda^{-1}\xi + \lambda)\,\mathrm{d}\xi \right| \\ &= \left| \int e^{i\phi_2(\eta,x,t)} \psi_0(-\eta)\,\mathrm{d}\eta \right|,\end{aligned}$$

where

$$\phi_2(\eta,x,t) = -(x-t^\kappa)\lambda\eta + \lambda^m t|\lambda+\eta|^m.$$

By a Taylor expansion,

$$\begin{aligned}(\lambda+\eta)^m &= \lambda^m(1+\lambda^{-1}\eta)^m \\ &= \lambda^m \left(1 + m\lambda^{-1}\eta + \frac{m(m-1)}{2}\lambda^{-2}\eta^2 + O(\lambda^{-3}|\eta|^3)\right) \\ &= \lambda^m + m\lambda^{-(1-m)}\eta + \frac{m(m-1)}{2}\lambda^{-(2-m)}\eta^2 + O(\lambda^{-(3-m)}|\eta|^3),\end{aligned}$$

and it follows that

$$\begin{aligned}\phi_2(\eta,x,t) &= -\lambda x\eta + \lambda t^\kappa \eta + \lambda^{2m} t + m\lambda^{-(1-2m)} t\eta \\ &\quad + \frac{m(m-1)}{2}\lambda^{-(2-2m)} t\eta^2 + O(\lambda^{-(3-2m)} t|\eta|^3) \\ &= \lambda^{2m} t + \lambda(-x + t^\kappa + m\lambda^{-(2-2m)} t)\eta \\ &\quad + \frac{m(m-1)}{2}\lambda^{-(2-2m)} t\eta^2 + O(\lambda^{-(3-2m)} t|\eta|^3).\end{aligned}$$

For $x \in (0, \frac{1}{100})$, we can choose $t(x)$ such that $x = t(x)^\kappa + m\lambda^{-(2-2m)} t(x)$. In fact, if we consider the function $\tau(t) = t^\kappa + m\lambda^{-(2-2m)} t$, then $\tau : [0,\infty) \to [0,\infty)$ is a strictly increasing bijection and

$$0 = \tau^{-1}(0) < \tau^{-1}(x) = t(x) < \tau^{-1}(\tfrac{1}{100}) < \frac{\lambda^{2-2m}}{100m}.$$

Therefore, for such choice of $(x, t(x))$, it follows that

$$|\phi_2(\eta, x, t(x)) - \lambda^{2m} t(x)| \lesssim 0 + \frac{1}{100} + O(\lambda^{-1}) \le \frac{1}{2},$$

which implies that

$$\begin{aligned}|S_t f_2(\gamma(x,t(x)))| &\sim \left| \int \cos(\phi_2(\eta,x,t(x)) - \lambda^{2m} t(x))\psi_0(-\eta)\,\mathrm{d}\eta \right| \\ &\gtrsim \chi_{(0,\frac{1}{100})}(x).\end{aligned}$$

Hence,

$$\left\| \sup_{t\in I} |S_t f_2(\gamma(\cdot,t))| \right\|_{L^2(I,\mathrm{d}\mu)} \gtrsim 1.$$

On the other hand,

$$\begin{aligned}\|f_2\|_{H^s} &= \left(\int (1+|\xi|^2)^s |\lambda^{-1}\psi_0(\lambda^{-1}\xi + \lambda)|^2\,\mathrm{d}\xi \right)^{\frac{1}{2}} \\ &\lesssim \lambda^{2s}\lambda^{-\frac{1}{2}}.\end{aligned}$$

Therefore, combining the calculations above implies that

$$1 \lesssim \lambda^{2s} \lambda^{-\frac{1}{2}}.$$

As $\lambda \to \infty$, it is necessary that

$$s \geq \frac{1}{4}.$$

This ends the proof that $s \geq \max\left\{\frac{1}{4}, \frac{1-\alpha}{2}, \frac{1-m\alpha\kappa}{2}\right\}$ is necessary for (1.16) to hold. $\square$

Chapter 2

Sharp bilinear estimates for the Klein–Gordon equation

2.1 Introduction

This chapter is based on work with Jayson Cunanan.

In this chapter, we slightly shift our interest from the Schrödinger equation to the Klein–Gordon equation, which is often considered to be a "hybrid object" of the Schrödinger and the wave equations. Here, we study the null-form type estimates for the Klein–Gordon equation with the optimal constant. The Klein–Gordon equation is given by

$$\begin{cases} \partial_{tt} - \Delta_x u + u = 0, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \end{cases}$$

Formally, for functions f and g, if we let f_+ and f_- be given in terms of the initial conditions by

$$u_0 = f_+ + f_-, \quad u_1 = i\langle D\rangle(f_+ - f_-)$$

then one can write $u = u_+ + u_-$, where

$$\begin{aligned} u_\pm(t,x) &= e^{\pm it\langle D\rangle} f_\pm(x) \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t\langle\xi\rangle)} \widehat{f_\pm}(\xi)\, \mathrm{d}\xi. \end{aligned}$$

(One may be able to see that the Klein–Gordon equation is connected to the wave and Schrödinger equations at this point since $\langle\xi\rangle \sim |\xi|$ for $|\xi| \gg 1$ and $\langle\xi\rangle \sim 1 + \frac{1}{2}|\xi|^2$ for $|\xi| \ll 1$, respectively.)

Strichartz estimates are a family of the inequalities that have a crucial role in the theory of nonlinear dipersive PDE. For the Klein–Gordon equation the Strichartz estimates have the form

$$\|e^{it\langle D\rangle} u_0\|_{L^q(\mathbb{R}^{d+1})} \leq C\|u_0\|_{H^\alpha(\mathbb{R}^d)}$$

for the valid triples (d,q,α), called KG-admissible satisfying

$$\frac{1}{2} \leq \alpha \leq \frac{d}{2}, \quad \frac{2d+4}{d} \leq q \leq \frac{2d+2}{d-2\alpha}.$$

It is often useful to consider a generalized version

$$\|e^{it\phi_s(D)}u_0\|_{L^q(\mathbb{R}^{d+1})} \le C\|u_0\|_{H^\alpha(\mathbb{R}^d)} \tag{2.1}$$

for KG-admissible triples, where we recall that $\phi_s(r) = (s^2 + r^2)^{\frac{1}{2}}$. This type of estimates as well as those variations such as that with the mixed-norm and the null-form has been a major subject in the study of nonlinear dispersive equations. One advantage of the perspective of (2.1) is that its special case when $s = 0$ coincides with the situation for the wave equation. Although our focus here is rather the Klein–Gordon propagator, we shall first take a look at the wave case, $s = 0$, and see a bit of history.

Let $s = 0$ and consider

$$\|e^{itD}f\|_{L^q_t L^r_x} \le C\|f\|_{\dot{H}^{\frac{d}{2}-\frac{d}{r}-\frac{1}{q}}}$$

for $q \ge 2$ and $(d, q, r) \ne (3, 2, \infty)$. This estimate except the endpoint case was proved by Jean Ginibre and Giorgio Velo [64] and later the endpoint case by Markus Keel and Terence Tao [87]. The Sobolev exponent on the right-hand side is due to homogeneity. The reader may also note that the order of the mixed norm on the left-hand side is reversed from the maximal inequality we have considered in Chapter 1; taking the integral in space first and then time. In certain situations, the optimal constant for (2.1) is known: Let $q = r = 4$. In 2006, pioneering work of Damiano Foschi [61] established the optimality of the constant $C = \mathbf{F}(0,3)^{\frac{1}{4}}$. Here, for general $d \ge 2$ and $\beta \ge \max\{\frac{1-d}{4}, \frac{2-d}{2}\}$ we note,

$$\mathbf{F}(\beta, d) := 2^{d-3+4\beta}\pi^{-\frac{d}{2}}\frac{\Gamma(\frac{d}{2})\Gamma(\frac{d-1}{2}+2\beta)}{(d-2+2\beta)\Gamma(\frac{3d-5}{2}+2\beta)}.$$

Neal Bez and Keith Rogers built on Foschi's work and proved $C = \mathbf{F}(0,5)^{\frac{1}{4}}$ is optimal when $d = 5$ via the bilinear estimate

$$\|e^{itD}f\overline{e^{itD}g}\|^2_{L^2(\mathbb{R}^{d+1})} \le \mathbf{W}(0,d)\int_{(\mathbb{R}^{d+1})^2}|\widehat{f}(\eta_1)|^2|\widehat{g}(\eta_2)|^2|\eta_1||\eta_2|K_0^{\mathrm{BR}}(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2, \tag{2.2}$$

where

$$\mathbf{W}(\beta, d) := 2^{\frac{-5d+1}{2}+2\beta}\pi^{\frac{-5d+1}{2}}\frac{\Gamma(\frac{d-1}{2}+2\beta)}{\Gamma(d-1+2\beta)},$$

and

$$K_\beta^{\mathrm{BR}}(\eta_1,\eta_2) = (|\eta_1||\eta_2| - \eta_1\cdot\eta_2)^{\frac{d-3}{2}+\beta}.$$

The optimality of $\mathbf{F}(0,4)^{\frac{1}{4}}$ when $d = 4$ in (2.1) in was proved by Bez and Chris Jeavons by using polar coordinates and techniques from the theory of spherical harmonics in addition to using (2.2). Recently, in 2016, Bez, Jeavons and Tohru Ozawa established the bilinear estimates

$$\||\Box|^\beta(e^{itD}f\overline{e^{itD}g})\|^2_{L^2(\mathbb{R}^{d+1})} \le \mathbf{W}(\beta,d)\int_{(\mathbb{R}^d)^2}|\widehat{f}(\eta_1)|^2|\widehat{g}(\eta_2)|^2|\eta_1||\eta_2|K_\beta^{\mathrm{BR}}(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2, \tag{2.3}$$

from which it quickly follows (via polar coordinates) that $\mathbf{F}(\beta,d)^{\frac{1}{2}}$ is the optimal constant in the corresponding estimate

$$\||\Box|^\beta(e^{itD}f\overline{e^{itD}g})\|_{L^2(\mathbb{R}^{d+1})} \le \mathbf{F}(\beta,d)^{\frac{1}{2}}\|f\|_{\dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)}\|g\|_{\dot{H}^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)} \tag{2.4}$$

whenever $d \geq 2$ and $\beta > \beta_d := \max\{\frac{1-d}{4}, \frac{2-d}{2}\}$ if we restrict to radially symmetric data f and g. The estimate (2.4) was motivated by the null-form type estimate

$$\|D^{\beta_0} D_-^{\beta_-} D_+^{\beta_+}(e^{itD} f \overline{e^{itD} g})\|_{L^2(\mathbb{R}^{d+1})} \leq C\|f\|_{\dot{H}^{\alpha_1}} \|g\|_{\dot{H}^{\alpha_2}}, \tag{2.5}$$

in the case of $(\beta_0, \beta_-, \beta_+, \alpha_-, \alpha_+) = (\frac{2-d}{2}, 0, 0, \frac{1}{4}, \frac{1}{4})$ for the propagator e^{itD} associated with the wave equation. The estimate combining with the corresponding $(++)$ case (while (2.5) is $(+-)$ case),

$$\|D^{\beta_0} D_-^{\beta_-} D_+^{\beta_+}(e^{itD} f e^{itD} g)\|_{L^2(\mathbb{R}^{d+1})} \leq C\|f\|_{\dot{H}^{\alpha_1}} \|g\|_{\dot{H}^{\alpha_2}} \tag{2.6}$$

has found important applications in study of nonlinear wave equations. This type of estimate has been studied back in work of Michael Beals [11] and work by Sergiu Klainerman and Matei Machedon [89, 90, 91]. A complete characterization of the admissible exponents $(\beta_0, \beta_-, \beta_+, \alpha_-, \alpha_+)$ for (2.5) and (2.6) were eventually obtained by Foschi and Klainerman [62]. Such a characterization when the $L^2_{t,x}$ norm on the left-hand side of (2.5) is replaced by $L^q_t L^r_x$ has also drawn great attention. Using bilinear Fourier restriction techniques, Jean Bourgain [34] made a breakthrough contribution, then Tom Wolff [146] and Tao [136] (in the endpoint case; see also Sanghyuk Lee [97] and Daniel Tataru [142]) completed the diagonal case $q = r$. For the non-diagonal case we refer readers to [101] due to Lee and Ana Vargas for a complete characterization when $d \geq 4$ and partial results when $d = 2$, 3. Soon later Lee, Rogers and Vargas [99] completed $d = 3$, but a gap between necessary and sufficient conditions still remains when $d = 2$.

Before moving onto the Klein–Gordon case and introducing our main results, we shall make small remarks on the analogous results for the Schrödinger equation. The study pursuing the optimal constants for the Schrödinger equation can be traced back to the work of Ozawa and Yoshio Tsutsumi in 1998 [124]. This result has been extended in a natural ways by Emanuel Carneiro [38] and Fabrice Planchon and Luis Vega [125]. The unification of those results can be found in a recent paper by the aforementioned authors Bez, Jeavons in collaboration with Jonathan Bennett and Nikolaos Pattakos [20].

In addition to the above, the related literature on sharp Strichartz estimates is large; the author would like the interested reader to consult with the survey article by Foschi and Diogo Oliveira e Silva [63]. From a methodological view point, specifically for this thesis, work of Bennett, Bez in collaboration with Anthony Carbery and Dirk Hundertmark [18] and [20] applied the heat-flow monotonicity method which we will employ in a different study in Chapter 3.

Finally, in the context of the Klein–Gordon equation, René Quilodrán [126] appropriately developed Foschi's argument in [61] when $\alpha = \frac{1}{2}$ and proved the sharp Strichartz estimate (2.1) for $(d, q) = (2, 4)$, $(2, 6)$, $(3, 4)$, the endpoint cases of the KG-admissible range when $\alpha = \frac{1}{2}$; when $(d, q) = (3, 4)$, in particular, the optimal constant coincides with $\mathbf{F}(0, 3)^{\frac{1}{4}}$. In [126], he also proved that there is no extremiser which attains (2.1) with the optimal constant for $(d, q) = (2, 4)$, $(2, 6)$, $(3, 4)$. Later, Carneiro, Oliveira e Silva and Mateus Sousa [39] further revealed the nature of (2.1) for $d = 1$, 2 by answering the questions raised in [126]; in particular, they found the best constant in (2.1) for $(d, q, \alpha) = (1, 6, \frac{1}{2})$ and absence of the extremisers. Meanwhile, they also established there exist extremisers in the non-endpoint cases in low dimensions $d = 1$, 2. A subsequent study by the same authors in collaboration with Betsy Stovall [40] proved the analogous results in the non-endpoint cases for higher dimensions $d \geq 3$ by using some tools from bilinear restriction theory.

In [84], Jeavons obtained the following refined Strichartz estimate in five spatial dimensions

$$\|e^{it\phi_s(D)}f\|_{L^4(\mathbb{R}^{5+1})} \leq \mathbf{F}(0,5)^{\frac{1}{4}} \left(\|\phi_s(D)f\|_{L^2(\mathbb{R}^5)}^4 - s^2\|\phi_s(D)^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^5)}^4\right)^{\frac{1}{4}}, \qquad (2.7)$$

which recovers the inequality (2.4) when $(\beta, d) = (0, 5)$ in the limit $s \to 0$. Moreover, by simply omitting the negative second term, it follows that

$$\|e^{it\phi_1(D)}f\|_{L^4(\mathbb{R}^{5+1})} \leq \mathbf{F}(0,5)^{\frac{1}{4}}\|f\|_{H^1(\mathbb{R}^5)},$$

where the constant $\mathbf{F}(0,5)^{\frac{1}{4}} = (24\pi^2)^{-\frac{1}{4}}$ is still sharp.

