Cylinders in normal surfaces over algebraically non-closed fields with an application to cylindricity of del Pezzo fibrations

(非代数閉体上に定義された曲面内のシリンダーとその del Pezzo ファイブレーションへの応用)

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Chapter 1

Introduction

1.1 Anti-canonical polar cylinders in del Pezzo surfaces

Let k be a field (this is not necessarily algebraically closed), and let X be an algebraic variety defined over k. An open subset U of X is called an $\mathbb{A}^r_k$-*cylinder* if U is isomorphic to $\mathbb{A}^r_k \times Z$ for some variety Z over k. When the rank r of cylinder U is not important, U is just said to be a *cylinder*.

Certainly, cylinders are geometrically simple objects, however, they are known to have a variety of applications. As an example, there exists its application to unipotent group actions on affine cones over polarized varieties. In order to explain it, we shall define polarized cylinders in normal projective varieties as follows:

Definition 1.1.1. Let k be an algebraically closed field of characteristic zero, let X be a normal projective variety over k, let H be an ample $\mathbb{Q}$-divisor on X, and let $U \simeq \mathbb{A}^1_k \times Z$ be a cylinder in X such that Z is affine. Then we say that U is an H-*polar cylinder* if there exists an effective $\mathbb{Q}$-divisor D on X such that $D \sim_{\mathbb{Q}} H$ and $U = X \backslash \mathrm{Supp}(D)$.

The following theorem plays an important role in connecting polarized cylinders in projective varieties and unipotent group actions on affine cones:

Theorem 1.1.2 ([37, 40])**.** Let k be an algebraically closed field of characteristic zero, let X be a normal projective variety over k and H be an ample $\mathbb{Q}$-divisor on X. Then X contains an H-polar cylinder if and only if the affine cone:

$$\mathrm{Cone}_H(X) := \mathrm{Spec}\left(\bigoplus_{i \geq 0} H^0(X, \mathscr{O}_X(iH))\right)$$

admits an effective $\mathbb{G}_a$-action.

We will give the application using Theorem 1.1.2 later. To do so, we consider the existing condition of anti-canonical polar cylinders in del Pezzo surfaces. In what follows, let S be a del Pezzo surface defined over an algebraically closed field of characteristic zero. In other words, S is a normal projective surface such that its anti-canonical divisor $-K_S$ is ample.

If S is smooth, then the existing condition of $(-K_S)$-polar cylinders is given by [37, 40, 11]. More precise, [37] presents that S contains an $(-K_S)$-polar cylinder provided that this degree $d := (-K_S)^2$ is more than or equal to 4. Furthermore, [40] shows that S does not contain any $(-K_S)$-polar cylinder if $d \leq 2$. Finally, [11] proves that S does not contain any $(-K_S)$-polar cylinder if $d = 3$. Thus, their results can be summarized the following theorem:

Theorem 1.1.3 ([37, 40, 11])**.** Let S be a smooth del Pezzo surface over an algebraically closed field of characteristic zero and let d be the degree of S, i.e., $d = (-K_S)^2 \in \{1, \dots, 9\}$. Then S contains an $(-K_S)$-polar cylinder if and only if $d \geq 4$.

Incidentally, it is also known that the existing condition of $(-K_S)$-polar cylinders provided that S has at most Du Val singularities.

Theorem 1.1.4 ([12])**.** Let S be a del Pezzo surface with at most Du Val singularities over an algebraically closed field of characteristic zero and let d be the degree of S, i.e., $d = (-K_S)^2 \in \{1, \dots, 9\}$. Then S does not contain any $(-K_S)$-polar cylinder if and only if one of the following three conditions holds:

- $d = 3$ and S allows no singular point;
- $d = 2$ and S allows only singular points of type A_1;
- $d = 1$ and S allows only singular points of types A_1, A_2, A_3, D_4.

This paper in Chapter 4 will consider cylinders in del Pezzo surfaces of rank one with at most Du Val singularities by using the result and some ideas of Theorem 1.1.4. Hence, Theorem 1.1.4 is the very important result in this paper.

Now, in order to explain an application of cylinders to unipotent group action, we state the following problem:

Problem 1.1.5 ([23])**.** Let V be the 3-dimensional affine variety defined over the complex number field $\mathbb{C}$ defined by:

$$V := (x^3 + y^3 + z^3 + w^3 = 0) \subseteq \mathbb{A}^4_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x, y, z, w]).$$

Then does V admit an effective $\mathbb{G}_a$-action?

Certainly, Problem 1.1.5 is a purely algebraic problem, however, this problem was solved by the geometric approach for the cylinder. We shall present this outline.

Let S be the cubic hypersurface in $\mathbb{P}^3_{\mathbb{C}}$ over $\mathbb{C}$ defined by:

$$S := (x^3 + y^3 + z^3 + w^3 = 0) \subseteq \mathbb{P}^3_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x, y, z, w]).$$

Then we notice that S is a smooth del Pezzo surface of degree 3, so that S contains never $(-K_S)$-polar cylinder by Theorem 1.1.3. Hence, the affine cone $\mathrm{Cone}_{(-K_S)}(S)$ admits no effective $\mathbb{G}_a$-action by Theorem 1.1.2. In particular, Problem 1.1.5 is false by virtue of:

$$\mathrm{Cone}_{(-K_S)}(S) \simeq (x^3 + y^3 + z^3 + w^3 = 0) \subseteq \mathbb{A}^4_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x, y, z, w]).$$

Therefore, cylinders in normal projective varieties receive a lot of attention recently from the viewpoint of unipotent group actions on affine cones over polarized varieties.

At the end of this section, we present several remarks on polarized cylinders:

Remark 1.1.6. Let S be a del Pezzo surface with at most Du Val singularities over an algebraically closed field of characteristic zero. If S is smooth, then it always contains an H-polar cylinder for some an ample $\mathbb{Q}$-divisor H on S. However, this does not hold unless S is smooth (see [5], [13, §3]).

Remark 1.1.7. Although not used in this paper, partial results are also known for the existing condition of polarized cylinders in smooth rational surfaces, which is an extension of Theorem 1.1.3. In other words, letting S be a smooth rational surface with $(-K_S)^2 \geq 3$ over an algebraically closed field of characteristic zero, for any ample $\mathbb{Q}$-divisor H on S except for $H \in \mathbb{Q}_{>0}[-K_S]$ if $(-K_S)^2 = 3$, there exists an H-polar cylinder in S (see [13, 53]).

Remark 1.1.8. In this section, we have mainly treated the application of polarized cylinders to unipotent group action. However, there are several other known applications for polarized cylinders. For example, Fano varieties containing anti-canonical polar cylinders also receive attention recently since the α-invariant of these varieties is strictly less than 1 (see [10, Theorem 1.26]).

1.2 Cylinders in Mori fiber spaces

The importance of finding cylinders in projective varieties is treated in the previous section 1.1. In this section, we thus discuss cylinders in higher dimensional normal projective varieties defined over the complex number field $\mathbb{C}$. For example, the classification of Fano threefolds of rank one containing the 3-dimensional affine space $\mathbb{A}^3_{\mathbb{C}}$ is known (see [25, 26]). Moreover, some examples of Fano threefolds and Fano fourfolds containing a cylinder are also known (see, e.g., [10, 37, 39, 60, 61, 62]). However, in general, it is not easy to decide whether a given projective variety contains a cylinder.

Now, we shall consider how to find cylinders in a projective variety X' by using the minimal model program (MMP, for short). Assume that X' contains a cylinder. A resolution of singularities of X' still contains a cylinder, in particular, its canonical divisor is not pseudo-effective. Then by virtue of [6, Corollary 1.3.3], X' is birational to a suitable Mori fiber space (MFS, for short) $f : X \to Y$ by means of minimal model program (MMP, for short). Conversely, assuming that a normal projective variety X' admits a process of MMP $\theta : X' \dashrightarrow X$ is MFS which contains a cylinder, it follows that so does the initial X' by [20, Lemma 9]. Thus, in some sense, it is important and essential to try to find cylinders contained in MFS. In this paper, as a special and ideal situation, we shall consider finding a vertical $\mathbb{A}^s_{\mathbb{C}}$-cylinder with respect to MFS $f : X \to Y$ over $\mathbb{C}$, where vertical cylinders are defined as follows:

Definition 1.2.1 ([19])**.** Let $f : X \to Y$ be a dominant projective morphism of relative dimension $s \geq 1$ defined over $\mathbb{C}$. For an integer r with $1 \leq r \leq s$, an $\mathbb{A}^r_{\mathbb{C}}$-cylinder $U \simeq \mathbb{A}^r_{\mathbb{C}} \times Z$ in X is called a *vertical* $\mathbb{A}^r_{\mathbb{C}}$*-cylinder* with respect to f if there exists a morphism $g : Z \to Y$ (of relative dimension $s - r$) such that the restriction of f to U coincides with $g \circ pr_Z$.

It is known the following lemma about the existing condition of vertical cylinders:

Lemma 1.2.2 ([19, Lemma 3])**.** Let $f : X \to Y$ be a dominant projective morphism of relative dimension $s \geq 1$ defined over $\mathbb{C}$, and let r be an integer with $1 \leq r \leq s$. Then f admits a vertical $\mathbb{A}^r_{\mathbb{C}}$-cylinder if and only if the generic fiber X_η, which is defined over the function field $\mathbb{C}(Y) = \mathbb{C}(\eta)$ of the base variety, contains an $\mathbb{A}^r_{\mathbb{C}(Y)}$-cylinder.