As part of the study of sharp bilinear estimates for the Fourier extension operator and inspired by work of Ozawa and Tsutsumi [124], David Beltran and Vega [14] very recently presented the following sharp estimate associated to the Klein–Gordon propagator

$$\begin{aligned} &\|D^{\frac{2-d}{2}}(e^{it\phi_s(D)}f\overline{e^{it\phi_s(D)}g})\|_{L^2(\mathbb{R}^d\times\mathbb{R})}^2 \qquad (2.8)\\ &\leq (2\pi)^{1-3d}\int_{(\mathbb{R}^d)^2}|\widehat{f}(\eta_1)|^2|g(\eta_2)|^2\phi_s(|\eta_1|)\phi_s(|\eta_2|)K^{\mathrm{BV}}(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2, \end{aligned}$$

where

$$K^{\mathrm{BV}}(\eta_1,\eta_2) = \int_{\mathbb{S}^{d-1}} \frac{\phi_s(|\eta_1|)+\phi_s(|\eta_2|)}{(\phi_s(|\eta_1|)+\phi_s(|\eta_2|))^2 - ((\eta_1+\eta_2)\cdot\theta)^2}\,\mathrm{d}\sigma(\theta).$$

The estimate (2.8) has some interesting connections to well-known results. For example, as we shall see in more detail later, (2.8) leads null-form type estimates by appropriately estimating the kernel. In particular, when $d = 2$ the Strichartz estimate

$$\|e^{it\phi_1(D)}f\|_{L^4(\mathbb{R}^{2+1})} \leq 2^{-\frac{1}{4}}\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \qquad (2.9)$$

with the optimal constant quickly follows from (2.8). We also note that the approach taken by Beltran–Vega [14], which in turn built on work of Planchon–Vega [125] rested on interplay with geometric operators such as the Radon transform or, more generally, the k-plane transform. For related work in this context of interaction with geometrically-defined operators, we also refer the reader to work of Bennett, Bez, Taryn C. Flock, Susana Gutiérrez and Marina Iliopoulou [21] and Bennett and Shohei Nakamura [26].

In this chapter, we establish the following new bilinear estimates for the Klein–Gordon propagator. Let

$$\mathcal{K}_a^b(\eta_1,\eta_2) := \frac{\left(\phi_s(|\eta_1|)\phi_s(|\eta_2|) - \eta_1\cdot\eta_2 - s^2\right)^b}{(\phi_s(|\eta_1|)\phi_s(|\eta_2|) - \eta_1\cdot\eta_2 + s^2)^a}.$$

Theorem 2.1.1. *For $d \geq 2$ and $\beta > \frac{1-d}{4}$, we have the estimate*

$$\begin{aligned} &\||\Box - (2s)^2|^\beta(e^{it\phi_s(D)}f\overline{e^{it\phi_s(D)}g})\|_{L^2(\mathbb{R}^{d+1})}^2 \qquad (2.10)\\ &\leq \mathbf{KG}(\beta,d)\int_{\mathbb{R}^{2d}}|\widehat{f}(\eta_1)|^2|\widehat{g}(\eta_2)|^2\phi_s(|\eta_1|)\phi_s(|\eta_2|)\mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2, \end{aligned}$$

with the optimal constant

$$\mathbf{KG}(\beta,d) := 2^{\frac{-5d+1}{2}+2\beta}\pi^{\frac{-5d+1}{2}}\frac{\Gamma(\frac{d-1}{2}+2\beta)}{\Gamma(d-1+2\beta)}.$$

In the case when $s \to 0$, certain sharp bilinear estimates for solutions to the wave equation with the operator $|\Box|^\beta$ has been deeply studied by Bez–Jeavons–Ozawa [30]. One may note that, when $d = 2$, a slightly larger range of β is valid in Theorem 2.1.1 than one for the corresponding result (2.3) for the wave case in [30]. In order to prove Theorem 2.1.1, we employ their argument and adapt it into the context of Klein–Gordon equation. As a consequence of Theorem 2.1.1, we will generate null-form type estimates of the form

$$\| |\Box - (2s)^2|^\beta |e^{it\phi_s(D)} f|^2 \|_{L^2(\mathbb{R}^{d+1})} \leq C \|\phi_s(D)^\alpha f\|^2_{L^2(\mathbb{R}^d)} \tag{2.11}$$

for certain pairs (α, β) with the optimal constant.

2.2 Some connections to recent results and corollaries

2.2.1 Wave regime

For $d \geq 4$, the kernel K^{BV} can be estimated[1] as

$$\begin{aligned} K^{\mathrm{BV}}_d(\eta_1, \eta_2) &\leq \frac{|\mathbb{S}^{d-1}|}{\phi_s(|\eta_1|) + \phi_s(|\eta_2|)} \int_{-1}^1 \left(1 - \left|\frac{\eta_1 + \eta_2}{\phi_s(|\eta_1|) + \phi_s(|\eta_2|)}\right|^2 \lambda^2\right)^{-1} (1 - \lambda^2)^{\frac{d-3}{2}} \, \mathrm{d}\lambda \\ &\leq \frac{C}{\phi_s(|\eta_1|) + \phi_s(|\eta_2|)} \end{aligned}$$

for some absolute constant C since $|\eta_1 + \eta_2| \leq \phi_s(|\eta_1|) + \phi_s(|\eta_2|)$. Then, it follows from the arithmetic-geometric mean that

$$\phi_s(|\eta_1|)\phi_s(|\eta_2|) K^{\mathrm{BV}}_d(\eta_1, \eta_2) \leq C \phi_s(|\eta_1|)^{\frac{1}{2}} \phi_s(|\eta_2|)^{\frac{1}{2}},$$

and hence the null-form type estimate

$$\|D^{\frac{2-d}{2}} (e^{it\phi_s(D)} f \overline{e^{it\phi_s(D)} g})\|_{L^2(\mathbb{R}^d \times \mathbb{R})} \leq C \|\phi_s(D)^{\frac{1}{4}} f\|_{L^2(\mathbb{R}^d)} \|\phi_s(D)^{\frac{1}{4}} g\|_{L^2(\mathbb{R}^d)} \tag{2.12}$$

holds. When $s \to 0$, the estimate (2.12) yields (2.5) in the case of $(\beta_0, \beta_-, \beta_+, \alpha_-, \alpha_+) = (\frac{2-d}{2}, 0, 0, \frac{1}{4}, \frac{1}{4})$ for the propagator e^{itD} associated with the wave equation.

As a means of comparing our bilinear estimate (2.10) with (2.8), we note that using the trivial estimate

$$\frac{\phi_s(|\eta_1|)\phi_s(|\eta_2|) - \eta_1 \cdot \eta_2 - s^2}{\phi_s(|\eta_1|)\phi_s(|\eta_2|) - \eta_1 \cdot \eta_2 + s^2} \leq 1, \tag{2.13}$$

we estimate our kernel as

$$\mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2} + 2\beta}(\eta_1, \eta_2) \leq \mathcal{K}_0^{\frac{d-3}{2} + 2\beta}(\eta_1, \eta_2). \tag{2.14}$$

For $\beta \geq \frac{3-d}{4}$, it follows that

$$\begin{aligned} &\| |\Box - (2s)^2|^\beta (e^{it\phi_s(D)} f \overline{e^{it\phi_s(D)} g}) \|_{L^2(\mathbb{R}^{d+1})} \\ &\quad \leq C \|\phi_s(D)^{\frac{d-1}{4} + \beta} f\|_{L^2(\mathbb{R}^d)} \|\phi_s(D)^{\frac{d-1}{4} + \beta} g\|_{L^2(\mathbb{R}^d)} \end{aligned} \tag{2.15}$$

[1] We observe that $d \geq 4$ is important here. For $d = 3$, the estimate (2.12) actually does not hold. The counterexample has been given by Foschi [60] for the wave equation, and the same argument appropriately adapted works for the Klein–Gordon propagator.

for some absolute constant C, which, as in the discussion for the Beltran–Vega bilinear estimate, places Theorem 2.1.1 in the framework of null-form type estimates. If we *formally* set $\beta = \frac{2-d}{4}$ in (2.15) to get data with regularity whose order is $\frac{1}{4}$ as in (2.12), the order of "smoothing" from $|\Box - (2s)^2|^\beta$ becomes $2\beta = \frac{2-d}{2}$, which is compatible with (2.12). Unfortunately, $\frac{2-d}{4}$ is outside the range $\beta \geq \frac{3-d}{4}$ and, in fact, as we shall see in Proposition 2.4.4, $\beta \geq \frac{3-d}{4}$ is *a necessary condition* for (2.15).

Corollary 2.2.1. *Let $d \geq 2$. For $\beta \in [\frac{2-d}{4}, \frac{3-d}{4}] \cup [\frac{5-d}{4}, \infty)$, the optimal constant in* (2.15) *for radially symmetric f and g is $\mathbf{F}(\beta, d)^{\frac{1}{2}}$, but there does not exist a non-trivial pair of functions (f, g) that attains equality.*

We remark that it is a result of the *homogeneity* of the kernel $\mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2}+\beta}$ when $s = 0$ that one can immediately deduce the optimality of $\mathbf{F}(\beta, d)^{\frac{1}{2}}$ in (2.4) for radial data from (2.3). In contrast, our concern is the case $s > 0$ and the lack of homogeneity in the kernel causes significant difficulty in this regard, and this can be seen as responsible for smaller range of β in Corollary 2.2.1 compared with the analogous result in [30]. We prove Corollary 2.2.1 by first making use of our bilinear estimate (2.10); somewhat surprisingly given that (2.10) is a sharp inequality, we shall prove (Proposition 2.4.3) that it is impossible to obtain the optimality of $\mathbf{F}(\beta, d)^{\frac{1}{2}}$ in (2.4) for radial data and $\beta \in (\frac{3-d}{4}, \frac{5-d}{4})$ once one makes use of (2.10) as a first step.

The special case of Theorem 2.1.1 generalizes one of the current results (2.1) in the case of $(d, q, \alpha) = (3, 4, \frac{1}{2})$ by Quilodrán and (2.7) by Jeavons [84].

Corollary 2.2.2. *Let $d \geq 2$. Then, the estimate* (2.11) *holds with the optimal constant $C = \mathbf{F}(\beta, d)^{\frac{1}{2}}$ for $(\alpha, \beta) = (\frac{1}{2}, \frac{3-d}{4})$ and $(\alpha, \beta) = (1, \frac{5-d}{4})$, but there are no non-trivial extremisers. Furthermore, when $(\alpha, \beta) = (1, \frac{5-d}{4})$, we have the refined Strichartz estimate*

$$\begin{aligned} &\| |\Box - (2s)^2|^{\frac{5-d}{4}} |e^{it\phi_s(D)} f|^2 \|_{L^2(\mathbb{R}^{d+1})} \\ &\leq \mathbf{F}(\tfrac{5-d}{4}, d)^{\frac{1}{2}} \left(\|\phi_s(D) f\|_{L^2(\mathbb{R}^d)}^4 - s^2 \|\phi_s(D)^{\frac{1}{2}} f\|_{L^2(\mathbb{R}^d)}^4 \right)^{\frac{1}{2}}, \end{aligned} \tag{2.16}$$

where the constant is optimal and there are no non-trivial extremisers.

2.2.2 Non-wave regime

One may examine the Beltran–Vega bilinear estimate (2.8) from a somewhat different perspective to that taken in our earlier discussion which led to (2.12). For $d \geq 2$ the kernel K^{BV} can also be reinterpreted by

$$\begin{aligned} &K^{\mathrm{BV}}(\eta_1, \eta_2) \\ &= |\mathbb{S}^{d-2}| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\phi_s(|\eta_1|) + \phi_s(|\eta_2)}{2(\phi_s(|\eta_1|)\phi_s(|\eta_2|) - \eta_1 \cdot \eta_2 + s^2) + |\eta_1 + \eta_2|^2 \cos\theta} (\cos\theta)^{d-2} \, \mathrm{d}\theta, \end{aligned}$$

then by applying the trivial bound $\cos\theta \leq 1$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and another key relationship

$$\phi_s(|\eta_1)\phi_s(|\eta_2|) - \eta_1 \cdot \eta_2 \geq s^2, \tag{2.17}$$

we have

$$K^{\mathrm{BV}}(\eta_1, \eta_2) \leq 2^{-1}\pi |\mathbb{S}^{d-2}| s^{-1}.$$

Thus, the inequality (2.8) directly implies

$$\|D^{\frac{2-d}{2}}|e^{it\phi_s(D)}f|^2\|^2_{L^2(\mathbb{R}^{d+1})} \leq 2^{-1}s^{-1}\|\phi_s(D)^{\frac{1}{2}}f\|^4_{L^2(\mathbb{R}^d)}. \tag{2.18}$$

By comparison with (2.12), the regularity level on the initial data has increased to $H^{\frac{1}{2}}$ but this has allowed for a wider range of d which, in particular, includes $d = 2$ in which case (2.18) coincides with the sharp $H^{\frac{1}{2}} \to L^4_{x,t}$ Strichartz estimate (2.9) obtained by Quilodrán. Note that, in the non-wave regime, we are not allowed to let $s \to 0$ because of the factor s^{-1} appearing in the constant.

On the other hand, Theorem 2.1.1 also yields (2.9) as a special case of the following family of sharp null-form type estimates valid in all dimensions $d \geq 2$. Indeed, since we have another kernel estimate

$$\mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(\eta_1, \eta_2) \leq 2^{-\frac{1}{2}}\mathcal{K}_0^{\frac{d-2}{2}+2\beta}(\eta_1, \eta_2)s^{-1} \tag{2.19}$$

via (2.17), we immediately deduce the following from Theorem 2.1.1.

Corollary 2.2.3. *Let $d \geq 2$. Then the estimate* (2.11) *holds with the optimal constant*

$$C = \left(\frac{2^{-d+1}\pi^{\frac{-d+2}{2}}}{s\Gamma(\frac{d}{2})}\right)^{\frac{1}{2}}$$

for $(\alpha, \beta) = (\frac{1}{2}, \frac{2-d}{4})$, but there are no non-trivial extremisers. Furthermore, when $(\alpha, \beta) = (1, \frac{4-d}{4})$, we have the refined Strichartz estimate

$$\begin{aligned} &\||\Box - (2s)^2|^{\frac{4-d}{4}}|e^{it\phi_s(D)}f|^2\|^2_{L^2(\mathbb{R}^{d+1})} \\ &\leq \left(\frac{2^{-d+1}\pi^{\frac{-d+2}{2}}}{s\Gamma(\frac{d+2}{2})}\right)^{\frac{1}{2}}\left(\|\phi_s(D)f\|^4_{L^2(\mathbb{R}^d)} - s^2\|\phi_s(D)^{\frac{1}{2}}f\|^4_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}}, \end{aligned} \tag{2.20}$$

where the constant is optimal and there are no non-trivial extremisers.

One may note that (2.20) provides a sharp form of the following refined Strichartz inequality in the analogous manner of (2.7) when $d = 4$:

$$\|e^{it\phi_1(D)}f\|_{L^4(\mathbb{R}^{4+1})} \leq \left(\frac{1}{16\pi}\right)^{\frac{1}{4}}(\|f\|^4_{H^1(\mathbb{R}^4)} - \|f\|^4_{H^{\frac{1}{2}}(\mathbb{R}^4)})^{\frac{1}{4}},$$

however we have unable to conclude whether the constant $(\frac{1}{16\pi})^{\frac{1}{4}}$ continues to be optimal if we drop the second term on the right-hand side.

2.3 Proof of Theorem 2.1.1

Although some steps require additional care due to the extra parameter s, broadly speaking Theorem 2.1.1 can be proved by adapting the argument for wave propagators presented in [30], whose techniques are originated in [19] (see also [20]). The key tool here is the following Lorentz transform given by L; for $(t, x) \in \mathbb{R} \times \mathbb{R}^d$

$$L\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma(t - \zeta \cdot x) \\ x + (\frac{\gamma - 1}{|\zeta|^2}\zeta \cdot x - \gamma t)\zeta \end{pmatrix},$$

where $\zeta := -\frac{\xi}{\tau}$ and $\gamma := \frac{\tau}{(\tau^2-|\xi|^2)^{\frac{1}{2}}}$. Let us first introduce two lemmas whose proof come later in this section.