Let $f : X \to Y$ be a MFS over $\mathbb{C}$. Notice that the generic fiber X_η of f is a Fano variety of rank one defined over $\mathbb{C}(Y)$. In particular, the dimension of X_η over $\mathbb{C}(Y)$ is less than that of X unless Y is a point. However, the base field $\mathbb{C}(Y)$ is not algebraically closed unless Y is a point. Hence, in order to find a vertical cylinder with respect to MFS, the following problem is essential to consider:

Problem 1.2.3. Let V be a normal Fano variety of rank one defined over a field k of characteristic zero (this is not necessarily algebraically closed), and let r be an integer with $1 \leq r \leq \dim_k(V)$. In which case does V contain an $\mathbb{A}^r_k$-cylinder?

Let V be a Fano variety of rank one over a field k of characteristic zero. In the case of $\dim_k(V) = 1$, the above problem is quite easy. Indeed, V contains an $\mathbb{A}^1_k$-cylinder if and only if V has a k-rational point. In case of $\dim_k(V) = 2$, then V is a del Pezzo surface. Dubouloz and Kishimoto provided the following theorem about the existing condition of smooth del Pezzo surfaces of rank one:

Theorem 1.2.4 ([19, Theorem 1])**.** Let k be a field of characteristic zero, let S be a smooth del Pezzo surface of rank one over k and let d be the degree of S, i.e., $d = (-K_S)^2 \in \{1, \ldots 6, 8, 9\}$. Then:

(1) S contains an $\mathbb{A}^1_k$-cylinder if and only if $d \geq 5$ and S has a k-rational point.

(2) S contains an $\mathbb{A}^2_k$-cylinder if and only if $d \geq 8$ and S has a k-rational point.

Remark 1.2.5. Any $\mathbb{A}^2_k$-cylinder in a surface is clearly isomorphic to the affine plane $\mathbb{A}^2_k$.

Remark 1.2.6. Let S be a del Pezzo surface of rank one (this is not necessarily smooth) over a field k of characteristic zero. Then $U_{\overline{k}}$ is an $(-K_{S_{\overline{k}}})$-polar cylinder for any cylinder U in S (if it exists) since $\mathrm{Cl}(S)_{\mathbb{Q}}$ is generated by only $-K_S$. In particular, in Theorem 1.2.4, we immediately see that any smooth del Pezzo surface of rank one with degree ≤ 3 contains no cylinder (but the case of degree 4 is not easy).

Remark 1.2.7. Although not used in this paper, partial results in the case of dimension 3 in Problem 1.2.3 are also known (see [21, 51]).

In this paper, we shall extend Theorem 1.2.4 to the singular case. In other words, we will give a partial answer to Problem 1.2.3 when V is of dimension 2 and singular. We explain the details in the next section.

1.3 Main results

In this paper, we shall mainly give the existing condition of cylinders in normal surfaces over algebraically non-closed fields. Furthermore, we apply these results to the following fibrations:

Definition 1.3.1. Let $f : X \to Y$ be a dominant projective morphism of relative dimension two between normal varieties defined over the complex number field $\mathbb{C}$. Then:

(1) f is a *weak del Pezzo fibration* if the total space X has only $\mathbb{Q}$-factorial terminal singularities and the generic fiber X_η of f is a weak del Pezzo surface, which is minimal over the rational function field $\mathbb{C}(Y)$.

(2) f is a *generically canonical* (resp. *klt*, *lc*) *del Pezzo fibration* if the generic fiber X_η of f is a Du Val (resp. a log, an lc) del Pezzo surface of rank one over the rational function field $\mathbb{C}(Y)$.

Remark 1.3.2. We present two remark about Definition 1.3.1:

(1) Note that a del Pezzo fibration means a MFS of relative dimension two normal varieties defined over $\mathbb{C}$. Hence, letting $f : X \to Y$ be a weak del Pezzo fibration over $\mathbb{C}$, f is a del Pezzo fibration if and only if the generic fiber X_η of f is of rank one over $\mathbb{C}(Y)$.

(2) In Definition 1.3.1 (1), we may omit the assumption that X has only $\mathbb{Q}$-factorial terminal singularities. Indeed, we will not use this assumption in this paper.

The author studied the existing condition of vertical cylinders with respect to fibrations as in Definition 1.3.1 ([65, 66, 67]). Then his results are summarized in the following Table:

Singularities	Terminal	Canonical	Log terminal	Log canonical
Vertical $\mathbb{A}^1$-cylinder	[19]	[66]	?	?
Vertical $\mathbb{A}^2$-cylinder	[19]	[67]	[67]	[67]

In what follows, we detailly state the author's results in three subsections separately.

1.3.1 Cylinders in weak del Pezzo fibrations

Notice that Du Val del Pezzo surfaces have a one-to-one correspondence to weak del Pezzo surfaces via minimal resolutions (see Subsection 2.4.2). Hence, in order to consider Du Val del Pezzo surfaces of rank one containing a cylinder defined over algebraically non-closed fields, we shall study minimal weak del Pezzo surfaces containing a cylinder defined over algebraically non-closed fields. Indeed, this consideration is necessary to determine the existing condition of cylinders in Du Val del Pezzo surfaces of rank one over algebraically non-closed fields (for details, see Section 4.4). As the results, we obtain the following two Theorems 1.3.3 and 1.3.4:

Theorem 1.3.3 ([65, Theorem 1.6]). Let k be a field of characteristic zero, let $\widetilde{S}$ be a weak del Pezzo surface, whose $-K_S$ is not ample, defined over k and let d be the degree of $\widetilde{S}$, i.e., $d := (-K_{\widetilde{S}})^2$. Then $\widetilde{S}$ is minimal over k if and only if $\rho_k(\widetilde{S}) = 2$ and the type of $\widetilde{S}$ is one of the following (for the definition of the type of $\widetilde{S}$, see Section 2.4.2):

- $d = 8$ and A_1-type;
- $d = 4$ and $(2A_1)_<$-type;
- $d = 2$ and A_1, A_2 or $(4A_1)_>$-type;
- $d = 1$ and $2A_1$ or $2A_2$-type.

Theorem 1.3.4 ([65, Theorem 1.7]). Let k be a field of characteristic zero, let $\widetilde{S}$ be a minimal weak del Pezzo surface with $\rho_k(\widetilde{S}) > 1$ defined over k and let d be the degree of $\widetilde{S}$, i.e., $d := (-K_{\widetilde{S}})^2$. Then the following assertions hold:

(1) $\widetilde{S}$ contains an $\mathbb{A}^1_k$-cylinder if and only if $d = 8$ and $\widetilde{S}$ is endowed with a structure of Mori conic bundle admitting a section defined over k.

(2) $\widetilde{S}$ contains the affine plane $\mathbb{A}^2_k$ if and only if $d = 8$ and $\widetilde{S}$ has a k-rational point.

We shall present some corollaries of Theorem 1.3.4. Let the notation be the same as in Theorem 1.3.4, assume further that $d = 8$. At first, consider the case that $-K_{\widetilde{S}}$ is not ample, i.e., $\widetilde{S}$ is a k-form of the Hirzebruch surface $\mathbb{F}_2$ of degree 2. Then any Mori conic bundle $\pi : S \to B$ such that $\pi_{\overline{k}}$ admits the minimal section, which is defined over k. Hence, we obtain:

Corollary 1.3.5 ([65, Corollary 4.5]). Let the notation be the same as in Theorem 1.3.4, assume further that $d = 8$. If $-K_{\widetilde{S}}$ is not ample, then $\widetilde{S}$ always contains an $\mathbb{A}^1_k$-cylinder.

Next, consider the case that $k = \mathbb{R}$. It is known the classification of smooth real del Pezzo surfaces ([63]). In particular, we know that any weak del Pezzo surface, which is an $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, of rank two is always endowed with a structure of Mori conic bundle admitting a section defined over $\mathbb{R}$ ([63], see also [47, Lemma 3.2]). By this fact and Corollary 1.3.5, we obtain:

Corollary 1.3.6 ([65, Example 4.6]). Let the notation be the same as in Theorem 1.3.4, assume further that $d = 8$. If $k = \mathbb{R}$, then $\widetilde{S}$ always contains an $\mathbb{A}^1_{\mathbb{R}}$-cylinder.

Incidentally, consider the case that k is a C_1-field. Hence, $\widetilde{S}$ always has k-rational point by virtue of [29, Theorem 3.12], so that we obtain:

Corollary 1.3.7. Let the notation be the same as in Theorem 1.3.4, assume further that $d = 8$. If k is a C_1-field, then $\widetilde{S}$ always contains the affine plane $\mathbb{A}^2_k$.