Lemma 2.3.1. *For η_1, $\eta_2 \in \mathbb{R}^d$ and $\beta > \frac{1-d}{4}$, define*

$$\begin{aligned} &J^{2\beta}(\eta_1,\eta_2) \\ &\quad := \int_{\mathbb{R}^{2d}} \frac{|\phi_s(|\eta_1|)\phi_s(|\eta_4|) - \eta_1\cdot\eta_4 - s^2|^{2\beta}}{\phi_s(|\eta_3|)\phi_s(|\eta_4|)} \delta\begin{pmatrix} \tau - \phi_s(|\eta_3|) - \phi_s(|\eta_4|) \\ \xi - \eta_3 - \eta_4 \end{pmatrix} \mathrm{d}\eta_3\mathrm{d}\eta_4, \end{aligned} \tag{2.21}$$

where $\tau = \phi_s(|\eta_1|) + \phi_s(|\eta_2|)$ and $\xi = \eta_1 + \eta_2$. Then, we have

$$J^{2\beta}(\eta_1,\eta_2) = (2\pi)^{\frac{d-1}{2}} \frac{\Gamma(\frac{d-2}{2}+2\beta)}{\Gamma(d-1+2\beta)} \mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(\eta_1,\eta_2).$$

Lemma 2.3.2. *Let η_1, $\eta_2 \in \mathbb{R}^d$. Set*

$$\xi = \eta_1 + \eta_2, \qquad \tau = \phi_s(|\eta_1|) + \phi_s(|\eta_2|)$$

and $\eta \in \mathbb{R}^d$ satisfying

$$2\phi_s(|\eta|) = (\tau^2 - |\xi|^2)^{\frac{1}{2}}.$$

Then, there exists $\omega_* \in \mathbb{S}^{d-1}$ depending only on η_1, η_2 and $|\eta|$ such that

$$\begin{pmatrix} \phi_s(|\eta_1|) \\ -\eta_1 \end{pmatrix} \cdot L\begin{pmatrix} \phi_s(|\eta|) \\ \eta \end{pmatrix} - s^2 = |\eta|^2\left(1 + \frac{\eta}{|\eta|}\cdot\omega_*\right).$$

Proof of Theorem 2.1.1. Let $u(t,x) = e^{it\phi_s(D)}f(x)$ and $v(t,x) = e^{it\phi_s(D)}g(x)$. By the expressions $\widetilde{u}(\tau,\xi) = 2\pi\delta(\tau-\phi_s(|\xi|))\widehat{f}(\xi)$ and $\widetilde{\overline{v}}(\tau,\xi) = 2\pi\delta(\tau+\phi_s(|\xi|))\overline{\widehat{g}}(-\xi)$, Plancherel's theorem, and appropriately relabeling the variables, one can deduce

$$\begin{aligned} &(2\pi)^{3(d+1)}\||\Box - (2s)^2|^\beta(u\overline{v})\|^2_{L^2(\mathbb{R}^{d+1})} \\ &= \int_{\mathbb{R}^{d+1}} |\tau^2 - |\xi|^2 - (2s)^2|^{2\beta}|\widetilde{u}*\widetilde{\overline{v}}(\xi,\tau)|^2\,\mathrm{d}\tau\mathrm{d}\xi \\ &= 2^{2\beta}\int_{\mathbb{R}^{4d}} |\phi_s(|\eta_1|)\phi_s(|\eta_4|) - \eta_1\cdot\eta_4 - s^2|^{2\beta} \frac{F(\eta_1,\eta_2)\overline{F(\eta_3,\eta_4)}}{(\phi_s(|\eta_1|)\phi_s(|\eta_2|)\phi_s(|\eta_3|)\phi_s(|\eta_4|))^{\frac{1}{2}}} \\ &\quad\times\delta\begin{pmatrix} \phi_s(|\eta_1|) - \phi_s(|\eta_2|) - \phi_s(|\eta_3|) + \phi_s(|\eta_4|) \\ \eta_1 + \eta_2 - \eta_3 - \eta_4 \end{pmatrix} \mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3\mathrm{d}\eta_4. \end{aligned}$$

Here,

$$F(\eta_1,\eta_2) := \widehat{f}(\eta_1)\widehat{g}(\eta_2)\phi_s(|\eta_1|)^{\frac{1}{2}}\phi_s(|\eta_2|)^{\frac{1}{2}}.$$

If we define $\Psi = \Psi_s(\eta_1,\eta_2,\eta_3,\eta_4) = \left(\frac{\phi_s(|\eta_1|)\phi_s(|\eta_2|)}{\phi_s(|\eta_3|)\phi_s(|\eta_4|)}\right)^{\frac{1}{2}}$, then by the arithmetic-geometric mean we have

$$F(\eta_1,\eta_2)\Psi^{\frac{1}{2}}F(\eta_3,\eta_4)\Psi^{-\frac{1}{2}} \le \frac{1}{2}\left(|F(\eta_1,\eta_2)|^2\Psi + |F(\eta_3,\eta_4)|^2\Psi^{-1}\right)$$

so that

$$\frac{F(\eta_1,\eta_2)F(\eta_3,\eta_4)}{(\phi_s(|\eta_1|)\phi_s(|\eta_2|)\phi_s(|\eta_3|)\phi_s(|\eta_4|))^{\frac{1}{2}}} \le \frac{1}{2}\left(\frac{|F(\eta_1,\eta_2)|^2}{\phi_s(|\eta_3|)\phi_s(|\eta_4|)} + \frac{|F(\eta_3,\eta_4)|^2}{\phi_s(|\eta_1|)\phi_s(|\eta_2|)}\right). \tag{2.22}$$

Here, the equality holds if and only if

$$\phi_s(|\eta_1|)\phi_s(|\eta_2|)\widehat{f}(\eta_1)\widehat{g}(\eta_2) = \phi_s(|\eta_3|)\phi_s(|\eta_4|)\widehat{f}(\eta_3)\widehat{g}(\eta_4)$$

almost everywhere on the support of the delta measure, which is satisfied by, for instance, $f = g = f_a$ with $a > 0$ that is given by

$$\widehat{f_a}(\xi) = \frac{e^{-a\phi_s(|\xi|)}}{\phi_s(|\xi|)}. \tag{2.23}$$

Therefore,

$$\begin{aligned}&\left(2^{3d+3+2\beta}\pi^{3d+3}\right)^{-1} \||\Box - (2s)^2|^\beta (u\overline{v})\|^2_{L^2(\mathbb{R}^{d+1})} \\ &\quad \le \frac{1}{2}\int_{\mathbb{R}^{2d}} F(\eta_1,\eta_2)J^{2\beta}(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2 + \frac{1}{2}\int_{\mathbb{R}^{2d}} F(\eta_3,\eta_4)J^{2\beta}(\eta_4,\eta_3)\,\mathrm{d}\eta_3\mathrm{d}\eta_4,\end{aligned}$$

which implies (2.10) by applying Lemma 2.3.1. One may note that the constant in (2.10) is sharp since we only apply the inequality (2.22) in the proof.

To see the lower bound imposed on β for (2.10), we shall let f and g be its extremisers (2.23) with $a = 1$. Then, by the polar coordinates,

$$\begin{aligned}&\||\Box - (2s)^2|^\beta (e^{it\phi_s(\sqrt{-\Delta})}f\overline{e^{it\phi_s(\sqrt{-\Delta})}g})\|^2_{L^2(\mathbb{R}^{d+1})} \\ &\quad = \mathbf{KG}(\beta,d)\int_0^\infty\int_0^\infty |\widehat{f}(r_1)|^2|\widehat{g}(r_2)|^2\phi_s(r_1)^{\frac{d-1}{2}+2\beta}\phi_s(r_2)^{\frac{d-1}{2}+2\beta} \\ &\qquad\qquad \times \Theta_{\frac{1}{2}}^{\frac{d-3}{2}+2\beta}(r_1,r_2)r_1^{d-1}r_2^{d-1}\,\mathrm{d}r_1\mathrm{d}r_2,\end{aligned} \tag{2.24}$$

where

$$\Theta_a^b(r_1,r_2) := \int_{(\mathbb{S}^{d-1})^2} \frac{\left(1 - \frac{r_1r_2\theta_1\cdot\theta_2}{\phi_s(r_1)\phi_s(r_2)} - \frac{s^2}{\phi_s(r_1)\phi_s(r_2)}\right)^b}{\left(1 - \frac{r_1r_2\theta_1\cdot\theta_2}{\phi_s(r_1)\phi_s(r_2)} + \frac{s^2}{\phi_s(r_1)\phi_s(r_2)}\right)^a}\,\mathrm{d}\sigma(\theta_1)\mathrm{d}\sigma(\theta_2).$$

For $\beta \ge \frac{3-d}{4}$, by the argument in the subsequent section associated with Lemma 2.4.1, it follows readily that (2.24) is bounded. For more delicate case when $\beta < \frac{3-d}{4}$, observe that

$$s^2 \le \phi_s(r_1)\phi_s(r_2) - r_1r_2\lambda + s^2 \le 3\phi_s(r_1)\phi_s(r_2)$$

and

$$\phi_s(r_1)\phi_s(r_2) - r_1r_2\lambda - s^2 \sim \frac{s^2}{\phi_s(r_1)\phi_s(r_2)}|r_1 - r_2|^2.$$

Then, for $\beta \in [\frac{2-d}{4}, \frac{3-d}{4})$, the right-hand side is bounded if $\beta \in [\frac{2-d}{4}, \frac{3-d}{4})$. For $\beta \in (\frac{1-d}{4}, \frac{2-d}{4})$, the right-hand side is essentially bounded above by

$$\int_0^\infty\int_0^\infty H(r_1)H(r_2)|r_1 - r_2|^{d-2+4\beta}\,\mathrm{d}r_1\mathrm{d}r_2, \tag{2.25}$$

which is bounded by the dual form of Hardy–Littlewood–Sobolev inequality if $-1 < d - 2 + 4\beta < 0$, or equivalently $\beta \in (\frac{1-d}{4}, \frac{2-d}{4})$. Here,$H(r) = e^{-2\phi_s(r)}\phi_s(r)^p r^{d-1}$ for some $p \in \mathbb{R}$, which belongs to Schwartz class. Similarly, the right-hand side of (2.24) is bounded below by (2.25) with different p from before so that $\beta \in (\frac{1-d}{4}, \frac{2-d}{4})$ is necessary for (2.24) to be bounded. $\square$

We now prove the aforementioned lemmas.

Proof of Lemma 2.3.1. Let $\tau = \phi_s(|\eta_1|) + \phi_s(|\eta_2|)$ and $\xi = \eta_1 + \eta_2$. It is well known that the measure $\frac{\delta(\sigma - \phi_s(|\eta|))}{\phi_s(|\eta|)}$ for $(\sigma, \eta) \in \mathbb{R} \times \mathbb{R}^d$ is invariant under the Lorentz transform L, $|\det L| = 1$, and

$$L\begin{pmatrix} (\tau^2 - |\xi|^2)^{\frac{1}{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \tau \\ \xi \end{pmatrix}.$$

The change of variables $\binom{\sigma_j}{\eta_j} \mapsto L\binom{\sigma_j}{\eta_j}$ for $j = 3, 4$ gives

$$\begin{aligned}
&J^{2\beta}(\eta_1, \eta_2) \\
&= \int_{\mathbb{R}^{2(d+1)}} \left| \begin{pmatrix} \phi_s(|\eta_1|) \\ -\eta_1 \end{pmatrix} \cdot \begin{pmatrix} \sigma_4 \\ \eta_4 \end{pmatrix} - s^2 \right|^{2\beta} \\
&\quad \times \frac{\delta(\sigma_3 - \phi_s(|\eta_3|))}{\phi_s(|\eta_3|)} \frac{\delta(\sigma_4 - \phi_s(|\eta_4|))}{\phi_s(|\eta_4|)} \delta\begin{pmatrix} \tau - \sigma_3 - \sigma_4 \\ \xi - \eta_3 - \eta_4 \end{pmatrix} \mathrm{d}\sigma_3 \mathrm{d}\sigma_4 \mathrm{d}\eta_3 \mathrm{d}\eta_4 \\
&= \int_{\mathbb{R}^d} \left| \begin{pmatrix} \phi_s(|\eta_1|) \\ -\eta_1 \end{pmatrix} \cdot L\begin{pmatrix} \phi_s(|\eta|) \\ \eta \end{pmatrix} - s^2 \right|^{2\beta} \frac{1}{\phi_s(|\eta|)^2} \delta(2\phi_s(|\eta|) - (\tau^2 - |\xi|^2)^{\frac{1}{2}})\, \mathrm{d}\eta.
\end{aligned}$$

By Lemma 2.3.2 and switching to polar coordinates,

$$J^{2\beta}(\eta_1, \eta_2) = \int_0^\infty \left(\int_{\mathbb{S}^{d-1}} (1 + \theta \cdot \omega_*)^{2\beta} \,\mathrm{d}\sigma(\theta) \right) \frac{r^{4\beta}}{\phi_s(r)^2} \delta(2\phi_s(r) - (\tau^2 - |\xi|^2)^{\frac{1}{2}}) r^{d-1} \,\mathrm{d}r.$$

Now, one can find a rotation R such that $R^{\mathrm{T}}\omega_* = e_1 = (1, 0, \dots, 0)^{\mathrm{T}}$ so that

$$\int_{\mathbb{S}^{d-1}} (1 + \theta \cdot \omega_*)^{2\beta} \,\mathrm{d}\theta = 2^{d-2+2\beta} |\mathbb{S}^{d-2}| B\left(\tfrac{d-1}{2} + 2\beta, \tfrac{d-1}{2}\right),$$

where B denotes the beta function given by

$$B(z, w) = \int_0^1 \lambda^{z-1}(1 - \lambda)^{w-1} \,\mathrm{d}\lambda$$

for z, $w \in \mathbb{C}$ whose real parts are strictly positive. For the remaining radial integration, one can perform the change of variables $2\phi_s(r) \mapsto \nu$ in order to get

$$\int_0^\infty \frac{r^{4\beta}}{\phi_s(r)^2} \delta(2\phi_s(r) - (\tau^2 - |\xi|^2)^{\frac{1}{2}}) r^{d-1} \,\mathrm{d}r = 2^{\frac{-d+1}{2} - 2\beta} \mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2} + 2\beta}(\eta_1, \eta_2)$$

and hence

$$J^{2\beta}(\eta_1, \eta_2) = 2^{\frac{d-3}{2}} |\mathbb{S}^{d-2}| B(\tfrac{d-1}{2} + 2\beta, \tfrac{d-1}{2}) \mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2} + 2\beta}(\eta_1, \eta_2).$$

Finally, simplifying the constant by the formulae

$$|\mathbb{S}^{d-1}| = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{2.26}$$

and

$$B(\tfrac{d-1}{2} + 2\beta, \tfrac{d-1}{2}) = \frac{\Gamma(\frac{d-1}{2} + 2\beta)\Gamma(\frac{d-1}{2})}{\Gamma(d - 1 + 2\beta)},$$

we are done. □

Proof of Lemma 2.3.2. Observe first that

$$L\begin{pmatrix}\phi_s(|\eta|)\\ \eta\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\tau + \frac{\xi\cdot\eta}{\phi_s(|\eta|)}\\ 2\eta + \xi(1 + \frac{\xi\cdot\eta}{(\tau+2\phi_s(|\eta|))\phi_s(|\eta|)})\end{pmatrix}.$$

Then, a direct calculation gives

$$\begin{pmatrix}\phi_s(|\eta_1|)\\ -\eta_1\end{pmatrix}\cdot L\begin{pmatrix}\phi_s(|\eta|)\\ \eta\end{pmatrix} = (\phi_s(|\eta|))^2\left(1 + \frac{\eta}{|\eta|}\cdot|\eta|z\right),$$

where

$$z = \frac{(\phi_s(|\eta|)+\phi_s(|\eta_1|))\eta_2 - (\phi_s(|\eta|)+\phi_s(|\eta_2|))\eta_1}{\phi_s(|\eta|)^2(\phi_s(|\eta_1|)+\phi_s(|\eta_2|)+2\phi_s(|\eta|))}.$$

Since we have the relation $2\phi_s(|\eta|) = \phi_s(|\eta_1|)\phi_s(|\eta_2|) - \eta_1\cdot\eta_2 + s^2$, the numerator of z can be simplified by

$$\begin{aligned}
&([\phi_s(|\eta|)+\phi_s(|\eta_1|)]\eta_2 - [\phi_s(|\eta|)+\phi_s(|\eta_2|)]\eta_1)^2\\
&= [\phi_s(|\eta|)+\phi_s(|\eta_1|)]^2|\eta_2|^2 + [\phi_s(|\eta|)+\phi_s(|\eta_2|)]^2|\eta_1|^2\\
&\quad - 2[\phi_s(|\eta|)+\phi_s(|\eta_1|)][\phi_s(|\eta|)+\phi_s(|\eta_2|)]\eta_2\cdot\eta_1\\
&= [\phi_s(|\eta|)+\phi_s(|\eta_1|)]^2\phi_s(|\eta_2|)^2 + [\phi_s(|\eta|)+\phi_s(|\eta_2|)]^2\phi_s(|\eta_1|)^2\\
&\quad - 2[\phi_s(|\eta|)+\phi_s(|\eta_2|)][\phi_s(|\eta|)+\phi_s(|\eta_1|)](\phi_s(|\eta_1|)\phi_s(|\eta_2|) - 2\phi_s(|\eta|)^2)\\
&\qquad - s^2([\phi_s(|\eta|)+\phi_s(|\eta_1|)]^2 + [\phi_s(|\eta|)+\phi_s(|\eta_2|)]^2\\
&\qquad\quad + 2[\phi_s(|\eta|)+\phi_s(|\eta_1|)][\phi_s(|\eta|)+\phi_s(|\eta_2|)])\\
&= \left(\phi_s(|\eta|)^2 - s^2\right)[\phi_s(|\eta_1|)+\phi_s(|\eta_2|)+2\phi_s(|\eta|)]^2,
\end{aligned}$$

and so it follows that

$$|z| = \frac{|\eta|}{\phi_s(|\eta|)^2}.$$

Therefore,

$$\begin{pmatrix}\phi_s(|\eta_1|)\\ -\eta_1\end{pmatrix}\cdot L\begin{pmatrix}\phi_s(|\eta|)\\ \eta\end{pmatrix} - s^2 = |\eta|^2\left(1 + \frac{\eta}{|\eta|}\cdot\omega_*\right),$$

where we have set $\omega_* = \frac{z}{|z|}$. ☐

2.4 On estimate (2.15)

2.4.1 Estimate (2.15) with explicit constant

In this subsection, we prove (2.15) for radially symmetric data f and g for $\beta > \frac{2-d}{4}$ and an explicit constant $C < \infty$; for $\beta = [\frac{2-d}{4}, \frac{3-d}{4}] \cup [\frac{5-d}{4}, \infty)$, this explicit constant coincides with $\mathbf{F}(\beta,d)^{\frac{1}{2}}$. In order to complete the proof of Corollary 2.2.1, we need to show the sharpness of $\mathbf{F}(\beta,d)^{\frac{1}{2}}$ for $\beta \in [\frac{2-d}{4}, \frac{3-d}{4}] \cup [\frac{5-d}{4}, \infty)$, and the non-existence of extremisers; for these arguments, we refer the reader to Section 2.5.

Lemma 2.4.1. *Let $a+b > -1$, $b > -1$ and $\kappa \in [0,1]$. Define*

$$h^{a,b}(\kappa) := \int_{-1}^{1}(1-\kappa\lambda)^a(1-\lambda^2)^b\,\mathrm{d}\lambda.$$

Then,

$$\sup_{\kappa\in[0,1]} h^{a,b}(\kappa) < \infty.$$

Moreover, for $a \in (-\infty, 0] \cup [1,\infty)$

$$\sup_{\kappa\in[0,1]} h^{a,b}(\kappa) = h^{a,b}(1) = 2^{a+2b+1}B(a+b+1, b+1).$$

Proof of Lemma 2.4.1. By the Lebesgue dominated convergence theorem,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\kappa} h^{a,b}(\kappa) &= -a\kappa \int_{-1}^{1} (1-\kappa\lambda)^{a-1}\lambda(1-\lambda^2)\,\mathrm{d}\lambda \\ &= a\kappa \int_0^1 \left((1+\kappa\lambda)^{a-1} - (1-\kappa\lambda)^{a-1}\right)\lambda(1-\lambda^2)^b\,\mathrm{d}\lambda\end{aligned}$$

Thus,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\kappa} h^{a,b}(\kappa) \geq 0 & \text{if } a \in (-\infty, 0] \cup [1,\infty), \\ \frac{\mathrm{d}}{\mathrm{d}\kappa} h^{a,b}(\kappa) < 0 & \text{if } a \in (0,1). \end{cases}$$

For $a \in (-\infty, 0] \cup [1,\infty)$,

$$\sup_{\kappa\in[0,1]} h^{a,b}(\kappa) = h^{a,b}(1) = \int_{-1}^{1} (1-\lambda)^a (1-\lambda^2)^b\,\mathrm{d}\lambda$$

and the change of variables $1+\lambda \mapsto 2\lambda$ gives

$$\int_{-1}^{1} (1-\lambda)^a(1-\lambda^2)^b\,\mathrm{d}\lambda = 2^{a+2b+1}B(a+b+1, b+1) < \infty$$

if $a+b>0$ and $b>-1$. Similarly, for $a \in (0,1)$,

$$h^{a,b}(\kappa) \leq h^{a,b}(0) = 2^{2b+1}B(b+1,b+1) < \infty$$

if $b > -1$. □

Let f, g be radially symmetric. By Theorem 2.1.1, we have

$$\begin{aligned}&\| |\Box - (2s)^2|^{\beta} (e^{it\phi_s(\sqrt{-\Delta})} f \overline{e^{it\phi_s(\sqrt{-\Delta})} g}) \|_{L^2(\mathbb{R}^{d+1})}^2 \\ &\leq \mathbf{KG}(\beta, d) \int_0^\infty \int_0^\infty |\widehat{f}(r_1)|^2 |\widehat{g}(r_2)|^2 \phi_s(r_1)^{\frac{d-1}{2}+2\beta} \phi_s(r_2)^{\frac{d-1}{2}+2\beta} \Theta_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(r_1,r_2) r_1^{d-1} r_2^{d-1}\,\mathrm{d}r_1\mathrm{d}r_2,\end{aligned} \tag{2.27}$$

where

$$\Theta_a^b(r_1,r_2) := \int_{(\mathbb{S}^{d-1})^2} \frac{\left(1 - \frac{r_1r_2\theta_1\cdot\theta_2}{\phi_s(r_1)\phi_s(r_2)} - \frac{s^2}{\phi_s(r_1)\phi_s(r_2)}\right)^b}{\left(1 - \frac{r_1r_2\theta_1\cdot\theta_2}{\phi_s(r_1)\phi_s(r_2)} + \frac{s^2}{\phi_s(r_1)\phi_s(r_2)}\right)^a} \mathrm{d}\sigma(\theta_1)\mathrm{d}\sigma(\theta_2).$$

We divide the range of β into $\beta \in [\frac{2-d}{4}, \frac{3-d}{4}]$ and $\beta \in [\frac{5-d}{4}, \infty)$ and treat these cases differently. First, let us consider $\beta \in [\frac{5-d}{4}, \infty)$ as the easier case. By applying the fundamental kernel estimate (2.14), we have

$$\Theta_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(r_1,r_2) \leq \Theta_0^{\frac{d-3}{2}+2\beta}(r_1,r_2) = |\mathbb{S}^{d-1}||\mathbb{S}^{d-2}| h^{\frac{d-3}{2}+2\beta, \frac{d-3}{2}}(\kappa)$$

with $\kappa = \frac{r_1 r_2}{\phi_s(r_1)\phi_s(r_2)}$. Since $d-3+2\beta \geq 1$, Lemma 2.4.1 implies that

$$\sup_{\kappa\in[0,1)} h^{\frac{d-3}{2}+2\beta,\frac{d-3}{2}}(\kappa) = h^{\frac{d-3}{2}+2\beta,\frac{d-3}{2}}(1),$$

and hence

$$\sup_{r_1,r_2>0} \Theta_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(r_1,r_2) \leq 2^{\frac{3d-7}{2}+2\beta}|\mathbb{S}^{d-1}||\mathbb{S}^{d-2}|B\left(d-2+2\beta,\tfrac{d-1}{2}\right),$$

which yields (2.15) with $C = \mathbf{F}(\beta,d)^{\frac{1}{2}}$.

For $\beta \in [\frac{2-d}{4},\frac{3-d}{4}]$, in which case $\frac{d-2}{2}+2\beta \in [0,\frac{1}{2}]$, the basic idea of our argument is the same as above but it requires a few more steps. Let

$$\Xi(\nu,\upsilon) := \int_{-1}^{1} \frac{(1-\nu-\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{d-2}{2}+2\beta}}{(1+\nu-\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{1}{2}}}\,\mathrm{d}\mu(\lambda)$$

with ν and υ satisfying

$$\nu\in[0,1], \qquad \upsilon^2 \leq 1-\nu^2,$$

and $\mathrm{d}\mu(\lambda) = (1-\lambda^2)^{\frac{d-3}{2}}\,\mathrm{d}\lambda$. Then, from (2.27), it suffices to show

$$\Xi(\nu,\upsilon) \leq \Xi(0,\upsilon) \leq \Xi(0,0). \tag{2.28}$$

In order to show the first inequality of (2.28), we establish monotonicity in ν on $\big[0,\sqrt{\frac{1-\upsilon^2}{2}}\big]$, and calculate directly for $\nu \in \big[\sqrt{\frac{1-\upsilon^2}{2}},\sqrt{1-\upsilon^2}\big]$. Indeed, it simply follows that

$$\begin{aligned}\partial_\nu \Xi(\nu,\upsilon) \leq &-\left(\frac{d-2}{2}+2\beta\right)\int_0^1 \frac{(1-\nu-\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{d-4}{2}+2\beta}}{(1+\nu-\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{1}{2}}}\left(1-\frac{\nu}{\sqrt{1-\nu^2-\upsilon^2}}\lambda\right)\mathrm{d}\mu(\lambda)\\ &-\frac{1}{2}\int_0^1 \frac{(1-\nu-\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{d-2}{2}+2\beta}}{(1+\nu-\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{3}{2}}}\left(1+\frac{\nu}{\sqrt{1-\nu^2-\upsilon^2}}\lambda\right)\mathrm{d}\mu(\lambda)\\ &-\left(\frac{d-2}{2}+2\beta\right)\int_0^1 \frac{(1-\nu+\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{d-4}{2}+2\beta}}{(1+\nu+\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{1}{2}}}\left(1+\frac{\nu}{\sqrt{1-\nu^2-\upsilon^2}}\lambda\right)\mathrm{d}\mu(\lambda)\\ &-\frac{1}{2}\int_0^1 \frac{(1-\nu+\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{d-2}{2}+2\beta}}{(1+\nu+\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{3}{2}}}\left(1-\frac{\nu}{\sqrt{1-\nu^2-\upsilon^2}}\lambda\right)\mathrm{d}\mu(\lambda),\end{aligned}$$

which is non-positive since

$$1-\frac{\nu}{\sqrt{1-\nu^2-\upsilon^2}}\lambda \geq 0$$

for $\nu \in \big[0,\sqrt{\frac{1-\upsilon^2}{2}}\big]$. On the other hand, for $\nu \in \big[\sqrt{\frac{1-\upsilon^2}{2}},\sqrt{1-\upsilon^2}\big]$, which imposes $0 \leq \sqrt{1-\nu^2-\upsilon^2} \leq \nu$, it follows that

$$\begin{aligned}\Xi(\nu,\upsilon) &= \int_0^1 \frac{(1-\nu-\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{d-2}{2}+2\beta}}{(1+\nu-\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{1}{2}}}\,\mathrm{d}\mu(\lambda) + \int_0^1 \frac{(1-\nu+\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{d-2}{2}+2\beta}}{(1+\nu+\sqrt{1-\nu^2-\upsilon^2}\lambda)^{\frac{1}{2}}}\,\mathrm{d}\mu(\lambda)\\ &\leq \int_0^1 2\,\mathrm{d}\mu(\lambda)\\ &\leq \int_0^1 (1-\sqrt{1-\upsilon^2}\lambda)^{\frac{d-3}{2}+2\beta}\,\mathrm{d}\mu(\lambda) + \int_0^1 (1+\sqrt{1-\upsilon^2}\lambda)^{\frac{d-3}{2}+2\beta}\,\mathrm{d}\mu(\lambda)\\ &= \Xi(0,\upsilon).\end{aligned}$$

Here, the last inequality is given by the arithmetic-geometric mean:

$$\frac{1}{2}\Big((1-\sqrt{1-\upsilon^2}\lambda)^{\frac{d-3}{2}+2\beta}+(1+\sqrt{1-\upsilon^2}\lambda)^{\frac{d-3}{2}+2\beta}\Big)\geq\big(1-(1-\upsilon^2)\lambda^2\big)^{\frac{d-3}{4}+\beta}\geq 1.$$

Since the second inequality of (2.28) can be readily proved by Lemma 2.4.1, we have (2.15) with $C=\mathbf{F}(\beta,d)^{\frac{1}{2}}$ for $\beta\in[\frac{2-d}{4},\frac{3-d}{4}]$ as well.

2.4.2 Threshold of our argument for $\beta\in(\frac{3-d}{4},\frac{5-d}{4})$

Although $C=\mathbf{F}(\beta,d)^{\frac{1}{2}}$ will be shown to be optimal for $\beta\in[\frac{2-d}{4},\frac{3-d}{4}]\cup[\frac{5-d}{4},\infty)$ in the case of radial data, it remains unclear whether this continues to be true for $\beta\in(\frac{3-d}{4},\frac{5-d}{4})$; here we establish that there is no way to obtain the constant $\mathbf{F}(\beta,d)^{\frac{1}{2}}$ if one first makes use of Theorem 2.1.1. In order to show that, we shall invoke the following useful result for the beta function due to Agarwal–Barnett–Dragmir [1]:

Lemma 2.4.2 ([1])**.** *Let m, p and $k\in\mathbb{R}$ satisfy $m,p>0$, and $p>k>-m$. If we have*

$$k(p-m-k)>0$$

then

$$B(p,m)>B(p-k,m+k)$$

holds.

Proposition 2.4.3. *Let $d\geq 2$ and $\beta\in(\frac{3-d}{4},\frac{5-d}{4})$. Then there exist radially symmetric f and g such that*

$$\begin{aligned}\mathbf{KG}(\beta,d)\int_{\mathbb{R}^{2d}}|\widehat{f}(\eta_1)|^2|\widehat{g}(\eta_2)|^2\phi_s(|\eta_1|)\phi_s(|\eta_2|)\mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2\\ >\mathbf{F}(\beta,d)\|\phi_s(D)^{\frac{d-1}{4}+\beta}f\|^2_{L^2(\mathbb{R}^d)}\|\phi_s(D)^{\frac{d-1}{4}+\beta}g\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

holds.

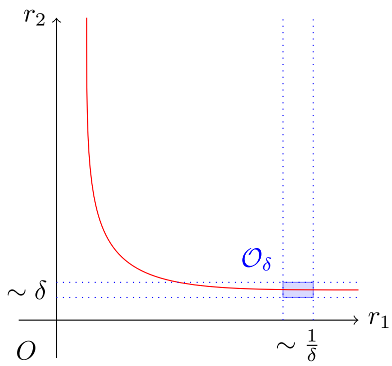


Figure 2.1: The set $\mathcal{O}_\delta$ along the curve $r_1=r_2^{-1}$.

Proof of Proposition 2.4.3. Let $0 < \delta \ll 1$ and

$$A = \left\{ \xi \in \mathbb{R}^d \,:\, \frac{1}{2} < |\xi| < 2 \right\}.$$

Define $f = f_A$ and $g = g_A$ so that for $\xi \in \mathbb{R}^d$

$$\widehat{f_A}(\xi) = \chi_A(\tfrac{\xi}{\delta}) \quad \text{and} \quad \widehat{g_A}(\xi) = \chi_A(\delta\xi),$$

where χ_A is the characteristic function of A. By use of polar coordinates

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} |\widehat{f}(\eta_1)|^2 |\widehat{g}(\eta_2)|^2 \phi_s(|\eta_1|)\phi_s(|\eta_2|)\mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2 \\
&\quad = \int_{\mathcal{O}_\delta} |\widehat{f}(r_1)|^2 |\widehat{g}(r_2)|^2 (\phi_s(r_1)\phi_s(r_2))^{\frac{d-1}{2}+2\beta}\Theta_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(r_1r_2)^{d-1}\,\mathrm{d}r_1\mathrm{d}r_2.
\end{aligned}$$

Here, the set $\mathcal{O}_\delta$ is defined by

$$\mathcal{O}_\delta = \left\{ (r_1, r_2) : \frac{1}{2\delta} < r_1 < \frac{2}{\delta}, \frac{\delta}{2} < r_2 < 2\delta \right\}.$$

Now, for $(r_1, r_2) \in \mathcal{O}_\delta$, taking the limit $\delta \to 0$ so that $\phi_s(r_1) \to \infty$ and $\phi_s(r_2) \to s$ and invoking the Legendre duplication formula

$$\Gamma(z)\Gamma(z + \tfrac{1}{2}) = 2^{1-2z}\pi^{\frac{1}{2}}\Gamma(2z), \tag{2.29}$$

we obtain

$$\Theta_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(r_1, r_2) \to |\mathbb{S}^{d-1}|^2.$$

Therefore, for sufficiently small $\delta > 0$,

$$\begin{aligned}
&\mathbf{KG}(\beta, d)\int_{\mathbb{R}^{2d}} |\widehat{f}(\eta_1)|^2 |\widehat{g}(\eta_2)|^2 \phi_s(|\eta_1|)\phi_s(|\eta_2|)\mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2 \\
&\quad = (2\pi)^{2d}\mathbf{KG}(\beta, d)\|\phi_s(\sqrt{-\Delta})^{\frac{d-1}{4}+\beta} f\|_{L^2(\mathbb{R}^d)}^2 \|\phi_s(\sqrt{-\Delta})^{\frac{d-1}{4}+\beta} g\|_{L^2(\mathbb{R}^d)}^2,
\end{aligned}$$

and it is enough to show

$$(2\pi)^{2d}\mathbf{KG}(\beta, d) > \mathbf{F}(\beta, d). \tag{2.30}$$

By the formula (2.26) and the definitions of constants, this can be simplified as

$$B(\tfrac{3d-5}{4} + \beta, \tfrac{3d-3}{4} + \beta) > B(d - 2 + 2\beta, \tfrac{d}{2})$$

which, if fact, follows from Lemma 2.4.2 by letting $p = \frac{3d-5}{4} + \beta$, $m = \frac{3d-3}{4} + \beta$ and $k = \frac{3-d}{4} - \beta$. □

2.4.3 Contributions of radial symmetry

Here, we observe for general (not necessary radially symmetric) data f and g the inequality (2.15) holds only if $\beta \geq \frac{3-d}{4}$, in other words, the radial symmetry condition on f and g widens the range of the regularity parameter β. The proof is based on the Knapp type argument in [62] where they proved $\beta_- \geq \frac{3-d}{4}$ is necessary for (2.5) to hold.