Finally, by using Theorem 1.3.4 and Corollaries 1.3.5 and 1.3.7 combined with Lemma 1.2.2, we obtain the following corollary about the existing condition of vertical cylinders with respect to weak del Pezzo fibrations:

Corollary 1.3.8 ([65, Corollaries 1.8 and 1.9])**.** Let $f : X \to Y$ be a weak del Pezzo fibration over $\mathbb{C}$, let X_η be the generic fiber of f, and d be the degree of f, i.e., $d := (-K_{X_\eta})^2$. Then we have the following assertions:

(1) If f is not a del Pezzo fibration, i.e., $\rho_{\mathbb{C}(Y)}(X_\eta) > 1$, then we obtain:
- $d = 1, 2, 4$ or 8;
- f admits a vertical $\mathbb{A}^1_{\mathbb{C}}$-cylinder if and only if $d = 8$ and X_η is endowed with a structure of Mori conic bundle admitting a section defined over $\mathbb{C}(Y)$;
- f admits a vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinder if and only if $d = 8$ and X_η has a $\mathbb{C}(Y)$-rational point.

(2) If X is a threefold, which is the equivalence that Y is a curve, and $d = 8$, then f always admits a vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinder.

In Corollary 1.3.8 (2), we note that this assertion follows form Corollary 1.3.7 and the Tsen's theorem (see, e.g., [54, Lemma 12.3.1]). In other words, for a dominant morphism $f : X \to C$ over $\mathbb{C}$, the function field $\mathbb{C}(C)$ is a C_1-field if C is a curve.

1.3.2 Cylinders in canonical del Pezzo fibrations

Let S be a Du Val del Pezzo surface of rank one over a field k of characteristic zero. As a generalization of Theorem 1.2.4 in the sense of singularities, we will consider the existing condition under which S contains a cylinder. Based on previous works (Theorems 1.1.4 and 1.2.4), it may seem that this condition can be only determined by the degree of S, the existence of k-rational points on S, and singularity type on $S_{\overline{k}}$. However, as Theorem 1.3.9 shows, we know that the treatment of singularity type on Du Val del Pezzo surfaces of rank one actually turns out to be very subtle. For instance, even in the case that two Du Val del Pezzo surfaces are of rank one over a field k of characteristic zero whose base extensions over $\overline{k}$ are mutually isomorphic, it may be that exactly only one of them contains a cylinder (see Example 1.3.10).

Theorem 1.3.9 ([66, Theorems 1.4, 1.5 and 1.6])**.** Let k be a field of characteristic zero, let S be a Du Val del Pezzo surface of rank one defined over k and let d be the degree of S, i.e., $d := (-K_S)^2$. Then[*1]:

(1) In case of $d \geq 5$, then S always contains a cylinder.

(2) In case of $d = 3$ or 4, then S contains a cylinder if and only if $S_{\overline{k}}$ has a singular point defined over k, which is not of type A_1^{++} over k.

(3) In case of $d = 1$ or 2, then:

  (i) If $d = 2$ (resp. $d = 1$) and $S_{\overline{k}}$ has a singular point of type A_6, A_7, D_n or E_n (resp. type A_8, D_6, D_7, D_8, E_7 or E_8), then S contains a cylinder.

[*1] For the singularities notation, see Section 4.1.

(ii) If $d = 2$ (resp. $d = 1$) and $S_{\overline{k}}$ has a singular point of type $(A_5)''$ (resp. type $(A_7)''$)[*2], say x, then S contains a cylinder if and only if x is not of type A_5^{++} (resp. type A_7^{++}) over k.

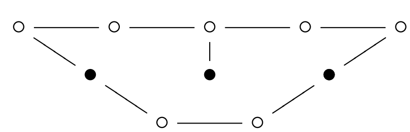
(iii) If $d = 2$ (resp. $d = 1$) and $S_{\overline{k}}$ allows only singular points of type A_1 (resp. types A_1, A_2, A_3 and D_4), then S does not contain any cylinder.

(iv) If S does not satisfy any condition on singularities of (i), (ii) and (iii) above, then S contains a cylinder if and only if $S_{\overline{k}}$ has a singular point defined over k, which is of type A_n^-, D_n^- or E_n^- over k.

Example 1.3.10 ([66, Example 6.4])**.** Let S_n be the quadratic hypersurface in the weighted projective space $\mathbb{P}(1,1,1,2)$ over a rational function field $\mathbb{C}(t)$ defined by:

$$S_n := (t^n w^2 + x^2 z^2 + x z^3 = 0) \subseteq \mathbb{P}(1,1,1,2) = \mathrm{Proj}(\mathbb{C}(t)[x,y,z,w]),$$

where $n \in \mathbb{Z}$. Then S_n is a del Pezzo surface of degree 2. Moreover, $S_{n,\overline{\mathbb{C}(t)}}$ has exactly two singular points $p_1 := [0:0:1:0]$ and $p_2 := [1:0:0:0]$ in $\mathbb{P}(1,1,1,2)$ of types A_2 and $(A_5)'$, respectively. In particular, p_1 and p_2 are k-rational, and $S_{n,\overline{\mathbb{C}(t)}}$ is of rank one, namely, so is S_n. Then we know that S_n contains a cylinder if and only if n is even. Indeed, this fact can be shown as follows: Let $\sigma : \widetilde{S}_n \to S_n$ be the minimal resolution over $\mathbb{C}(t)$. Then we see that $\widetilde{S}_{n,\overline{\mathbb{C}(t)}}$ contains some reduced curves, whose union is defined over k, corresponding to the following weighted dual graph (see Example 4.3.11):



Here "$\bullet$" and "$\circ$" mean a (-1)-curve and a (-2)-curve on $\widetilde{S}_{n,\overline{\mathbb{C}(t)}}$, respectively. By Theorem 1.3.9 (4)(iv) combined with above the weighted dual graph, S contains a cylinder if and only if p_1 is of type A_2^- over $\mathbb{C}(t)$. By easy computation, we see that the local equation of exceptional set of $p_1 \in S_n$ can be written $t^n u^2 + v^2 \in \mathbb{C}(t)[u,v]$ for some regular parameters u and v. Note that p_1 is of type A_2^- over $\mathbb{C}(t)$ on S_n if and only if $t^n u^2 + v^2$ is reducible over $\mathbb{C}(t)[u,v]$. In particular, this is equivalent that n is even.

Let $f : X \to Y$ be a generically canonical del Pezzo fibration over $\mathbb{C}$ and let X_η be the generic fiber of f. By virtue of Theorem 1.3.9 combined with Lemma 1.2.2, we can give a condition under which f admits a vertical $\mathbb{A}^1_{\mathbb{C}}$-cylinder depending on degree of f, i.e., $(-K_{X_\eta})^2$, and singularities in X_η over $\mathbb{C}(Y)$.

1.3.3 Compactification of the affine plane over non-closed fields

Let k be a field of characteristic zero. As the next target of Theorem 1.3.9, we shall deal with the following problems:

Problem 1.3.11. In which case does a del Pezzo surface of rank one with at most Du Val singularities (and more generally, with singularities worse than Du Val singularities) over k contain the affine plane $\mathbb{A}^2_k$?

[*2]Note that a singular point of type $(A_5)''$ (resp. $(A_7)''$) on a Du Val del Pezzo surface of degree 2 (resp. 1) admits at most one point and is automatically k-rational.

In order to give the solution to the above problem, we consider compactifications of the affine plane into lc del Pezzo surfaces of rank one over k. In the case of $k = \overline{k}$, [56] classifies compactifications of the affine plane into Du Val del Pezzo surfaces of rank one, furthermore, [42, 45] classify compactifications of the affine plane into lc del Pezzo surfaces of rank one. More precisely, [42] and [45] give the classification of compactifications of the affine plane into log del Pezzo surfaces and lc del Pezzo surfaces of rank one over $\mathbb{C}$, respectively. Hence, we shall generalize their reworks to the case where the base field is of characteristic zero without assuming that it is algebraically closed. As a result, we obtain the following theorem:

Theorem 1.3.12 ([67])**.** Let (S, Δ) be an lc compactification of the affine plane over a field k of characteristic zero (see Definition 5.1.2, for this definition), let $\sigma : \widetilde{S} \to S$ be the minimal resolution and let $\widetilde{\Delta}$ be the reduced effective divisor on $\widetilde{S}$ defined by $\widetilde{\Delta} := \sigma^*(\Delta)_{\mathrm{red.}}$. Then we have the following:

(1) $\widetilde{\Delta}_{\overline{k}}$ is an SNC-divisor.

(2) The following three assertions about the number of singularities hold:

  (i) $\sharp\mathrm{Sing}(S) \geq 1$. In other words, $S_{\overline{k}}$ has a singular point, which is k-rational.

  (ii) $\sharp\mathrm{Sing}(S) \leq 2$. Moreover, $\sharp\mathrm{Sing}(S) = 2$ if and only if $\sharp\mathrm{Sing}(S_{\overline{k}}) = 2$.

  (iii) $\sharp\mathrm{Sing}(S_{\overline{k}}) = 1$ or $\rho_{\overline{k}}(S_{\overline{k}}) + 1$.

(3) In case of $\rho_{\overline{k}}(S_{\overline{k}}) > 1$, the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is one of the graphs (1)–(52) in Appendix A.2.

Remark 1.3.13. In Theorem 1.3.12, if $\rho_{\overline{k}}(S_{\overline{k}}) = 1$, then the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is classified (see [42, Appendix C] and [45, Fig. 1]).

By applying Theorem 1.3.12, we can determine the condition whether lc del Pezzo surfaces of rank one contain the affine plane $\mathbb{A}^2_k$ as follows:

Theorem 1.3.14 ([67])**.** Let k be a field of characteristic zero, let S be an lc del Pezzo surface of rank one defined over k such that $\mathrm{Sing}(S_{\overline{k}}) \neq \emptyset$, and let $\sigma : \widetilde{S} \to S$ be the minimal resolution over k. Then the following are equivalent:

(A) S contains the affine plane $\mathbb{A}^2_k$.

(B) There exists a reduced effective divisor $\widetilde{\Delta}$ on $\widetilde{S}$ such that the exceptional locus of σ is included in $\mathrm{Supp}(\widetilde{\Delta})$, any irreducible component of $\widetilde{\Delta}_{\overline{k}}$ is a rational curve and the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is one of the graphs in [42, Appendix C], [45, Fig. 1] or Appendix A.2.

Now, we shall focus on Du Val del Pezzo surfaces. By Theorem 1.3.14, we can determine the condition whether Du Val del Pezzo surfaces contain the affine plane $\mathbb{A}^2_k$ depending only on degrees and singularity types as follows:

Theorem 1.3.15 ([67])**.** Let k be a field of characteristic zero, let S be a Du Val del Pezzo surface of rank one defined over k such that $\mathrm{Sing}(S_{\overline{k}}) \neq \emptyset$, and let d be the degree of S, i.e., $d := (-K_S)^2$. Then S contains the affine plane $\mathbb{A}^2_k$ if and only if one of the following two conditions holds:

- $d = 8$ and S contains a singular point of type A_1^+ over k (see Section 4.1, for this notation);
- $d = 5$ or 6;

- $d \leq 4$ and the pair of degree d and singularity type of $S_{\overline{k}}$ is one of the following:

$$(4, D_5),\ (4, D_4),\ (4, A_2 + 2A_1),\ (4, A_2),\ (3, E_6),\ (3, D_4),\ (2, E_7),\ (2, E_6),\ (2, A_6),\ (1, E_8).$$

Remark 1.3.16. We state some remarks of Theorem 1.3.15:

(1) In the case of $k = \overline{k}$, it is known that the condition for a Du Val del Pezzo surface of rank one to contain the affine plane can be determined only by the singularities type (see, e.g., [56]). However, it is not true unless k is algebraically closed (see Example 5.6.4).

(2) Notice that the "if part" in Theorem 1.3.15 also follows from arguments of the proof of Theorem 1.3.9 (see Remarks 4.2.8 and 4.4.6). In other words, Theorem 1.3.15 particularly asserts that the "only if part" in Theorem 1.3.15 is also true.

By Theorem 1.3.14 combined with Lemma 1.2.2, we can give a condition under which any generically lc del Pezzo fibration $f : X \to Y$ over $\mathbb{C}$ admits a vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinder. In particular, by virtue of Theorem 1.3.15 combined with Lemma 1.2.2, the existing condition of vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinders with respect to a generically canonical del Pezzo fibration $f : X \to Y$ over $\mathbb{C}$ can be determined only degree and singularities of $X_{\eta,\overline{\mathbb{C}(Y)}}$, where X_η is the generic fiber of f. For details, see Subsection 5.5.2.

1.4 Organization of this paper

In Chapter 2, we shall summarize basic properties. More precisely, we review five topics in this chapter as follows. In Section 2.1, we deal with basic properties on weak del Pezzo surfaces defined over algebraically closed fields. In Section 2.2, we treat some facts about algebraic varieties over algebraically non-closed fields. In particular, we recall Galois actions, forms of the projective spaces, which are called the *Severi-Brauer varieties*, and the classification of the minimal model of smooth projective surfaces over a field of characteristic zero. In Section 2.3, we define terminal, canonical, log terminal and log canonical singularities. Moreover, we recall the properties of singularities on normal algebraic surfaces over an algebraically closed field of characteristic zero. In Section 2.4, we provide an overview of the classification of weak del Pezzo surfaces. In Section 2.5, we prepare some useful facts about cylinders in normal projective surfaces over a field of characteristic zero. Moreover, we also introduce the variant of Corti's inequality since this inequality is used to prove a few facts.

In Chapter 3, we prove Theorems 1.3.3 and 1.3.4. In Section 3.1, we show that any minimal del Pezzo surface of rank two and of degree ≤ 4 defined over a field of characteristic zero is endowed with the structure of two Mori conic bundles. In Section 3.2, we classify minimal weak del Pezzo surfaces. As a corollary, for any minimal weak del Pezzo surfaces of rank two and of degree d, we obtain either $d = 8$ or $d \leq 4$. In Section 3.3, we determine the existing condition of cylinders in a minimal weak del Pezzo surface $\widetilde{S}$ over a field of characteristic zero. This proof will be divided according to the degree d, more precisely the case of $d = 8$ and the case of $d \leq 4$, separately. In the case of $d = 8$, we show that $\widetilde{S}$ contains a cylinder by using the property of $\mathbb{A}^1$-bundle. In the case that $d \leq 4$, we show that $\widetilde{S}$ does not contain any cylinder by using a variant of Corti's inequality combined with facts in Section 3.1.

In Chapter 4, we prove Theorem 1.3.9. In other words, we determine the existing condition of cylinders in any Du Val del Pezzo surface S of rank one over a field of characteristic zero. In Section 4.1, in order to precisely state our main result in this chapter, we prepare the more detailed notation on Du Val singularities. Note that the proof of our main result will be divided according to the degree d, more precisely the case of $d \geq 3$ and the case of $d \leq 2$, separately. In Section 4.2, we treat the case of $d \geq 3$. However, the case of $d \leq 2$ is a little tricky. Thus, in Section 4.3, we prepare some results. In Section 4.4, we treat the case of $d \leq 2$ by using facts in Section 4.3. In Section 4.5, we shall provide some examples of cylinders in Du Val del Pezzo surfaces of rank one over a field of characteristic zero and vertical cylinders with respect to generically canonical del Pezzo fibrations.

In Chapter 5, we prove Theorems 1.3.12, 1.3.14 and 1.3.15. The main idea of the proof of this theorem is to classify compactifications of the affine plane. In Section 5.1, we thus prepare the notation on this and present previous works. In Section 5.2, we also prepare some properties of twigs, where a twig is a special kind of weighted dual graph, which will play an important role in proving Theorem 1.3.12. In Section 5.3, we show Theorem 1.3.12 (1) and (2). These results will play an important role in the next Section 5.4. In Section 5.4, we show Theorem 1.3.12 (3). In other words, we classify the weighted dual graph of the boundary divisors on the minimal resolution of lc compactifications of the affine plane. In Section 5.5, we shall show Theorems 1.3.14 and 1.3.15 as an application of Theorem 1.3.12. Moreover, we will yield a criterion for generically canonical del Pezzo fibrations to contain vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinders in terms of degree and singular type of generic fibers (see Corollary 5.5.4). Finally, in Section 5.6, we will give various remarks about Theorem 1.3.12.

In Appendix A, we summarize two kind lists as follows. Appendix A.1 summarizes the list of the classification of weak del Pezzo surfaces over algebraically closed fields over characteristic zero for the readers' convenience. Appendix A.2 summarizes the list of configurations of all lc compactifications of the affine plane defined over algebraically non-closed fields in terms of weighted dual graphs.

Conventions. We employ the notation basically as in [54].

In this paper, a *del Pezzo surface* means a normal projective surface such that its anti-canonical divisor is ample. Also, *weak del Pezzo surface* means a smooth projective surface such that its anti-canonical divisor is nef and big. On the other hand, throughout, the *rank* of a del Pezzo surface means the rank of its Neron-Severi group.

Now, we state the notation on weighted dual graphs (see, e.g., [55, p. 52], for the definition). For any weighted dual graph, a vertex $\circ$ with the number m corresponds to an m-curve (see also the following Notation). Exceptionally, we omit this label if $m = -2$, moreover, we omit this label and use the vertex $\bullet$ (resp. $\diamond$) instead of $\circ$ if $m = -1$ (resp. $m = 0$).