Proposition 2.4.4. *Let $\beta < \frac{3-d}{4}$. For any $C_* > 0$, there exists $f, g \in H^{\frac{d-1}{4}+\beta}(\mathbb{R}^d)$ such that*

$$\begin{aligned} &\| |\Box - (2s)^2|^\beta (e^{it\phi_s(\sqrt{-\Delta})} f \overline{e^{it\phi_s(\sqrt{-\Delta})} g}) \|^2_{L^2(\mathbb{R}^{d+1})} \\ &\quad > C_* \|\phi_s(\sqrt{-\Delta})^{\frac{d-1}{4}+\beta} f\|^2_{L^2(\mathbb{R}^d)} \|\phi_s(\sqrt{-\Delta})^{\frac{d-1}{4}+\beta} g\|^2_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{2.31}$$

Proof of Proposition 2.4.4. For $\eta_1 \in \mathbb{R}^d$ (similarly, for $\eta_2 \in \mathbb{R}^d$), we set indices $(1), \dots, (d)$ to indicate components of vectors, namely, $\eta_1 = (\eta_{1(1)}, \dots, \eta_{1(d)})$. Also, denote $\eta_1' = (\eta_{1(2)}, \dots, \eta_{1(d)}) \in \mathbb{R}^{d-1}$ and $\eta_1'' = (\eta_{1(3)}, \dots, \eta_{1(d)}) \in \mathbb{R}^{d-2}$. Now, for large $L > 0$, eventually sent to infinity, define sets $\mathfrak{F}$ and $\mathfrak{G}$ by

$$\mathfrak{F} = \{\eta \in \mathbb{R}^d : L \le \eta_{(1)} \le 2L, 1 \le \eta_{(2)} \le 2, |\eta''| \le 1\}$$

and

$$\mathfrak{G} = \{\eta \in \mathbb{R}^d : L \le \eta_{(1)} \le 2L, -1 \le \eta_{(2)} \le -2, |\eta''| \le 1\}.$$

For such f and g

$$\left| |\Box - (2s)^2|^\beta (e^{it\phi_s(D)} f(x) \overline{e^{it\phi_s(D)} g(x)}) \right| \sim \left| \int_{\mathfrak{F}} \int_{\mathfrak{G}} e^{i\Phi_s(x,t:\eta_1,\eta_2)} \mathcal{K}^0_{-\beta}(\eta_1, \eta_2) \, \mathrm{d}\eta_1 \mathrm{d}\eta_2 \right|,$$

where

$$\Phi_s(x, t : \eta_1, \eta_2) = x \cdot (\eta_1 - \eta_2) + t(\phi_s(|\eta_1|) - \phi_s(|\eta_2|)).$$

Now, we follow the idea of Knapp's example to derive a lower bound. From the setting (see also Figure 2.2) we have $|\eta_1| \sim |\eta_2| \sim \phi_s(|\eta_1|) \sim \phi_s(|\eta_2|) \sim |\eta_1 + \eta_2| \sim L$, $\theta \sim L^{-1}$ for $(\eta_1, \eta_2) \in \mathfrak{F} \times \mathfrak{G}$, $|\phi_s(|\eta_1|) - \eta_{1(1)}| \sim |\eta_1'|^2 |\eta_1|^{-1} \sim |\phi_s(|\eta_2|) - \eta_{2(1)}| \sim |\eta_2'|^2 |\eta_2|^{-1} \sim L^{-1}$, $|\eta_{1(1)} - \eta_{2(1)}| \sim 1$ and $|\eta_1' + \eta_2'| \lesssim 1$. Then, it follows that

$$(\phi_s(|\eta_1|)\phi_s(|\eta_2|))^2 - (\eta_1 \cdot \eta_2 - s^2)^2 \sim s^2 |\eta_1 + \eta_2|^2 + |\eta_1|^2 |\eta_2|^2 \sin^2 \theta \sim L^2 \tag{2.32}$$

and hence

$$\mathcal{K}^0_{-\beta}(\eta_1, \eta_2) \sim \left(\frac{(\phi_s(|\eta_1|)\phi_s(|\eta_2|))^2 - (\eta_1 \cdot \eta_2 - s^2)^2}{\phi_s(|\eta_1|)\phi_s(|\eta_2|) + \eta_1 \cdot \eta_2 + s^2} \right)^\beta \sim 1.$$

Moreover, for the phase, then it follows that

$$\begin{aligned} &|\Phi_s(x, t : \eta_1, \eta_2)| \\ &\quad = |t(\phi_s(|\eta_1|) - \eta_{1(1)} - \phi_s(|\eta_2|) + \eta_{2(1)}) + (x + t)(\eta_{1(1)} - \eta_{2(1)}) + x' \cdot (\eta_1' + \eta_2')| \\ &\quad \le |t| L^{-1} + |x + t| L + |x'| < \frac{\pi}{3} \end{aligned}$$

for (x, t) in a slab $\mathfrak{R} = [-L^{-1}, L^{-1}] \times [-1, 1]^{d-1} \times [-L, L]$ whose volume is the order of 1. Hence,

$$|\Box - (2s)^2|^\beta (e^{it\phi_s(D)} f(x) \overline{e^{it\phi_s(D)} g(x)}) \gtrsim |\mathfrak{F}||\mathfrak{G}| \chi_{\mathfrak{R}}(x, t)$$

and so

$$\| |\Box - (2s)^2|^\beta (e^{it\phi_s(D)} f(x) \overline{e^{it\phi_s(D)} g(x)}) \gtrsim |\mathfrak{F}|^2 |\mathfrak{G}|^2 |\mathfrak{R}| \sim |\mathfrak{F}|^2 |\mathfrak{G}|^2.$$

On the other hand, we have

$$\|\phi_s(\sqrt{-\Delta})^{\frac{d-1}{4}+\beta} f\|^2_{L^2(\mathbb{R}^d)} \|\phi_s(\sqrt{-\Delta})^{\frac{d-1}{4}+\beta} g\|^2_{L^2(\mathbb{R}^d)} \lesssim L^{d-1+4\beta} |\mathfrak{F}||\mathfrak{G}|.$$

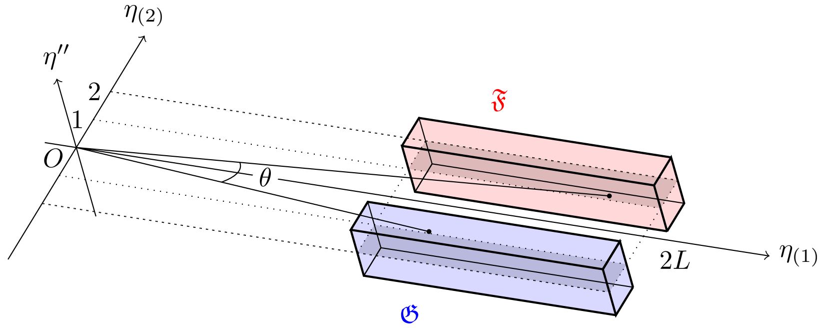


Figure 2.2: The sets $\mathfrak{F}$ and $\mathfrak{G}$, which are sent away from the origin along $\eta_{(1)}$-axis.

Therefore, it is implied that

$$|\mathfrak{F}|^2|\mathfrak{G}|^2 \lesssim L^{d-1+4\beta}|\mathfrak{F}||\mathfrak{G}|.$$

The fact $|\mathfrak{F}| \sim |\mathfrak{G}| \sim L$ and letting $L \to \infty$ result in the desired necessary condition

$$\frac{3-d}{4} \leq \beta.$$

□

2.5 Sharpness of constants

It is straightforward that the estimates (2.11) with claimed constants in Corollaries 2.2.2 and 2.2.3 when $(\alpha, \beta) = (\frac{1}{2}, \frac{3-d}{4})$ and $(\alpha, \beta) = (\frac{1}{2}, \frac{2-d}{4})$ coincides with the results obtained by applying the kernel estimates (2.14) and (2.19) to (2.10), respectively. We will see in the forthcoming sections the sharpness of those constants. To obtain the estimate (2.16) and (2.20), we require the additional fact that

$$\int_{\mathbb{R}^{2d}} f(x)f(y)x \cdot y \,\mathrm{d}x\mathrm{d}y \geq 0.$$

Indeed, in the wave regime, after we apply the kernel estimate (2.14) to (2.10), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} |\widehat{f}(\eta_1)|^2|\widehat{f}(\eta_2)|^2\phi_s(|\eta_1|)\phi_s(|\eta_2|)\mathcal{K}_0^1(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2 \\
&\quad\leq \int_{\mathbb{R}^{2d}} |\widehat{f}(\eta_1)|^2|\widehat{f}(\eta_2)|^2\phi_s(|\eta_1|)\phi_s(|\eta_2|)(\phi_s(|\eta_1|)\phi_s(|\eta_2|) - s^2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2,
\end{aligned}$$

which immediately yields (2.16). Similarly, one can deduce (2.20) in the non-wave regime. Finally, the estimate (2.11) with $C = \mathbf{F}(\frac{5-d}{4}, d)^{\frac{1}{2}}$ when $(\alpha, \beta) = (1, \frac{5-d}{4})$ is obtained by further estimating the kernel of (2.16) as

$$\phi_s(|\eta_1|)\phi_s(|\eta_2|) - s^2 \leq \phi_s(|\eta_1|)\phi_s(|\eta_2|).$$

Again, we will see the sharpness of constants below. Of course, by a similar argument to the above, one can easily obtain the estimate (2.11) with

$$C = \frac{2^{-d+1}\pi^{\frac{-d+2}{2}}}{s\Gamma(\frac{d+2}{2})} \tag{2.33}$$

when $(\alpha, \beta) = (1, \frac{4-d}{4})$ from (2.20) in the non-wave regime, and it is natural to hope that the constant is still optimal. We do not, however, know whether or not the constant (2.33) is optimal, which will become clear from the following argument on the sharpness of constants.

In the rest of Section 2.5, we focus on completing our proof of Corollaries 2.2.1, 2.2.2 and 2.2.3 by proving that the stated constants are optimal and non existence of non-trivial extremisers. We achieve optimality of constants by considering the functions f_a given by (2.23); this is a natural guess given that such functions are extremisers for (2.10), as shown in our proof of Theorem 2.1.1. Before proceeding, we introduce the following useful notation.

$$\mathrm{N}_a(\beta) := \int_{4as}^{\infty} e^{-\rho} \int_0^{(2a)^{-1}\sqrt{\rho^2-(4as)^2}} \frac{(\rho^2-(2ar)^2-(4as)^2)^{\frac{d-2}{2}+2\beta}}{\rho^2-(2ar)^2} r^{d-1}\,\mathrm{d}r\mathrm{d}\rho$$

and

$$\mathrm{D}_a(\beta, b) := \left(\int_{2as}^{\infty} e^{-\rho}\rho^b(\rho^2-(2as)^2)^{\frac{d-2}{2}}\,\mathrm{d}\rho\right)^2.$$

2.5.1 Wave regime

We shall consider (2.11) with $(\alpha, \beta) = (\frac{d-1}{4} + \beta, \beta)$ for $\beta \in [\frac{3-d}{4}, \infty)$. Let f_a satisfy (2.23). Then, we have

$$\| |\Box - (2s)^2|^\beta |e^{it\phi_s(D)} f_a|^2 \|_{L^2(\mathbb{R}^{d+1})}^2 = 2^{\frac{-3d+7}{2}-2\beta} |\mathbb{S}^{d-1}| \mathbf{KG}(\beta, d)(2a)^{-2d+5-4\beta}\mathrm{N}_a(\beta)$$

and

$$\|\phi_s(D)^{\frac{d-1}{4}+\beta} f_a\|_{L^2(\mathbb{R}^d)}^4 = (2\pi)^{-2d}|\mathbb{S}^{d-1}|^2(2a)^{-3d+5-4\beta}\mathrm{D}_a(\beta, \tfrac{d-3}{2}+2\beta), \tag{2.34}$$

and so it is enough to show

$$\lim_{a\to 0} \frac{\| |\Box - (2s)^2|^\beta |e^{it\phi_s(D)} f_a|^2 \|_{L^2(\mathbb{R}^{d+1})}^2}{\|\phi_s(D)^{\frac{d-1}{4}+\beta} f_a\|_{L^2(\mathbb{R}^d)}^4} = \lim_{a\to 0} (2a)^d C(\beta, d) \frac{\mathrm{N}_a(\beta)}{\mathrm{D}_a(\beta, \frac{d-3}{2}+2\beta)} = \mathbf{F}(\beta, d), \tag{2.35}$$

where

$$C(\beta, d) = 2^{-2(d-2)}\pi^{\frac{-d+1}{2}} \frac{\Gamma(\frac{d-1}{2}+2\beta)}{\Gamma(d-1+2\beta)}.$$

Since we have, by appropriate change of variables,

$$\mathrm{N}_a(\beta) = e^{-4as}(2a)^{-d} \int_0^\infty e^{-\rho}\rho^{\frac{3}{2}d-2+2\beta}(\rho+8as)^{\frac{3}{2}d-2+2\beta}$$
$$\times \int_0^1 \frac{(1-\nu^2)^{d-2+2\beta}\nu^{d-1}}{(\rho+4as)^2(1-\nu^2)+(4as)^2\nu^2}\,\mathrm{d}\nu\mathrm{d}\rho$$

and

$$\mathrm{D}_a(\beta, \tfrac{d-3}{2} + 2\beta) = e^{-4as}\left(\int_0^\infty e^{-\rho}(\rho + 2as)^{\frac{d-3}{2}+2\beta}\rho^{\frac{d-2}{2}}(\rho + 4as)^{\frac{d-2}{2}}\,\mathrm{d}\rho\right)^2,$$

one may deduce

$$\lim_{a\to 0}(2a)^d\frac{\mathrm{N}_a(\beta)}{\mathrm{D}_a(\beta, \frac{d-3}{2} + 2\beta)} = \frac{\Gamma(3d-5+4\beta)B(d-2+2\beta, \frac{d}{2}))}{2\Gamma(\frac{3d-5}{2} + 2\beta)^2},$$

which leads to (2.35).

In order to show the constant $\mathbf{F}(\frac{5-d}{4}, d)^{\frac{1}{2}}$ is sharp in (2.16), we apply a similar calculation. In particular, one may note that the right-hand side of (2.16) can be written as

$$(2\pi)^{-2d}|\mathbb{S}^{d-1}|^2(2a)^{-2d}(\mathrm{D}_a(\beta, 1) - (2as)^2\mathrm{D}_a(\beta, 0)), \tag{2.36}$$

instead of (2.34). One can also see the second term is negligible in the sense of the optimal constant since it vanishes while a tends to 0.