In what follows, letting k be a field of characteristic zero, we state the conventions on varieties defined over k. Let X be an algebraic variety X defined over a field k. Then $X_{\overline{k}}$ denotes the base extension of X to the algebraic closure $\overline{k}$ of k, i.e., $X_{\overline{k}} := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$. Moreover, we say that X is *geometrically rational* if $X_{\overline{k}}$ is rational. When X is a smooth projective surface, we say that X is *minimal* over k (or simply, *k-minimal*) if any birational morphism $f : X \to Y$ from X to a smooth projective surface Y defined over k is an isomorphism. Letting X' be an algebraic variety defined over $\overline{k}$, we say that X is a *k-form* of X' if $X_{\overline{k}} \simeq X'$. Also, we write $\mathrm{Sing}(X) := \mathrm{Sing}(X_{\overline{k}}) \cap X(k)$, in other words, $\mathrm{Sing}(X)$ is the set of singularities on $X_{\overline{k}}$ defined over k. For a normal surface V over k, we say that V has at most Du Val singularities (resp. quotient singularities, log canonical singularities) if the base extension $V_{\overline{k}}$ has at most Du Val singularities (resp. quotient singularities, log canonical singularities). For a del Pezzo surface S over k, we say that S is a *Du Val* (resp. a *log*, an *lc*) *del Pezzo surface* if S has at most Du Val singularities (resp. quotient singularities, log canonical singularities) (see Section 2.3, for singularities). For $\pi : X \to Y$ a surjective morphism between algebraic varieties defined over k, we say that π is a $\mathbb{P}^1$-*fibration* (resp. $\mathbb{P}^1$-*bundle*) if a general fiber (resp. any fiber) of the base extension $\pi_{\overline{k}} : X_{\overline{k}} \to Y_{\overline{k}}$ is isomorphic to $\mathbb{P}^1_{\overline{k}}$. Moreover, we say that π is a *conic bundle* if any fiber of $\pi_{\overline{k}}$ is isomorphic to the plane conic (not necessarily irreducible). Let D be a reduced effective divisor on a variety defined over k. Then $D_{\overline{k}}$ denotes the base extension of D to the algebraic closure k. We say that D is an *SNC-divisor* if $D_{\overline{k}}$ has only simple normal crossings. Moreover, $\sharp D$ denotes the number of all irreducible components in $\mathrm{Supp}(D)$ over k. Note that if $\mathrm{Supp}(D)$ contains an irreducible component, which is not geometrically irreducible, then $\sharp D < \sharp D_{\overline{k}}$.

Notation. We will use the following notations:

- $\mathbb{Z}$: the set of all integers.
- $\mathbb{Q}$: the rational number field.
- $\mathbb{R}$: the real number field.
- $\mathbb{C}$: the complex number field.
- $\rho_k(X)$: the Picard number of a variety X defined over a field k.
- $\mathrm{Cl}(X)$: the divisor class group of a variety X.
- $\varphi^*(D)$: the total transform of a divisor D by a morphism φ.
- $\varphi_*^{-1}(D)$: the proper transform of a divisor D by a morphism φ.
- $\psi_*(D)$: the direct image of a divisor D by a morphism ψ.
- $(D \cdot D')$: the intersection number of two divisors D and D' on a surface.
- $(D)^2$: the self-intersection number of a divisor D on a surface.
- $\mathbb{F}_m$: the Hirzebruch surface of degree m.
- m-curve: a smooth projective rational curve defined over an algebraically closed field with self-intersection number m.
- $\mathbb{A}^1_{*,k}$: The affine line over k with one k-point removed, i.e., $\mathbb{A}^1_{*,k} := \mathrm{Spec}(k[t^{\pm 1}])$.
- $C_{(n)}$: A k-form of the affine line with n-times closed points removed.
- $\delta_{i,j}$: The Kronecker delta.

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Chapter 2

Preliminaries

2.1 Basic properties of weak del Pezzo surfaces

In this section, we review the basic but important properties of weak del Pezzo surfaces over an algebraically closed field of characteristic zero. We mainly refer to [18, §8]. Let V be a weak del Pezzo surface over an algebraically closed field of characteristic zero.

It seems that the following two lemmas are basic:

Lemma 2.1.1. With the notation as above, we obtain $(-K_V)^2 > 0$ and $(-K_V \cdot C) \geq 0$ for any curve C on V. Moreover, assuming that $-K_V$ is ample, we obtain $(-K_V \cdot C) > 0$ for any curve C on V.

Proof. This assertion follows from [49, Proposition 2.61] and [49, Theorem 1.37]. □

Lemma 2.1.2. With the notation as above, let C be an irreducible curve on V. Then the following assertions hold:

(1) If $(C)^2 < 0$, then C is either a (-1)-curve or a (-2)-curve.

(2) C is a (-2)-curve if and only if $(C \cdot -K_V) = 0$.

Proof. In (1), see [18, Lemma 8.1.3]. We shall prove (2). If C is a (-2)-curve, then $(C)^2 = -2$ and $p_a(C) = 0$, so that we easily obtain $(C \cdot -K_V) = 0$. Conversely, assume $(C \cdot -K_V) = 0$. Note that $(-K_V)^2 > 0$ since $-K_V$ is nef and big. Hence, we know $(C)^2 < 0$ by the Hodge index theorem (see, e.g., [54, Theorem 10.9]). Thus, C is either a (-1)-curve or a (-2)-curve by (1). However, we note that C is not a (-1)-curve. Indeed, if C is a (-1)-curve, we obtain $(C \cdot -K_V) = 1$ by virtue of $(C)^2 = -1$ and $p_a(C) = 0$. □

It is well known that any weak del Pezzo surface over an algebraically closed field of characteristic zero is rational. More strictly, the following lemma holds:

Lemma 2.1.3. With the notation as above, then V is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_2$ of degree 2, or a blow-up at most eight points, which may include infinitely near points, from $\mathbb{P}^2$.

Proof. See, e.g., [18, Theorem 8.1.15]. □

By Lemma 2.1.3, for a weak del Pezzo surface V, we see that $(-K_V)^2$ is an integer between 1 and 9.

Furthermore, we will use the following two lemmas after Chapters:

Lemma 2.1.4. With the notation as above, let D be a divisor on V such that $(D)^2 = -1$, $(D \cdot -K_V) = 1$ and $(D \cdot M) \geq 0$ for any (-2)-curve M on V. Then there exists a (-1)-curve E on V such that $D \sim E$.

Proof. See [18, Lemma 8.2.22]. □

Lemma 2.1.5. With the notation as above, the number of (-2)-curves on V is less than or equal to $9 - (-K_V)^2$.

Proof. See [18, Proposition 8.2.25]. □

2.2 Some properties of varieties over non-closed fields

In this section, we review standard facts on varieties over an algebraically non-closed field of characteristic zero. Some basic facts about (geometrically) rational surfaces over algebraically non-closed fields can be found in, e.g., [15, 32, 29] (see also [59, 50]). Let k be a field (this is not necessarily algebraically closed) of characteristic zero.

It seems that the Galois group actions are a useful tool for the studying of varieties defined over algebraically non-closed fields. In particular, the following lemma plays an important role in the studying:

Lemma 2.2.1 ([50, Exercise 1.8])**.** Let k'/k be a finite Galois extension. Note that the Galois group $\mathrm{Gal}(k'/k)$ acts on $\mathbb{A}^n_{k'} = \mathrm{Spec}(k'[x_1, \ldots, x_n])$ as follows:

$$\mathrm{Gal}(k'/k) \times \mathbb{A}^n_{k'} \ni (g, (a_1, \ldots, a_n)) \mapsto (g(a_1), \ldots, g(a_n)) \in \mathbb{A}^n_{k'}.$$

Let V be a closed algebraic subset of $\mathbb{A}^n_{k'}$. Then the following two conditions are equivalent:

- V can be defined by polynomials in $k[x_1, \ldots, x_n]$;
- V is invariant under the $\mathrm{Gal}(k'/k)$-action.

Next, recall the following proposition on Severi-Brauer varieties:

Proposition 2.2.2. Let V be a smooth algebraic variety over k satisfying $V_{\overline{k}} \simeq \mathbb{P}^n_{\overline{k}}$. Then $V \simeq \mathbb{P}^n_k$ if and only if V has a k-rational point.

Proof. See, e.g., [59, Proposition 4.5.10]. □

Example 2.2.3. Let C be the irreducible plane conic over $\mathbb{R}$ as follows:

$$C := (x^2 + y^2 + z^2 = 0) \subseteq \mathbb{P}^2_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[x, y, z]).$$

By $p_a(C_{\mathbb{C}}) = 0$, we cleary see that $C_{\mathbb{C}} \simeq \mathbb{P}^1_{\mathbb{C}}$. However, since C has no $\mathbb{R}$-rational point, we obtain $C \not\simeq \mathbb{P}^1_{\mathbb{R}}$.

Lemma 2.1.3 implies that minimal weak del Pezzo surfaces over an algebraically closed field is one of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$. On the other hand, minimal weak del Pezzo surfaces over an algebraically non-closed field are known as follows:

Proposition 2.2.4. Let $\widetilde{S}$ be a weak del Pezzo surface defined over k. If $\widetilde{S}$ is minimal over k, then one of the following assertions hold:

- $\widetilde{S}$ is a smooth del Pezzo surface of rank one;
- $\widetilde{S}$ is of rank two and is endowed with a structure of Mori conic bundle defined over k.

Proof. Since $\widetilde{S}$ is minimal and the canonical divisor K_S of $\widetilde{S}$ is not nef by the assumption, we obtain the assertion by [59, Theorem 9.3.20]. □

2.3 Classes of singularities

In this section, we review the four classes of singularities about minimal model programs. We will present these definitions in general dimensions, however, we will mainly treat singularities on normal algebraic surfaces later. In this paper, we only consider singularities over an algebraically closed field of characteristic zero. Throughout this section, we thus assume that all varieties are defined over an algebraically closed field k of characteristic zero. We refer to [1, 49, 55].

Definition 2.3.1 ([49, Notation 2.26 and Definition 2.28])**.** Let X be a normal projective variety and let $D = \sum_j d_j D_j$ be an effective $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier. For a birational morphism $f : \widetilde{X} \to X$, we write:

$$K_{\widetilde{X}} + f_*^{-1}(D) \equiv_{\mathbb{R}} f^*(K_X + D) + \sum_i a(E_i, X, D)E_i,$$

where each E_i is an irreducible component of the exceptional locus of f. Then the *discrepancy* of (X, D) is given by:

$$\operatorname{discrep}(X, D) := \inf_E \{a(E, X, D), \mid E \text{ is an exceptional divisor over } X\},$$

where E runs through all the irreducible exceptional divisors for all birational morphisms $f : \widetilde{X} \to X$ and through all the irreducible divisors of X.

Definition 2.3.2 (cf. [49, Definition 2.34])**.** Let X be a normal projective variety.

(1) Letting $D = \sum_j d_j D_j$ be a $\mathbb{Q}$-divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier, we say that (X, D) is *terminal* (resp. *canonical*, *kawamata log terminal*, *log canonical*) if $\operatorname{discrep}(X, D) > 0$ (resp. $\operatorname{discrep}(X, D) \geq 0$, $\operatorname{discrep}(X, D) > -1$ and $0 \leq d_j < 1$ for any j, $\operatorname{discrep}(X, D) \geq -1$).

(2) We say that X has at most *terminal singularities* (resp. *canonical singularities*, *log terminal singularities*, *log canonical singularities*) if $(X, 0)$ is terminal (resp. canonical, kawamata log terminal, log canonical).

In what follows, we shall consider two-dimensional singularities.

Theorem 2.3.3 (cf. [49, Theorem 4.5 (1)])**.** Let S be a normal algebraic surface. Then S has at most terminal singularities if and only if S is smooth.

Next, in order to deal with 2-dimensional canonical singularities, we recall Du Val singularities.

Definition 2.3.4 (e.g. [49, Definition 4.4])**.** Let S be a normal algebraic surface, let $x \in X$ be a singular point and let $\sigma : \widetilde{S} \to S$ be the minimal resolution. Then $x \in X$ is a *Du Val singular point* if any irreducible component E in the exceptional set of σ satisfies $(E \cdot -K_{\widetilde{S}}) = 0$.

Then it is known the following fact:

Theorem 2.3.5 (cf. [49, Theorem 4.5 (2)])**.** Let S be a normal algebraic surface. Then S has at most canonical singularities if and only if S has at most Du Val singular points.

[22] summarizes some properties on Du Val singularities. In particular, we will use the following facts in this paper later:

Lemma 2.3.6 ([22])**.** Let S be a normal algebraic surface, let $x \in S$ be a singular point, let $\sigma : \widetilde{S} \to S$ be the minimal resolution and let E be the exceptional set of σ. Then the following facts hold:

(1) Any irreducible component of E is a (-2)-curve.

(2) The dual graph of E is one of the following:

- Type A_n: $\circ \text{---} \cdots \text{---} \circ$;
- Type D_n: $\begin{array}{llll} \circ & & & \\ & \diagdown & & \\ & & \circ \text{---} \cdots \text{---} \circ & (n \geq 4); \\ & \diagup & & \\ \circ & & & \end{array}$
- Type E_6: $\begin{array}{ccccc} \circ \text{---} & \circ \text{---} & \circ \text{---} & \circ \text{---} & \circ \; ; \\ & & | & & \\ & & \circ & & \end{array}$
- Type E_7: $\begin{array}{cccccc} \circ \text{---} & \circ \text{---} & \circ \text{---} & \circ \text{---} & \circ \text{---} & \circ \; ; \\ & & | & & & \\ & & \circ & & & \end{array}$
- Type E_8: $\begin{array}{ccccccc} \circ \text{---} & \circ \text{---} & \circ \text{---} & \circ \text{---} & \circ \text{---} & \circ \text{---} & \circ \; . \\ & & | & & & & \\ & & \circ & & & & \end{array}$

Next, in order to deal with 2-dimensional log terminal singularities, we recall (2-dimensional) quotient singularities. We refer to [55, Chap I. §§5.3]. For details, see [8]. Let G be a finite subgroup of $\mathrm{GL}(2;k)$. Then G acts naturally on $\mathbb{A}^2_k$ as follows:

$$G \times \mathbb{A}^2_k \ni (\begin{bmatrix} a & b \\ c & d \end{bmatrix}, (x,y)) \mapsto (ax+by,\ cx+dy) \in \mathbb{A}^2_k.$$

Letting A be the coordinate ring[*1] of $\mathbb{A}^2_k$, then we have the following ring:

$$A^G := \{ f \in A \,|\, g \cdot f = f \text{ for any } g \in G \}.$$

Now, we say that the scheme $\mathbb{A}^2_k/G := \mathrm{Spec}(A^G)$ is called the *algebraic quotient scheme*. Moreover, the morphsim $\pi : \mathbb{A}^2_k \to \mathbb{A}^2_k/G$, which corresponds to the inclusion $A^G \hookrightarrow A$, is called the *quotient morphism*. Then the image via π of the point of origin 0 on $\mathbb{A}^2_k/G$ is denoted also by 0.

Definition 2.3.7 (cf. [55, Chap. I, §§5.3])**.** Let S be a normal algebraic surface and let $x \in S$ be a closed point. Let U be an affine open subset of S such that this coordinate ring R is a local ring with the only maximal ideal corresponding to x. Then we say that $x \in S$ is a *quotient singular point* if the completion of R is isomorphic to the completion of the local ring $\mathscr{O}_{\mathbb{A}^2_k/G,0}$ for some finite subgroup G of $\mathrm{GL}(2;k)$, where the notation of $\mathbb{A}^2_k/G$ and 0 are the same as above.

Then the following fact is known:

[*1]In this paper, a *ring* means a commutative ring with unit 1.

Theorem 2.3.8 (cf. [49, Proposition 4.18])**.** Let S be a normal algebraic surface. Then S has at most log terminal singularities if and only if S has at most quotient singular points.

[8] firstly classify quotient singularities. Furthermore, [1] gives elementary other proof of the classification. Their works are summarized as follows:

Lemma 2.3.9 (cf. [55, Chap. I, Lemma 5.3.3])**.** Let S be a normal algebraic surface, let $x \in S$ be a quotient singular point, let $\sigma : \widetilde{S} \to S$ be the minimal resolution and let E be the exceptional set of σ. Then the weight dual graph is one of the following:

(1) $\overset{-m_1}{\circ} \text{——} \cdots \text{——} \overset{-m_r}{\circ}$, where $m_i \geq 2$ for $i = 1, \ldots, r$;

(2)
$$\begin{array}{ccccccccc}
\overset{-m^{(1)}_{r_1}}{\circ} & \text{——} \cdots \text{——} & \overset{-m^{(1)}_{1}}{\circ} & \text{——} & \overset{-m}{\circ} & \text{——} & \overset{-m^{(2)}_{1}}{\circ} & \text{——} \cdots \text{——} & \overset{-m^{(2)}_{r_2}}{\circ} \\
 & & & & | & & & & \\
 & & & & \circ\ -m^{(3)}_1 & & & & \\
 & & & & | & & & & \\
 & & & & \vdots & & & & \\
 & & & & | & & & & \\
 & & & & \circ\ -m^{(3)}_{r_3} & & & &
\end{array}$$

such that three pairs of positive integers (n_1, q_1), (n_2, q_2) and (n_3, q_3) satisfy $0 < q_i < n_i$, $\gcd(n_i, q_i) = 1$ and:

$$\frac{n_i}{q_i} = m^{(i)}_1 - \cfrac{1}{m^{(i)}_2 - \cfrac{1}{m^{(i)}_3 - \cfrac{1}{\ddots - \cfrac{1}{m^{(i)}_{r_i-1} - \cfrac{1}{m^{(i)}_{r_i}}}}}}$$

for $i = 1, 2, 3$, where $\{n_1, n_2, n_3\} = \{2, 2, n\}$ $(n \geq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$.

Remark 2.3.10 ([24, (3.8)]). In general, the quotient singular point with respect to a finite cyclic subgroup of $\mathrm{GL}(2; k)$ is called a *cyclic quotient singular point.* Letting G be the cyclic subgroup of $\mathrm{GL}(2; k)$ given by:

$$G := \left\langle \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^q \end{bmatrix} \right\rangle \qquad \left(\zeta := \exp\left(\frac{2\pi\sqrt{-1}}{n}\right),\ 0 < q < n,\ \gcd(n, q) = 1\right),$$

then the weighted dual graph of the exceptional set of minimal resolution at the quotient singularity with respect to G is following:

$$\overset{-m_1}{\circ} \text{——} \cdots \text{——} \overset{-m_r}{\circ}$$

such that:

$$\frac{n}{q} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \cfrac{1}{\ddots - \cfrac{1}{m_{r_i-1} - \cfrac{1}{m_{r_i}.}}}}}$$

Hence, all cyclic quotient singularities up to isomorphic have a one-to-one correspondence to the set of rational numbers in the interval $(0, 1)$.