2.5.2 Non-wave regime

Let f_a satisfy (2.23). Note that in the non-wave regime the right-hand side of (2.11) is expressed as

$$\|\phi_s(D)^{\frac{d}{4}+\beta}f_a\|^4_{L^2(\mathbb{R}^d)} = (2\pi)^{-2d}|\mathbb{S}^{d-1}|^2(2a)^{-3d+4-4\beta}\mathrm{D}_a(\beta, \tfrac{d-2}{2} + 2\beta). \tag{2.37}$$

Then, as we have done above, reform $\mathrm{N}_a(\beta)$ and $\mathrm{D}_a(\beta, \frac{d-2}{2} + 2\beta)$ as follows by some appropriate change of variables:

$$\begin{aligned}\mathrm{N}_a(\beta) = e^{-4as}(2a)^{\frac{d}{2}-4+2\beta}\int_0^\infty e^{-\rho}\rho^{\frac{3}{2}d-2+2\beta}(\frac{\rho}{2a} + 4s)^{\frac{3}{2}d-2+2\beta}\\ \times\int_0^1 \frac{(1-\nu^2)^{d-2+2\beta}\nu^{d-1}}{(\frac{\rho}{2a} + 2s)^2(1-\nu^2) + (2s)^2\nu^2}\,\mathrm{d}\nu\mathrm{d}\rho\end{aligned}$$

and

$$\mathrm{D}_a(\beta, \tfrac{d-2}{2} + 2\beta) = e^{-4as}(2a)^{2d-4+4\beta}\left(\int_0^\infty e^{-\rho}(\frac{\rho}{2a} + s)^{\frac{d-2}{2}+2\beta}\rho^{\frac{d-2}{2}}(\frac{\rho}{2a} + 2s)^{\frac{d-2}{2}}\,\mathrm{d}\rho\right)^2.$$

First, we shall consider (2.11) with $(\alpha, \beta) = (0, \frac{2-d}{4})$. By a similar argument to the wave regime above, one can easily check that

$$\lim_{a\to\infty}(2a)^{d+1}\frac{\mathrm{N}_a(\frac{2-d}{4})}{\mathrm{D}_a(\frac{2-d}{4}, 0)} = 2^{d-3}s^{-1}$$

holds, from which it follows that

$$\begin{aligned}\lim_{a\to\infty}\frac{\||\Box - (2s)^2|^{\frac{2-d}{4}}|e^{it\phi_s(D)}f_a|^2\|^2_{L^2(\mathbb{R}^{d+1})}}{\|\phi_s(D)^{\frac{1}{2}}f_a\|^4_{L^2(\mathbb{R}^d)}} &= \lim_{a\to\infty}(2a)^{d+1}C(\tfrac{2-d}{4}, d)\frac{\mathrm{N}_a(\frac{2-d}{4})}{\mathrm{D}_a(\frac{2-d}{4}, 0)}\\ &= \frac{2^{-d+1}\pi^{\frac{-d+2}{2}}}{s\Gamma(\frac{d}{2})}.\end{aligned}$$

Similarly, for the case when $(\alpha,\beta)=(1,\frac{4-d}{4})$, it is enough to show

$$\begin{aligned}
&\lim_{a\to\infty}\frac{\||\Box-(2s)^2|^{\frac{4-d}{4}}|e^{it\phi_s(D)}f_a|^2\|^2_{L^2(\mathbb{R}^{d+1})}}{\|\phi_s(D)f_a\|^4_{L^2(\mathbb{R}^d)}-(2as)^2\|\phi_s(D)^{\frac12}f_a\|^4_{L^2(\mathbb{R}^d)}}\\
&=\lim_{a\to\infty}(2a)^{d+1}C(\tfrac{4-d}{4},d)\frac{\mathrm{N}_a(\frac{4-d}{4})}{\mathrm{D}_a(\frac{4-d}{4},1)-(2as)^2\mathrm{D}_a(\frac{4-d}{4},0)}\\
&=\frac{2^{-d+1}\pi^{\frac{-d+2}{2}}}{s\Gamma(\frac{d+2}{2})}. \qquad (2.38)
\end{aligned}$$

In fact,

$$\begin{aligned}
&((2a)^{d-2}e^{-4as})^{-1}\left(\mathrm{D}_a(\tfrac{4-d}{4},1)-(2as)^2\mathrm{D}_a(\tfrac{4-d}{4},0)\right)\\
&=\left(\int_0^\infty e^{-\rho}(\rho+2as)\rho^{\frac{d-2}{2}}\left(\frac{\rho}{2a}+2s\right)^{\frac{d-2}{2}}\mathrm{d}\rho\right)^2\\
&\quad-\left((2as)\int_0^\infty e^{-\rho}\rho^{\frac{d-2}{2}}\left(\frac{\rho}{2a}+2s\right)^{\frac{d-2}{2}}\mathrm{d}\rho\right)^2\\
&=(2a)\left(\int_0^\infty e^{-\rho}\rho^{\frac{d-2}{2}}\left(\frac{\rho}{2a}+2s\right)^{\frac{d}{2}}\mathrm{d}\rho\right)\left(\int_0^\infty e^{-\rho}\rho^{\frac{d}{2}}\left(\frac{\rho}{2a}+2s\right)^{\frac{d-2}{2}}\mathrm{d}\rho\right)
\end{aligned}$$

implies

$$\lim_{a\to\infty}(2a)^{d+1}\frac{N_a(\frac{4-d}{4})}{\mathrm{D}_a(\frac{4-d}{4},1)-(2as)^2\mathrm{D}_a(\frac{4-d}{4},0)}=2^{d-2}s^{-1}$$

and so (2.38) follows.

In the contrast to the wave regime, here a is sent to ∞ and the second term of the denominator of (2.38) does not vanish so that we cannot follow the argument for the wave regime and do not know whether the constant (2.33) is still optimal for (2.11) when $(\alpha,\beta)=(1,\frac{4-d}{4})$.

2.5.3 Non-existence of an extremiser

Suppose there were non-trivial f and g that satisfy any of the statements in Corollary 2.2.3 with equality. From our proof via Theorem 2.1.1, it would be required that

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}}|\widehat{f}(\eta_1)|^2|\widehat{g}(\eta_2)|^2\phi_s(|\eta_1|)\phi_s(|\eta_2|)\mathcal{K}_{\frac12}^{\frac{d-2}{2}+2\beta}(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2\\
&=2^{-\frac12}s^{-1}\int_{\mathbb{R}^{2d}}|\widehat{f}(\eta_1)|^2|\widehat{g}(\eta_2)|^2\phi_s(|\eta_1|)\phi_s(|\eta_2|)\mathcal{K}_0^{\frac{d-2}{2}+2\beta}(\eta_1,\eta_2)\,\mathrm{d}\eta_1\mathrm{d}\eta_2
\end{aligned}$$

holds. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}}|\widehat{f}(\eta_1)|^2|\widehat{g}(\eta_2)|^2\phi_s(|\eta_1|)\phi_s(|\eta_2|)\left(\mathcal{K}_{\frac12}^{\frac{d-2}{2}+2\beta}(\eta_1,\eta_2)-2^{-\frac12}s^{-1}\mathcal{K}_0^{\frac{d-2}{2}+2\beta}(\eta_1,\eta_2)\right)\mathrm{d}\eta_1\mathrm{d}\eta_2\\
&=0
\end{aligned}$$

would hold. Since f, g are assumed to be non-trivial $\widehat{f}$, $\widehat{g} \neq 0$ on some set $\mathfrak{F} \times \mathfrak{G} \subseteq \mathbb{R}^{2d}$ with $|\mathfrak{F}|$, $|\mathfrak{G}| > 0$, it would be deduced that

$$\mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(\eta_1, \eta_2) - 2^{-\frac{1}{2}} s^{-1} \mathcal{K}_0^{\frac{d-2}{2}+2\beta}(\eta_1, \eta_2) = 0 \tag{2.39}$$

on $(\mathfrak{F} \times \mathfrak{G}) \setminus \mathfrak{N}$ where $\mathfrak{N} \subseteq \mathbb{R}^{2d}$ is a null set. However, (2.39) would hold only on the diagonal line $\{(\eta_1, \eta_2) : \eta_1 = \eta_2\}$ (the equality condition of (2.19)), which is a null set and so is $\{(\eta_1, \eta_2) : \eta_1 = \eta_2\} \cap (\mathfrak{F} \times \mathfrak{G})$. This is a contradiction.

For Corollary 2.2.1, Corollary 2.2.2, similar arguments above can be carried. In particular, for equality in the wave regime, the formula (2.39) might be replaced by

$$\mathcal{K}_{\frac{1}{2}}^{\frac{d-2}{2}+2\beta}(\eta_1, \eta_2) - \mathcal{K}_0^{\frac{d-3}{2}+2\beta}(\eta_1, \eta_2) = 0$$

on $(\mathfrak{F} \times \mathfrak{G}) \setminus \mathfrak{N}$, which would only occur when $s = 0$ (the equality condition of (2.13)).

Chapter 3

A new perspective on hypercontractivity

3.1 Introduction

This chapter is based on work of the author in collaboration with Yosuke Aoki, Jonathan Bennett, Neal Bez, Shuji Machihara, and Kosuke Matsuura in [2].

The hypercontractive nature of the heat semigroup associated with the free Hamiltonian plays an important role in order to show the total Hamiltonian is bounded from below. This may be interpreted in quantum physics as the stability of the concerned system, and further characterization of the physical ground state or the physical vacuum for certain models, such as Boson fields and Fermi fields. Hypercontractivity inequality extracted from the Markov property was one of celebrated tools when Nelson [117] explored the Euclidean Field Theory. Aside from those fruitful applications in quantum physics, hypercontractivity inequalities have also kept engaging researchers by its purely mathematical charm up to the present day, about seventy years later after [117]. In this chapter, we consider the well-known hypercontractivity of the Ornstein–Uhlenbeck semigroup (and later the significantly more general setting of Markov semigroups). Here, we give a new "closure property" perspective on the hypercontractive inequality, which is closely associated with the heat-flow monotonicity method mentioned in Chapter 2.

We shall begin with the familiar Lebesgue measure space setting: One of the fundamental properties of the Laplacian $\Delta = \sum_{j=1}^{n} \partial_j^2$ is

$$\int_{\mathbb{R}^d} (\Delta f(x))\, g(x)\, \mathrm{d}x = -\int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x)\, \mathrm{d}x, \tag{3.1}$$

for compactly supported smooth functions f and g by invoking the divergence theorem and the fact that

$$\nabla \cdot (g \nabla f) = \nabla g \cdot \nabla f + g(\nabla \cdot \nabla f).$$

Now, it is natural to ask what would play a role of Δ if $\mathrm{d}x$ was replaced by the Gauss measure $\mathrm{d}\gamma(x)$ given by

$$\mathrm{d}\gamma(x) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}}\, \mathrm{d}x.$$

Note that $\int_{\mathbb{R}^d} \mathbb{1}(x)\,\mathrm{d}\gamma(x) = 1$ so that $\mathrm{d}\gamma$ is a probability measure. It turns out that a direct calculation gives

$$\int_{\mathbb{R}^d} (Lf(x))\, g(x)\,\mathrm{d}\gamma(x) = -\int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g(x)\,\mathrm{d}\gamma(x).$$

Here, L is defined by

$$Lf(x) = \Delta f(x) - x\cdot\nabla f(x),$$

and is sometimes called the Ornstein–Uhlenbeck operator. In this context, one can think L as the "Laplacian" for the Gauss measure $\mathrm{d}\gamma$.

Now, let us introduce the Ornstein–Uhlenbeck semigroup $(e^{sL})_{s\geq 0}$ by

$$e^{sL}f(x) = \int_{\mathbb{R}^d} f(e^{-s}x + (1-e^{-2s})^{1/2}y)\,\mathrm{d}\gamma(y), \tag{3.2}$$

then this can be the solution to the "heat equation" determined by L

$$\begin{cases} \partial_s u(s,x) = Lu(s,x), \\ u(0,x) = f(x), \end{cases}$$

for sufficiently nice initial data f. For $p\in[1,\infty]$, the Ornstein–Uhlenbeck semigroup satisfies the contraction property derived by Fubini's theorem and Hölder's inequality, namely,

$$\|e^{sL}f\|_{L^p(\mathrm{d}\gamma)} \leq \|f\|_{L^p(\mathrm{d}\gamma)}.$$

Furthermore, the Ornstein–Uhlenbeck semigroup has an even stronger contractive nature found by Edward Nelson called *hypercontractivity.* In the paper published in 1966 [117], he proved the special case of the inequality (3.4) below when $(p,q)=(2,4)$. After some improvements by several authors, in [119], Nelson himself finally completed his celebrated result as follows.

Theorem (Nelson's hypercontractivity inequality)**.** *Let* $1<p<q\leq\infty$ *and* $s\geq 0$. *If*

$$e^{2s} = \frac{q-1}{p-1} \tag{3.3}$$

then

$$\|e^{sL}f\|_{L^q(\mathrm{d}\gamma)} \leq \|f\|_{L^p(\mathrm{d}\gamma)} \tag{3.4}$$

holds for all $f\in L^p(\mathrm{d}\gamma)$.

The hypercontractivity inequality in the framework of the Ornstein–Uhlenbeck semigroup $(e^{sL})_{s\geq 0}$ is the model case of what we explore in this chapter. As seen later, the story above can be vastly generalized and understood in the language of the Markov semigroup and the so-called diffusion property. Aside from the numerous applications in quantum physics, from a purely mathematical perspective, the hypercontractivity inequality has been considered to be important by its remarkable connections to other famous inequalities in analysis. For the earlier results and futher background in quantum physics, we recommend the interested reader to visit [42, 65, 66, 69, 70, 78, 118, 127, 130, 131] and work cited in there. The earliest discovery was the equivalence of hypercontractivity to certain logarithmic Sobolev inequalities, expressed as

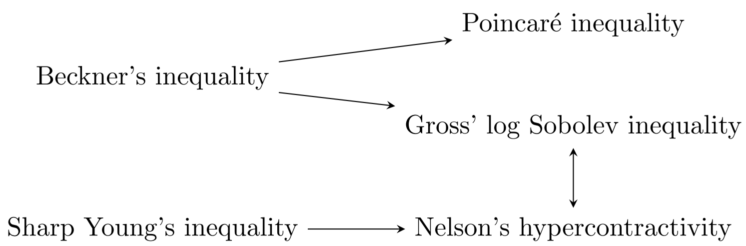
$$\int_{\mathbb{R}^d} |f(x)|^2\log|f(x)|\,\mathrm{d}\gamma(x) \leq \|\nabla f\|^2_{L^2(\mathrm{d}\gamma)} + \|f\|^2_{L^2(\mathrm{d}\gamma)}\log\|f\|_{L^2(\mathrm{d}\gamma)}.$$

Paul Federbush [59] first indicated their connection and later Leonard Gross [67] established the equivalence.

The logarithmic Sobolev inequality above itself has been generalized as Beckner's inequality[1] by William Beckner [13] (differentiate the following at $p = 2$); for $1 \leq p \leq 2$,

$$\|f\|^2_{L^2(\mathrm{d}\gamma)} \leq (2-p)\|\nabla f\|^2_{L^2(\mathrm{d}\gamma)} + \|f\|^2_{L^p(\mathrm{d}\gamma)},$$

which yields the so-called Poincaré inequality by letting $p = 1$. In a slightly different context, Beckner has made the observation that the hypercontractivity inequality is a special case of the sharp Young's inequality; this can be found in [12] where Beckner famously found the optimal constant for Young's convolution inequality.



The heat-flow monotonicity method has been successfully used to derive several sharp inequalities. With this method, we often flow the input function(s) according to an appropriate heat equation and obtain the desired inequality by comparing time at zero and infinity. We refer the reader to work of Eric Carlen, Elliott Lieb and Michael Loss [41] and Jonathan Bennett, Anthony Carbery, Michael Christ and Terence Tao [22] in the context of the Brascamp–Lieb inequality.

In [17], Bennett and Neal Bez have further developed an intriguing perspective on the heat flow monotonicity method by considering supersolutions and their algebraic closure properties. In other words, supersolutions are closed under certain operations. For example, suppose u_1 and u_2 are supersolutions to the heat equation

$$\partial_t u(t,x) = \Delta u(t,x),$$

and we denote this by

$$u_1, u_2 \in \mathfrak{S} := \{u : \partial_t u \geq \Delta u\}.$$

Then, the geometric mean $U = u_1^{1/2} u_2^{1/2}$ from u_1 and u_2 satisfies $U \in \mathfrak{S}$. The reader may refer to [17] for more detail. In a similar manner we show the following for the Ornstein–Uhlenbeck semigroup.