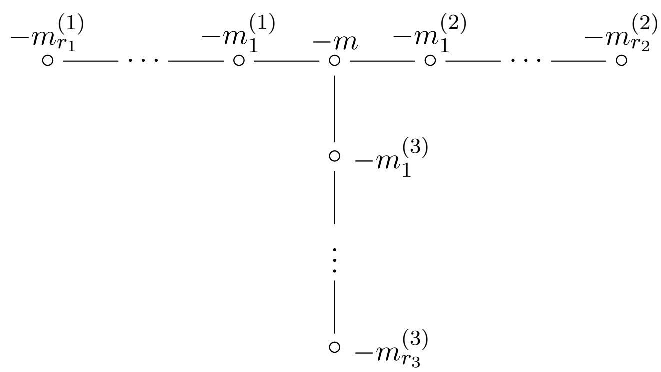
Notice that the explicit list of weighted dual graphs of the exceptional set of the minimal resolution of a quotient singular point is summarized in, e.g., [55, pp. 54–56] or [1, p. 57 (2)].

Next, 2-dimensional log canonical singularities are classified as follows:

Theorem 2.3.11 (cf. [1], [49, Theorem 4.7])**.** Let S be a normal algebraic surface, let $x \in S$ be a log canonical singular point, let $\sigma : \widetilde{S} \to S$ be the minimal resolution and let E be the exceptional set of σ. Then one of the following proprieties holds:

(0) x is a quotient singular point.

(1) The weighted dual graph of E is the following:

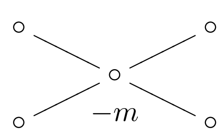


such that three pairs of positive integers (n_1, q_1), (n_2, q_2) and (n_3, q_3) satisfy $0 < q_i < n_i$, $\gcd(n_i, q_i) = 1$ and:

$$\frac{n_i}{q_i} = m^{(i)}_1 - \cfrac{1}{m^{(i)}_2 - \cfrac{1}{m^{(i)}_3 - \cfrac{1}{\ddots - \cfrac{1}{m^{(i)}_{r_i-1} - \cfrac{1}{m^{(i)}_{r_i}}}}}}$$

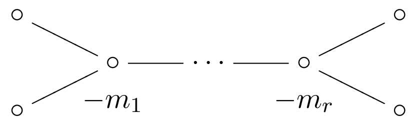
for $i = 1, 2, 3$, where $\{n_1, n_2, n_3\} = \{3, 3, 3\}$, $\{2, 4, 4\}$ or $\{2, 3, 6\}$.

(2) The weighted dual graph of E is as follows:



where $m \geq 3$.

(3) The weighted dual graph of E is as follows:



where $m_1, \ldots, m_r \geq 2$ and further $m_i \geq 3$ for some $i = 1, \ldots, r$.

(4) E is irreducible and either a smooth elliptic curve or a nodal curve.

(5) E consists of smooth rational curves such that this weighted dual graph is a cycle.

At the end of this section, we recall rational singularities.

Definition 2.3.12 ([4])**.** Let S be a normal algebraic surface and let $\sigma : \widetilde{S} \to S$ be a resolution. Then S has at most *rational singularities* if $R^1\sigma_*\mathscr{O}_S = 0$. (Note that this definition is known to be independent of the resolution.)

It seems that the following two theorems are basic but important facts.

Theorem 2.3.13 ([3, 4])**.** Let S be a normal protective surface. If S has at most rational singularities, then S is $\mathbb{Q}$-factorial, i.e., any Weil divisor on S is $\mathbb{Q}$-Cartier.

Theorem 2.3.14 (cf. [3, 4])**.** Let S be a normal algebraic surface and let $x \in S$ be a log canonical singular point. Then x is a rational singular point if and only if x satisfies one of the conditions (0), (1), (2) and (3) in Theorem 2.3.11.

2.4 Classification of weak del Pezzo surfaces

In this section, we recall a classification of weak del Pezzo surfaces over an algebraically closed field of characteristic zero, moreover, we define the *type* of weak del Pezzo surfaces. Almost all parts of this section depend on [17, §§2.1] and [18, §8] (see also [15, 9, 69]).

2.4.1

Let V be a weak del Pezzo surface defined over an algebraically closed field of characteristic zero, whose $-K_V$ is not ample, and let d be the degree of V, i.e., $d = (-K_V)^2$. If $d = 8$, then V is the Hirzebruch surface $\mathbb{F}_2$ of degree 2. Namely, we have the contraction $\sigma : V \to \mathbb{P}(1,1,2)$ of the minimal section. In what follows, we shall consider the case of $d \leq 7$. We prepare the following definition:

Definition 2.4.1 ([17, Definition 3], [12, Definition 2.8])**.** Letting V_1 and V_2 be two weak del Pezzo surfaces over an algebraically closed field of characteristic zero, we say that these surfaces have the *same type* if there is an isomorphism $\mathrm{Pic}(V_1) \simeq \mathrm{Pic}(V_2)$ preserving the intersection form that gives a bijection between their sets of classes of (-1)-curves and (-2)-curves.

By Lemma 2.1.3 and the assumption $d \leq 7$, we can take the following composition of blow-downs to $\mathbb{P}^2$:

$$\tau : V = V_d \overset{\tau_{9-d}}{\to} V_{d+1} \overset{\tau_{8-d}}{\to} \ldots \overset{\tau_2}{\to} V_8 \overset{\tau_1}{\to} V_9 = \mathbb{P}^2,$$

where τ_i is a contraction of a (-1)-curve on V_{9-i} for $i = 1, \dots, 9-d$. Let e_0 be the proper transform on V of a general line on $\mathbb{P}^2$ and let e_i be the total transform on V of the exceptional divisor of τ_i for $i = 1, \dots, 9-d$. Then we can write $\mathrm{Pic}(V) \simeq \bigoplus_{i=0}^{9-d} \mathbb{Z}e_i$ preserving the intersection form such that $(e_0)^2 = 1$, $(e_i)^2 = -1$ for $i > 0$ and $(e_i \cdot e_j) = 0$ for $i, j \geq 0$ with $i \neq j$. Let $R(V)$ be the subset of $\mathrm{Pic}(V)$ preserving the intersection form defined by:

$$R(V) := \{D \in \mathrm{Pic}(V) \mid (D)^2 = -2,\ (D \cdot -K_V) = 0\}.$$

By [18, Lemma 8.2.6 and Proposition 8.2.7], $R(V)$ is the root system of type A_1, $A_2 + A_1$, A_4, D_5 and E_{9-d} if $d = 7$, $d = 6$, $d = 5$, $d = 4$ and $d \leq 3$, respectively (see, e.g., [30, Chap. III], for the reference about root systems). By [18, Proposition 8.2.25], the number r of all (-2)-curves on V is less than $10 - d$, moreover, letting $M_1, \dots, M_r$ be all (-2)-curves on V, the sublattice $L(V)$, which is generated by $M_1, \dots, M_r$, in $R(V)$ is a root lattice of rank r corresponding to the intersection matrix with respect to these (-2)-curves. That is, $L(V)$ determines a subsystem of the root system $R(V)$. Indeed, this can be checked from the data in [15] for degree $d \geq 4$ and from [9] for $d = 3$. Moreover, [69] lists all cases with $d \leq 2$. Thus, noticing that the base field is of characteristic zero, $L(V)$ is one of the following according to the degree d:

- $d = 7$: the root system of type A_1;
- $d = 6$ (resp. $d = 5$, $d = 4$, $d = 3$): the subsystem of the root system of type $A_2 + A_1$ (resp. A_4, D_5, E_6);
- $d = 2$: the subsystem of the root system of type E_7 except for type of $7A_1$;
- $d = 1$: the subsystem of the root system of type E_8 except for types of $7A_1$, $8A_1$ and $D_4 + 4A_1$,

Remark 2.4.2. In this paper, we treat only algebraic varieties over a field of characteristic zero. Meanwhile, if the base field is algebraically closed and of characteristic two, there exists a weak del Pezzo surface V of degree 2 (resp. degree 1) such that $L(V)$ is the root system of type $7A_1$ (resp. $7A_1$, $8A_1$ or $D_4 + 4A_1$). For details, see [69] or [35].

2.4.2

Let k be a field (this is not necessarily algebraically closed) of characteristic zero, let $\widetilde{S}$ a weak del Pezzo surface defined over k and let d be the degree $\widetilde{S}$. Letting $\widetilde{S}_{\overline{k}}$ be the base extension of $\widetilde{S}$ to the algebraic closure $\overline{k}$, we obtain the root system $L(\widetilde{S}_{\overline{k}})$. Let $M_1, \dots, M_r$ be all (-2)-curves on $\widetilde{S}_{\overline{k}}$. Notice that the dual graph of $\sum_{i=1}^r M_i$ corresponds to type of $L(\widetilde{S}_{\overline{k}})$. Moreover, the union $\sum_{i=1}^r M_i$ is defined over k. Hence, we obtain the contraction $\sigma : \widetilde{S} \to S$ of $\sum_{i=1}^r M_i$ over k, so that S is a Du Val del Pezzo surface over k by Lemmas 2.1.2 (2) and 2.3.6. Conversely, for any Du Val del Pezzo surface S, its minimal resolution is a weak del Pezzo surface. Hence, types of singularities of Du Val del Pezzo surfaces have a one-to-one correspondence with types of root systems of their minimal resolution.