Theorem 3.1.1. *Let* $\infty > q > p > 1$ *and* $s > 0$ *be given by* $e^{2s} = \frac{q-1}{p-1}$. *Suppose* $u : (0,\infty)\times\mathbb{R}^d \to (0,\infty)$ *is such that* $u(t,\cdot)^{1/p}$, $\partial_t(u(t,\cdot)^{1/p})$, $\nabla(u(t,\cdot)^{1/p})$, $u(t,\cdot)^{-1/p}|\nabla(u(t,\cdot)^{1/p})|^2$ *and* $\Delta(u(t,\cdot)^{1/p})$ *are of polynomial growth locally uniformly in time* $t > 0$, *and satisfies*

$$\partial_t u \geq Lu.$$

[1] Beckner's inequality itself has been extended in a beautiful way by Ewain Gwynne and Elton P. Hsu [77] (see also [76]): for $1 \leq p < q$ and $2 \leq q$

$$\|f\|^2_{L^q(\mathrm{d}\gamma)} - \|f\|^2_{L^p(\mathrm{d}\gamma)} \leq (q-p)\|\nabla f\|^2_{L^q(\mathrm{d}\gamma)}.$$

Let $U : (0,\infty) \times \mathbb{R}^d \to (0,\infty)$ be given by

$$U(t,x)^{1/q} = e^{sL}(u(t,\cdot)^{1/p})(x). \tag{3.5}$$

Then $U(t,\cdot)^{1/q}$, $\partial_t(U(t,\cdot)^{1/q})$, $\nabla(U(t,\cdot)^{1/q})$, $U(t,\cdot)^{-1/q}|\nabla(U(t,\cdot)^{1/q})|^2$ and $\Delta(U(t,\cdot)^{1/q})$ are of polynomial growth locally uniformly in time $t > 0$, and

$$\partial_t U \geq LU.$$

The main point of this theorem is, of course, the closure property. Although the regularity conditions imposed on u are of a more technical nature, some care has been exercised to ensure that they are strong enough for the relevant terms in the statement of the theorem and its proof to be rigorously defined, and weak enough so that the regularity conditions are themselves preserved under the transformation $u \mapsto U$.

The following corollary shows that the closure property above underlies the key nature and, in fact, we immediately acquire a monotone quantity which provides the aforementioned Nelson's hypercontractivity.

Corollary 3.1.2. *Suppose u satisfies $\partial_t u = Lu$ with initial data a bounded and compactly supported nonnegative function on $\mathbb{R}^d$. Let $Q : (0,\infty) \to (0,\infty)$ be given by*

$$Q(t) = \int_{\mathbb{R}^d} \left(e^{sL}(u(t,\cdot)^{1/p})\right)^q (x)\, \mathrm{d}\gamma(x),$$

where $\infty > q > p > 1$ and $e^{2s} = \frac{q-1}{p-1}$. Then Q is nondecreasing on $(0,\infty)$.

Taking U as in (3.5), we have $Q(t) = \int U(t,\cdot)\,\mathrm{d}\gamma$ and by passing the time derivative through the integral, we may quickly obtain Corollary 3.1.2 from Theorem 3.1.1. In turn, the monotonicity of Q generates the well-known hypercontractivity inequality enjoyed by the Ornstein–Uhlenbeck semigroup. Indeed, taking u to satisfy $\partial_t u = Lu$ with initial data f^p, where f is a bounded and compactly supported nonnegative function on $\mathbb{R}^d$, the dominated convergence theorem implies that

$$\lim_{t\to 0} Q(t) = \|e^{sL} f\|^q_{L^q(\gamma)}$$

and

$$\lim_{t\to \infty} Q(t) = \|f\|^q_{L^p(\gamma)}. \tag{3.6}$$

The above arguments concerning non-negative functions are enough since we trivially have $\|e^{sL} f\|^q_{L^q(\gamma)} \leq \|e^{sL}|f|\|^q_{L^q(\gamma)}$.

3.2 Markov semigroups

3.2.1 Preliminaries

Let $p \in [1,\infty]$ and $(E,\mathcal{E},\mu)$ be a σ-finite measure space.

Definition 3.2.1 (Markov semigroup)**.** *Suppose the operator $(P_t)_{t\geq 0}$ acts on bounded measurable functions on $(E,\mathcal{E})$ and is given by*

$$P_s f(x) = \int_E f(y)\,\mathrm{d}\nu_{s,x}(y) \tag{3.7}$$

where $\nu_{s,x}$ is a non-negative measurable probability measure for $s \geq 0$ and $x \in E$. The family of such operators $(P_s)_{s\geq 0}$ is called a Markov semigroup if $(P_s)_{s\geq 0}$ enjoys:

(i) $P_0 = \mathrm{id}$ *(initial condition),*

(ii) $P_s \circ P_t = P_{s+t}$ *for all* $s, t \geq 0$ *(semigroup property),*

(iii) for each $f \in L^2(\mathrm{d}\mu)$*,* P_s *converges to* f *in* L^2 *as* $s \to 0$ *(continuity),*

(iv) for every $1 \leq p < \infty$ *and* $f \in L^p(\mathrm{d}\mu)$*,* $\|P_s f\|_{L^p(\mathrm{d}\mu)} \leq \|f\|_{L^p(\mathrm{d}\mu)}$ *(contraction property).*

Regarding the conditions (iii) and (iv), by Jensen's inequality, the contraction property holds for all bounded measurable functions. Hence, by density, the domains of $(P_s)_{s\geq 0}$ may be extended to $L^p(\mathrm{d}\mu)$ for $1 \leq p < \infty$ with the contraction property. One also may note that it immediately follows from the definition that $(P_s)_{s\geq 0}$ is positivity preserving and satisfies $P_s(\mathbb{1}) = \mathbb{1}$, where $\mathbb{1}$ is the constant function equal to 1.

Definition 3.2.2 (Ergodicity)**.** *A Markov semigroup* $(P_t)_{t\geq 0}$ *with a measure* μ *is said to be ergodic if for all* $f \in L^p(\mathrm{d}\mu)$

$$\lim_{t\to 0} \left\| P_t f - \int_{\mathbb{R}^d} f(x)\,\mathrm{d}\mu(x) \right\|_{L^p(\mathrm{d}\mu)} = 0.$$

The assumption of $(P_t)_{t\geq 0}$ being ergodic is crucial in our argument when Corollary 3.1.2 deduces hypercontractivity. It is easy to see ergodicity is being used to derive with (3.6).

Definition 3.2.3 (Symmetric Markov semigroup)**.** *Let* $(P_t)_{t\geq 0}$ *be a Markov semigroup on* $L^p(\mathrm{d}\mu)$ *and* μ *be the associated invariant measure. We call* $(P_t)_{t\geq 0}$ *a symmetric Markov semigroup with respect to* μ *if*

$$\int_{\mathbb{R}^d} P_t f(x) g(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^d} f(x) P_t g(x)\,\mathrm{d}\mu(x)$$

for all $f, g \in L^p(\mathrm{d}\mu)$ *and* $t \geq 0$*.*

Definition 3.2.4 (Infinitesimal generator)**.** *Let* $(P_t)_{t\geq 0}$ *be a Markov semigroup on* $L^p(\mathrm{d}\mu)$*. Consider the limit*

$$\lim_{t\to 0} \frac{P_t f - f}{t}$$

in $L^p(\mathrm{d}\mu)$*. When there exists such a limit, we denote the limit by* Lf *and call it the infinitesimal generator.*

An important property of L is

$$\partial_s P_s f = L P_s f = P_s[Lf], \tag{3.8}$$

which we will use several times throughout the chapter. By using the definition of a Markov semigroup (i), $P_s\mathbb{1}(x) - \mathbb{1}(x) = 0$ for all $s \geq 0$ and $x \in \mathbb{R}^d$ so that $L\mathbb{1} = 0$. The following are the key concepts to expose the nature of a Markov semigroup, on which our proof of Theorem 3.1.1 rely. These are somehow related, at least in the formal sense, to differential and Riemannian geometry, for which the reader may consult [8].

Definition 3.2.5. *The infinitesimal generator L is* diffusion *if, for all C^∞ functions ψ on $\mathbb{R}^n$, we have*

$$L\psi(f) = \sum_{j=1}^{n} \partial_j \psi(f) L f_j + \sum_{j,k=1}^{n} \partial^2_{j,k} \psi(f) \Gamma(f_j, f_k), \tag{3.9}$$

where $f = (f_1, \ldots, f_n)$.

Definition 3.2.6. *The bilinear forms $L^2(\mu) \times L^2(\mu) \to L^2(\mu)$ denoted by Γ and Γ_2 are defined by*

$$2\Gamma(f,g) = L(fg) - fL(g) - gL(f),$$
$$2\Gamma_2(f,g) = L\Gamma(f,g) - \Gamma(f, Lg) - \Gamma(g, Lf),$$

and called the carré du champ operator of L and the curvature operator of L, respectively. We write $\Gamma(f) = \Gamma(f,f)$ and similarly for Γ_2. Furthermore, if for Γ and Γ_2 there exists $c \in \mathbb{R}$ such that

$$\Gamma_2(f) \geq c\Gamma(f). \tag{3.10}$$

for all f in the respective domain of Γ and Γ_2, then we call c the curvature.

Under the diffusion condition, suppose Γ and Γ_2 also enjoy the curvature condition, then the following key gradient bound due to Dominique Bakry [4] holds (the curvature condition is actually *equivalent* to (3.11)).

Lemma 3.2.7. *If L is a diffusion and of curvature c, then*

$$\sqrt{\Gamma(P_s f)} \leq e^{-cs} P_s[\sqrt{\Gamma(f)}] \tag{3.11}$$

for all $s \geq 0$.

In the remainder of this section, we assume that $(P_s)_{s\geq 0}$ satisfies the diffusion condition (3.9) and the curvature condition (3.10) with curvature constant $c \in \mathbb{R}$. Also, we assume $\infty > q > p > 1$ and let s be defined by $e^{2cs} = \frac{q-1}{p-1}$.

Our goal is to present an abstract argument which yields the closure property

$$\partial_t u \geq Lu \qquad \Rightarrow \qquad \partial_t U \geq LU, \tag{3.12}$$

where $u : (0,\infty) \times E \to (0,\infty)$, and $U : (0,\infty) \times E \to (0,\infty)$ is given by

$$U(t,x)^{1/q} = P_s[u(t,\cdot)^{1/p}](x).$$

Consequently, if we additionally assume that μ is a probability measure and $(P_s)_{s\geq 0}$ is ergodic, then the monotonicity argument outlined in the previous section generates the associated hypercontractivity inequality

$$\|P_s f\|_{L^q(\mu)} \leq \|f\|_{L^p(\mu)} \tag{3.13}$$

from (3.12).

The inequality (3.13) is known in the above setting and is a fundamental component of a wider and celebrated theory. Systematic study of the curvature operator Γ_2 and associated curvature-dimension conditions (of which (3.10) is a particular case) go back to work of Bakry [5], and Bakry and Michel Émery [7]. Since then, a significant body of work of a highly geometric flavour has emerged on the analysis of Markov operators satisfying diffusion and curvature conditions; we refer the reader to the monograph by Bakry, Ivan Gentil and Michel Ledoux [8] and the lecture notes of Ledoux [102] for further details.

3.2.2 The closure property

For simplicity of the exposition, the following argument for (3.12) is based on certain formal considerations. For instance, we shall make multiple use of the identity

$$L(f^\lambda) = \lambda f^{\lambda-1} Lf + \lambda(\lambda-1) f^{\lambda-2}\Gamma(f) \tag{3.14}$$

for $\lambda > 0$. Observe that (3.14) formally follows from the diffusion property by taking $\psi(f) = f^\lambda$ (or in a rigorous sense in the case of, for example, the Ornstein–Uhlenbeck semigroup via direct computations).

Proceeding via the representation formula (3.7) and formally passing the time derivative through the integral, we have

$$\partial_t U(t,x) = \frac{q}{p} U(t,x)^{1-1/q} P_s[u(t,\cdot)^{1/p-1}\partial_t u(t,\cdot)](x). \tag{3.15}$$

Since u is a supersolution and by use of (3.14) we get

$$\begin{aligned}\partial_t U(x) &\geq \frac{q}{p} U(x)^{1-1/q} P_s[u^{1/p-1} Lu](x)\\ &= q\,U(x)^{1-1/q} P_s[Lu^{1/p}](x) + \frac{q}{pp'} U(x)^{1-1/q} P_s[u^{1/p-2}\Gamma(u)](x).\end{aligned}$$

Here we have dropped the dependence on the t variable since all operators are now acting in the spatial variable.

On the other hand, by a further application of (3.14),

$$LU(x) = q\,U(x)^{1-1/q} LP_s[u^{1/p}](x) + q(q-1)\,U(x)^{1-2/q}\Gamma(P_s[u^{1/p}])(x)$$

and, using that P_s and L formally commute, we thus have

$$\frac{1}{q} U(x)^{2/q-1}[\partial_t U - LU](x) \geq \frac{1}{pp'} U(x)^{1/q} P_s[u^{1/p-2}\Gamma(u)](x) - (q-1)\Gamma(P_s[u^{1/p}])(x).$$

However, by an application of Lemma 3.2.7 followed by the Cauchy–Schwarz inequality we have

$$(q-1)\Gamma(P_s[u^{1/p}])(x) \leq (q-1)e^{-2cs} P_s[u^{1/p}](x) P_s[u^{-1/p}\Gamma(u^{1/p})](x).$$

Applying the identity[2]

$$\Gamma(u^{1/p}) = \frac{1}{p^2} u^{2/p-2}\Gamma(u) \tag{3.16}$$

and using the relation $e^{2cs} = \frac{q-1}{p-1}$, it is clear from the above argument that $\partial_t U \geq LU$.

3.2.3 Ornstein–Uhlenbeck semigroup

In this section, we quickly check that Ornstein–Uhlenbeck semigroup enjoys the conditions in Section 3.2.1. In this case, $P_s f = e^{sL} f$ where $L = \Delta - x\cdot\nabla$, and a simple change of variables shows that (3.7) holds with

$$\mathrm{d}\nu_{s,x}(y) = \exp\left(-\frac{|\rho x - y|^2}{2(1-\rho^2)}\right)\frac{\mathrm{d}y}{[2\pi(1-\rho^2)]^{d/2}},$$

[2]The identity $\Gamma(\psi(u)) = \psi'(u)^2\Gamma(u)$ holds for smooth ψ as a result of the diffusion property (3.9) and thus (3.16) holds in a formal sense by taking $\psi(u) = u^{1/p}$. In the case of the Ornstein–Uhlenbeck semigroup, (3.16) may be rigorously verified by direct calculations.

where $\rho = e^{-s}$. Also, direct computations reveal that

$$\Gamma(f) = |\nabla f|^2 \tag{3.17}$$

and

$$\Gamma_2(f) = |D^2 f|^2 + |\nabla f|^2, \tag{3.18}$$

where $|D^2 f|^2 = \sum_{j,k=1}^d (\partial_{kj} f)^2)$ is the Frobenius norm of the Hessian of f. In fact,

$$\begin{aligned} 2\Gamma(f) &= L(f^2) - 2fLf \\ &= (2|\nabla f|^2 + 2f\Delta f - 2f\nabla f \cdot x) - (2f\Delta f - 2f\nabla f \cdot x) \\ &= 2|\nabla f|^2, \end{aligned}$$

which clearly shows (3.17). Similarly, one can show $\Gamma(f,g) = \nabla f \cdot \nabla g$. In addition, since we have

$$\Delta |\nabla f|^2 = 2\left(\sum_{j,k=1}^d \partial_k f \partial_{kjj} f + |D^2 f|^2 \right),$$

$$x \cdot \nabla |\nabla f|^2 = 2 \sum_{i,j=1}^d x_j \partial_j f \partial_{ij} f,$$

and

$$\begin{aligned} \Gamma(f, Lf) &= \nabla f \cdot \nabla(\Delta f - x \cdot \nabla f)) \\ &= \sum_{j,k=1}^d \partial_k f \partial_{kjj} f - \left(\sum_{j=1}^d |\nabla f|^2 + 2 \sum_{i,j=1}^d x_j \partial_j f \partial_{ij} f \right), \end{aligned}$$

the equality (3.18) then follows by combining the above;

$$\begin{aligned} 2\Gamma_2(f) &= \big(\Delta |\nabla f|^2 - x \cdot \nabla |\nabla f|^2\big) - \Gamma(f, Lf) \\ &= 2|D^2 f|^2 + 2|\nabla f|^2. \end{aligned}$$

Therefore, (3.9) and (3.10) with $c = 1$ hold. In this special case, the key estimate (3.11) can be too deduced significantly easily from the explicit formula (3.2); passing through each derivative ∂_j and using Cauchy–Schwarz inequality.

3.3 Proofs of main results

In this section, we write $P_s f = e^{sL} f$, where $L = \Delta - x \cdot \nabla$, and $B_s(x,y) = e^{-s}x + (1 - e^{-2s})^{1/2} y$.

3.3.1 Proof of Theorem 3.1.1

To begin, we observe that $U(t,x)$ is well-defined in a pointwise sense since our assumptions on u mean that $u(t,\cdot)^{1/p}$ is of polynomial growth for each fixed time.

In order to prove $\partial_t U \geq LU$, we run the argument in the previous section with $c = 1$. Rigorous justification of (3.15), at which point we passed the time derivative through the integral appearing in (3.7), is made using the fact that $\partial_t(u^{1/p})$ is of polynomial growth locally uniformly in time. Another formal step in the argument is the commutativity property

$$P_s[L(u^{1/p})](x) = LP_s[u^{1/p}](x).$$

Since $L = \Delta - x \cdot \nabla$, we may rigorously justify this since $\nabla(u^{1/p})$ and $\Delta(u^{1/p})$ are of polynomial growth locally uniformly in time $t > 0$. Finally, we note that the term $P_s[u^{1/p-2}\Gamma(u)](x)$ is well-defined in a pointwise sense thanks to the assumption that $u^{-1/p}|\nabla(u^{1/p})|^2$ is of polynomial growth locally uniformly in time. This completes the verification of the formal steps in the argument in the previous section.

It remains to check that the regularity conditions imposed on u result in U satisfying analogous regularity properties. Our assumption on u means that, for a fixed $t > 0$, there is a natural number N and compact interval $I \subset (0,\infty)$ containing t such that $\sup_{t' \in I} u(t',x)^{1/p} \lesssim_t \langle x \rangle^N$ for all $x \in \mathbb{R}^d$. Here, we are using the Japanese bracket notation $\langle x \rangle = (1+|x|^2)^{1/2}$. Thus, clearly we have

$$U(t',x)^{1/q} = \int u^{1/p}(t', B_s(x,y))\,\mathrm{d}\gamma(y) \lesssim_t \int \langle B_s(x,y)\rangle^N \,\mathrm{d}\gamma(y) \lesssim_{s,t} \langle x \rangle^N \tag{3.19}$$

for each $t' \in I$ and $x \in \mathbb{R}^d$, and it follows that $U^{1/q}$ is of polynomial growth locally uniformly in time.

For $\partial_t(U(t,\cdot)^{1/q})$, the assumption that $\partial_t(u^{1/p})$ is of polynomial growth locally uniformly in time means, by a routine application of the dominated convergence theorem,

$$\partial_t(U^{1/q})(t',x) = \int \partial_t(u^{1/p})(t', B_s(x,y))\,\mathrm{d}\gamma(y)$$

for all t' in an appropriate compact interval, and now estimating in a similar manner to (3.19) reveals that $\partial_t(U(t,\cdot)^{1/q})$ of polynomial growth locally uniformly in time. By similar considerations, the same conclusion also holds for $\nabla(U^{1/q})$ and $\Delta(U^{1/q})$. Finally, by the Cauchy–Schwarz inequality

$$\begin{aligned} |\nabla(U^{1/q})(t',x)|^2 &= \left| \int \nabla(u^{1/p})(t', B_s(x,y))\,\mathrm{d}\gamma(y) \right|^2 \\ &\leq U(t',x)^{1/q} \int \frac{|\nabla(u^{1/p}(t', B_s(x,y))|^2}{u^{1/p}(t', B_s(x,y))}\,\mathrm{d}\gamma(y) \end{aligned}$$

and the fact that $u^{-1/p}|\nabla(u^{1/p})|^2$ is of polynomial growth locally uniformly in time can be easily seen to induce the same property for $U^{-1/q}|\nabla(U^{1/q})|^2$.

3.3.2 Proof of Corollary 3.1.2

Suppose that f is bounded and nonnegative function with support inside $\{x \in \mathbb{R}^d : |x| \leq R\}$, and let $u(t,x) = P_t[f^p](x)$. By Theorem 3.1.1, it suffices to show that $u(t,\cdot)^{1/p}$, $\partial_t(u(t,\cdot)^{1/p})$, $\nabla(u(t,\cdot)^{1/p})$, $u(t,\cdot)^{-1/p}|\nabla(u(t,\cdot)^{1/p})|^2$ and $\Delta(u(t,\cdot)^{1/p})$ are of polynomial growth locally uniformly in time $t > 0$. Indeed, if this is the case, then $\partial_t U \geq LU$ where $U(t,x)^{1/q} = P_s(u(t,\cdot)^{1/p})(x)$ and therefore

$$Q'(t) = \frac{1}{q}Q(t)^{1-1/q} \int \partial_t U(t,x)\,\mathrm{d}\gamma(x) \geq \frac{1}{q}Q(t)^{1-1/q} \int LU(t,x)\,\mathrm{d}\gamma(x) = 0.$$

Note that we may use the dominated convergence theorem to justify the interchange of the time derivative and the integral in the above argument. Indeed, we know from

Theorem 3.1.1 that both $U(t,\cdot)^{1/q}$ and $\partial_t(U(t,\cdot)^{1/q})$ are of polynomial growth locally uniformly in $t>0$. By writing $\partial_t U = qU^{1-1/q}\partial_t(U^{1/q})$ and recalling that $q>1$, we see that the same property also holds for U, and this is sufficient to justify the interchange of time derivative and integral.

It remains verify the regularity claimed hypotheses for u. We first note that $u(t,x)^{1/p} \leq \|f\|_\infty$ obviously follows from (3.2). For $\partial_t(u^{1/p})$, we shall make use of the representation formula

$$u(t,x) = C(t)\int_{\mathbb{R}^d} f(y)^p \exp\left(-\frac{|\rho(t)x-y|^2}{2(1-\rho(t)^2)}\right)\mathrm{d}y,$$

where $C(t) = [2\pi(1-\rho(t)^2)]^{-d/2}$ and $\rho(t)=e^{-t}$. Using the assumption on the support of f, it easily follows that

$$|\partial_t u(t,x)| \lesssim_R (1-\rho(t)^2)^{-2}\langle x\rangle^2 u(t,x)$$

and therefore, since $\partial_t(u^{1/p}) = \frac{1}{p}\frac{\partial_t u}{u}u^{1/p}$, we see that $\partial_t(u^{1/p})$ is of polynomial growth locally uniformly in $t>0$. A similar argument reveals $|\nabla u(t,x)| \lesssim_R (1-\rho(t)^2)^{-1}\langle x\rangle u(t,x)$. From this we quickly obtain that $\nabla(u^{1/p})$ is of polynomial growth locally uniformly in $t>0$ and, via the identity $u^{-1/p}|\nabla(u^{1/p})|^2 = \frac{1}{p}u^{1/p}\frac{|\nabla u|^2}{u^2}$, the same conclusion too for $u^{-1/p}|\nabla(u^{1/p})|^2$. Finally, similar considerations show that $\Delta(u^{1/p})$ is also of polynomial growth locally uniformly in $t>0$.

3.4 Further remarks

3.4.1 Reverse hypercontractivity

A further appealing feature of our abstract argument in Section 3.3.1 is that it applies to exponents p and q in the setting of the reverse hypercontractivity inequality, thus providing a simple and unified approach to both forward and reverse forms. In general, we let U be given by

$$U(t,x) = \begin{cases} P_s[u(t,\cdot)^{1/p}]^q(x) & \text{if } p,q\neq 0,\\ P_s[e^{u(t,\cdot)}]^q(x) & \text{if } p=0, q\neq 0,\\ \log P_s[u(t,\cdot)^{1/p}](x) & \text{if } p\neq 0, q=0.\end{cases}$$

Then, at least in a formal sense, the following closure properties hold:

$$\begin{array}{lll} \partial_t u \geq Lu \;\Rightarrow\; \partial_t U \geq LU & & \text{for } 1<p<q<\infty, \text{ and for } -\infty<q<p<0,\\ \partial_t u \leq Lu \;\Rightarrow\; \partial_t U \leq LU & & \text{for } 0\leq q<p<1,\\ \partial_t u \leq Lu \;\Rightarrow\; \partial_t U \geq LU & & \text{for } -\infty<q<0\leq p<1. \end{array} \tag{3.20}$$

As a result, in each of the above cases, one may obtain the monotonicity of

$$Q(t) = \begin{cases} (\int U(t,\cdot)\,\mathrm{d}\mu)^{1/q} & \text{if } q\neq 0\\ \exp(\int U(t,\cdot)\,\mathrm{d}\mu) & \text{if } q=0\end{cases}$$

for solutions u of the diffusion equation $\partial_t u = Lu$ with nonnegative initial data.

The closure properties (3.20) for $q<p<1$ yield Borell's reverse form of the hypercontractivity inequality

$$\|P_s f\|_{L^q(\mu)} \geq \|f\|_{L^p(\mu)} \tag{3.21}$$

for positive functions.

In the special case of the Ornstein–Uhlenbeck semigroup, Christer Borell [32] observed that a unified approach to both forward and reverse hypercontractivity inequalities may be taken, whereby one first establishes a discrete "Boolean hypercontractivity inequality" (the forward form independently due to Aline Bonami [31] and Gross [67]) and then applies the central limit theorem. We note that our setting of a diffusion semigroup takes us outside the scope of discrete hypercontractivity inequalities.

3.4.2 Related work and extensions

As we have already alluded to, the closure properties perspective taken in [17], as well as the earlier work [16], gave inspiration to the present paper. In these papers one may find different examples of closure properties of supersolutions to certain diffusion equations in the context of euclidean spaces (with substantial influence from [41] and [24]).

We close the paper with a discussion on further related work in the literature and extensions to more general forms of hypercontractivity.

Yao-Zhong Hu [80] (see also [81]) has approached hypercontractivity, in dual form, by considering the quantity

$$\Lambda(t) = P_{T-t}[P_s[u(t,\cdot)^{1/p}] \cdot v(t,\cdot)^{1/q'}] \qquad (t \in [0,T])$$

where $u(t,\cdot) = P_t[f^p]$ and $v(t,\cdot) = P_t[g^{q'}]$ (here, and in what follows, we consider positive functions). It is shown in [80] that $\Lambda'(t) \geq 0$, yielding

$$P_T[P_s f \cdot g] \leq P_s[P_T[f^p]^{1/p}] \cdot P_T[g^{q'}]^{1/q'} \tag{3.22}$$

and, assuming ergodicity, taking the limit $T \to \infty$ yields the dual form of (3.13). As in the current paper, the approach taken by Hu is applicable in the abstract setting of a Markov semigroup which satisfies the diffusion and curvature conditions (3.9) and (3.10).

The inequality in (3.22) is reminiscent of so-called *local hypercontractivity inequalities*. In recent work of Bakry, Franoçis Bolley and Ivan Gentil in [6], it was shown that, for diffusion Markov semigroups, the curvature condition (3.10) is *equivalent* to the local hypercontractivity inequality

$$P_{T-s}[(P_s f)^q]^{1/q} \leq P_T[f^p]^{1/p},$$

where $s \in (0,T]$, $\infty > q > p > 1$ and

$$\frac{q-1}{p-1} = \frac{e^{2cT}-1}{e^{2c(T-s)}-1}.$$

As before, $c \in \mathbb{R}$ denotes the curvature constant from (3.10). Again, we see that the associated hypercontractivity inequality (3.13) follows by taking $T \to \infty$. We also remark that, much earlier in their fundamental paper [7], Bakry and Michel Émery established an analogous local form of the log-Sobolev inequality using a heat flow monotonicity argument.

In [80], Hu actually obtained a rather general form of hypercontractivity

$$\int_E P_s[\phi(f)](x)\psi(g(x))\,\mathrm{d}\mu(x) \leq \phi\Big(\int_E f\,\mathrm{d}\mu\Big)\psi\Big(\int_E g\,\mathrm{d}\mu\Big) \tag{3.23}$$

where ϕ, ψ are (sufficiently nice) nonnegative concave functions such that

$$(e^{-cs}\phi'(\sigma)\psi'(\tau))^2 \leq \phi(\sigma)\phi''(\sigma)\psi(\tau)\psi''(\tau) \tag{3.24}$$

for nonnegative σ, τ. One can easily verify that equality holds in (3.24) under the assumption $e^{2cs} = \frac{q-1}{p-1}$ when $\phi(\sigma) = \sigma^{1/p}$ and $\psi(\tau) = \tau^{1/q'}$, thus recovering the dual form of (3.13). The argument in [80] leading to (3.23) proceeds as described above for power-type ϕ and ψ, with the obvious modifications to the functional Λ.

We remark that our method too extends in a similar manner. Assume

$$e^{-2cs}\phi'(\sigma)^2\theta(\tau)\theta''(\tau) \geq \theta'(\tau)^2\phi(\sigma)\phi''(\sigma) \tag{3.25}$$

for nonnegative σ and τ, where ϕ and θ are (sufficiently nice) nonnegative, increasing and concave functions. Let

$$U(t,x) = \theta^{-1}(P_s[\phi(u(t,\cdot))])(x),$$

where u is positive and satisfies $\partial_t u \geq Lu$. Exactly as in our argument in Section 3.3.1, we quickly obtain

$$\partial_t U - LU \geq \theta'(\tau)^{-3}\big(\theta''(\tau)\Gamma(\theta(\tau)) - \theta'(\tau)^2 P_s[\phi''(u)\Gamma(u)]\big),$$

where $\theta(\tau) = P_s[\phi(u)]$. Using Lemma 3.2.7, the Cauchy–Schwarz inequality, and $\Gamma(\phi(u)) = \phi'(u)^2\Gamma(u)$, it follows that

$$\Gamma(\theta(\tau)) \leq -e^{-2cs}P_s[\phi''(u)\Gamma(u)]P_s\left[\frac{(\phi'(u))^2}{-\phi''(u)}\right].$$

Thus, using assumption (3.25), we may deduce $\partial_t U \geq LU$. As a result of this closure property, taking $u = P_t f$, we generate a generalised form of hypercontractivity

$$\mathbb{P}_\theta(P_s f) \leq \mathbb{P}_\phi(f) \tag{3.26}$$

where $\mathbb{P}_\theta(f) := \theta(\int_E \theta^{-1}(f)\,\mathrm{d}\mu)$. An advantage of our approach in the present paper is the avoidance of duality and it is conceivable that this may be fruitful in certain contexts.

One may readily verify that (3.25) holds if we take $\phi(\sigma) = \sigma^{1/p}$ and $\theta(\tau) = \tau^{1/q}$, under the condition $e^{2cs} = \frac{q-1}{p-1}$, and consequently we may view (3.26) as a generalisation of (3.13). Moreover, by a generalised form of Hölder's inequality we may obtain (3.23) from (3.26). To see this, note that

$$\int_E J(f(x), g(x))\,\mathrm{d}\mu(x) \leq J\bigg(\int_E f\,\mathrm{d}\mu, \int_E g\,\mathrm{d}\mu\bigg) \tag{3.27}$$

holds for concave functions J which are nondecreasing in each variable[3]. With $J(\sigma,\tau) = \theta(\sigma)\psi(\tau)$, the concavity condition becomes

$$\frac{\theta''(\sigma)\theta(\sigma)}{\theta'(\sigma)^2} \geq \frac{\psi'(\tau)^2}{\psi''(\tau)\psi(\tau)} \tag{3.28}$$

[3]This seems to be a fact of folklore type; a proof (based on a closure property of supersolutions to heat equations) may be found in [17]. Working in a probability measure space, a result of Janusz Matkowski [109] asserts that concavity of J is almost always necessary for (3.27).

for all σ, τ. Setting $\lambda = \inf_\tau \frac{\psi'(\tau)^2}{-\psi''(\tau)\psi(\tau)}$ and taking $\theta(\sigma) = \sigma^{\frac{1}{\lambda+1}}$, we see that (3.28) holds. It is also clear that (3.24) implies (3.25). Hence, combining (3.27) and (3.26), we deduce (3.23).

Reminiscent of (3.27), Hu's generalised form of (dualised) hypercontractivity (3.23) was extended in a different sense by Ledoux [103] to the setting

$$\int_E \int_E J(f(x), g(y))\, \mathrm{d}\nu_{s,y}(x)\mathrm{d}\mu(y) \leq J\bigg(\int_E f\, \mathrm{d}\mu, \int_E g\, \mathrm{d}\mu \bigg) \tag{3.29}$$

for J satisfying a so-called ρ-concavity condition, where $\rho = e^{-cs}$. Here, ρ-concavity refers to the semi-negative definiteness of the matrix

$$\begin{pmatrix} \partial_{11} J & \rho\partial_{12} J \\ \rho\partial_{12} J & \partial_{22} J \end{pmatrix}$$

and, rather interestingly, at least in the case of the Ornstein–Uhlenbeck semigroup, e^{-s}-concavity is *necessary* as well as sufficient (see [103]). Clearly the case $J(\sigma, \tau) = \phi(\sigma)\psi(\tau)$ reduces (3.29) to (3.23), and e^{-cs}-concavity to (3.24). Ledoux's approach was based on flowing the input functions f and g under the semigroup P_t and deriving monotonicity of the left-hand side of (3.29); the argument more closely resembles those in [41] and [24] rather those in the current paper.

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