Now, we say that the type of singularity type of $S_{\overline{k}}$ is called "Sing" of $\widetilde{S}$. Furthermore, we say that the number of (-1)-curves on $\widetilde{S}_{\overline{k}}$ is called "# Lines" of $\widetilde{S}$, where "# Lines" is finite by Lemma 2.1.4 and [18, Proposition 8.2.19]. In this paper, the triplet (d, Sing, # Lines) is called the *type* of $\widetilde{S}$. For two weak del Pezzo surfaces $\widetilde{S}_1$ and $\widetilde{S}_2$ over k, it is known that the types of $\widetilde{S}_1$ and $\widetilde{S}_2$ (in the sense of the above triplet) are the same if and only if $\widetilde{S}_1$ and $\widetilde{S}_2$

have the same type (in the sense of Definition 2.4.1). Moreover, it is known that all pairs (d, Sing) can uniquely determine the number of "# Lines" except for the following pairs:

$$\begin{aligned}(d, \mathrm{Sing}) =&(6, A_1), (4, A_3), (4, 2A_1),\\ &(2, A_5 + A_1), (2, A_5), (2, A_3 + 2A_1), (2, A_3 + A_1), (2, 4A_1), (2, 3A_1),\\ &(1, A_7), (1, A_5 + A_1), (1, 2A_3), (1, A_3 + 2A_1), (1, 4A_1).\end{aligned} \tag{2.4.1}$$

On the other hand, if the pair (d, Sing) is one of those in the list of (2.4.1), then it is known that there are exactly two possibilities of the number of "# Lines" ([15, 9, 69]).

2.4.3

Let k be a field of characteristic zero. For the simplify of the notation, we introduce the notation for types of weak del Pezzo surfaces instead of the triplet as follows: Let $\widetilde{S}$ be a weak del Pezzo surface over k such that the pair (d, X) of the degree and "Sing" of $\widetilde{S}$ is not in the list in (2.4.1). Then we say that $\widetilde{S}$ is of X-type. On the other hand, let $\widetilde{S}_1$ and $\widetilde{S}_2$ be two weak del Pezzo surfaces over k such that pairs of the degree and "Sing" of them are the same, and their common pair (d, X) is one of those in the list of (2.4.1). Moreover, assume that # Lines of $\widetilde{S}_1$ is strictly more than # Lines of $\widetilde{S}_2$. Then we say that $\widetilde{S}_1$ (resp. $\widetilde{S}_2$) is of $(X)_>$-type (resp. $(X)_<$-type). The detail is summarized in Appendix A.1, for the reader's convenience.

Example 2.4.3. Let us look at cases $(d, \mathrm{Sing}) = (4, 2A_1)$, $(2, 4A_1)$. There are two possibilities about # Lines for each of such cases as follows:

- In case of $(d, \mathrm{Sing}) = (4, 2A_1)$, if $\widetilde{S}$ is of $(2A_1)_>$-type (resp. $(2A_1)_<$-type), then # Lines $= 9$ (resp. # Lines $= 8$).
- In case of $(d, \mathrm{Sing}) = (2, 4A_1)$, if $\widetilde{S}$ is of $(4A_1)_>$-type (resp. $(4A_1)_<$-type), then # Lines $= 20$ (resp. # Lines $= 19$).

2.5 Basic properties of cylinders in normal projective surfaces

Let k be a field of characteristic zero. In this section, we prepare some basic facts about cylinders in normal projective surfaces.

We prepare two examples of cylinders in smooth rational surfaces over algebraically non-closed fields (cf. [19, Proposition 12]). Although not used in this paper, [13, §§4.1] present many examples of cylinders in smooth del Pezzo surfaces over algebraically closed fields.

Lemma 2.5.1. Let V be a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ containing a k-rational point, say p, let F_1 and F_2 be k-forms of irreducible curves of types $(1, 0)$ and $(0, 1)$ (see [28, Chap. II, Example 6.6.1], for the notation) passing through p, respectively, and let C be a geometrically irreducible curve on V passing through p such that $C \sim F_1 + F_2$. Then $V \backslash (F_1 \cup F_2 \cup C) \simeq \mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$.

Proof. Notice that $C_{\overline{k}}$ is a 2-curve. Let $\varphi : V' \to V$ be the blow-up at p, let E' be a reduced exceptional curve by φ, and let F'_1, F'_2 and C' be the proper transform of F_1, F_2 and C, respectively. Since $F'_{i,\overline{k}}$ is a (-1)-curve on V' for $i = 1, 2$ and $F'_1 + F'_2$ is defined over k, we thus obtain the contraction $\psi : V' \to V''$ of $F'_1 + F'_2$, so that V'' is a k-form of the projective plane $\mathbb{P}^2_{\overline{k}}$. Since E' contains a k-rational point, so does its image via ψ, in particular, we know that $V'' \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. On the other hand, $\psi_*(C')$ and $\psi_*(E')$ are distinct lines on $V'' \simeq \mathbb{P}^2_k$. Namely, $V'' \backslash (\psi_*(C') \cup \psi_*(E')) \simeq \mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$. Hence, we have $V \backslash (F_1 \cup F_2 \cup C) \simeq V' \backslash (F'_1 \cup F'_2 \cup C' \cup E') \simeq V'' \backslash (\psi_*(C') \cup \psi_*(E')) \simeq \mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$. □

Lemma 2.5.2. Let V be a k-form of $\mathbb{F}_2$ admitting a k-rational point, say p, let M and F be k-forms of the minimal section and the closed fiber passing through p of the structure morphism $V_{\overline{k}} \simeq \mathbb{F}_2 \to \mathbb{P}^1_{\overline{k}}$, respectively, let C_2 be a geometrically irreducible curve on V such that $C_2 \sim M+2F$ and let C_3 be a geometrically irreducible curve on V such that $C_3 \sim M+3F$ and $M \cap F \cap C_3 \neq \emptyset$. Then $V\backslash(M \cup F \cup C_2) \simeq V\backslash(M \cup F \cup C_3) \simeq \mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$.

Proof. Since V has a k-rational point p, we note that $V \simeq \mathbb{F}_2$ by using Lemma 2.2.2, i.e., $V \simeq \mathbb{P}(\mathscr{O}_{\mathbb{P}^1_k} \oplus \mathscr{O}_{\mathbb{P}^1_k}(2))$. Hence, $(V, M+F+C_2)$ is a minimal normal comapctification of $\mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$ by [68] or [43]. Indeed, M and F are rational curves over k by Lemma 2.2.2. Moreover, $C_2 \simeq \mathbb{P}^1_k$ since the intersection point of C_2 and F is k-rational. Namely, $V\backslash(M \cup F \cup C_2) \simeq \mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$.

By assumption, $M \cap F \cap C_3$ consists of a only one point, say q, defined over k. Let $\varphi : V' \to V$ be the blow-up at q, let E' be a reduced exceptional curve by φ, and let M', F' and C_3' be the proper transform of M, F and C_3, respectively. Since F' is a (-1)-curve on V' and defined over k, we thus obtain the contraction $\psi : V' \to V''$ of F', so that V'' is the Hirzebruch surface $\mathbb{F}_3$ of degree 3, i.e., $V \simeq \mathbb{P}(\mathscr{O}_{\mathbb{P}^1_k} \oplus \mathscr{O}_{\mathbb{P}^1_k}(3))$. Hence, $(V'', \psi_*(M') + \psi_*(E') + \psi_*(C_3'))$ is a minimal normal compactification of $\mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$ by [68] or [43]. Indeed, since $\psi_*(M')$, $\psi_*(E')$ and $\psi_*(C_3)$ admit a k-rational point, respectively, they are rational curves. Namely, $V\backslash(M \cup F \cup C_3) \simeq V'\backslash(M' \cup F \cup C_3' \cup E') \simeq V''\backslash(\psi_*(M') \cup \psi_*(E') \cup \psi_*(C_3')) \simeq \mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$. $\square$

The following fact seems to be well-known to experts, however, it will play an important role in later Chapters 4 and 5:

Lemma 2.5.3. Let V be a smooth projective surface over k and let U be a cylinder in V. Then the boundary divisor of U has no cycle.

Proof. Let us write $U \simeq \mathbb{A}^1_k \times Z$ for some curve Z, and let D be the boundary divisor of U, i.e., $V\backslash\mathrm{Supp}(D) = U$. If D has a cycle, then so does $D_{\overline{k}}$. Hence, we may assume $k = \overline{k}$. The closures in V of fibers of the projection $pr_Z : U \simeq \mathbb{A}^1_k \times Z \to Z$ yields a linear system on V, say $\mathscr{L}$, hence we have the rational map $\Phi_{\mathscr{L}} : V \dashrightarrow \overline{Z}$ to a projective model $\overline{Z}$ of the closure of Z in V. Note that $\mathrm{Bs}(\mathscr{L})$ consists of at most one point by the configure of $\mathscr{L}$. Let $\psi : \bar{V} \to V$ be the shortest succession of blow-ups the point on $\mathrm{Bs}(\mathscr{L})$ and its infinitely near points such that the proper transform of $\mathscr{L}$ is free of base points to give rise to a morphism $\bar{\varphi} := \Phi_{\mathscr{L}} \circ \psi : \bar{V} \to \overline{Z}$, where we shall define $\varphi := \Phi_{\mathscr{L}}$ if $\mathrm{Bs}(\mathscr{L}) = \emptyset$. Hence, $\bar{\varphi}$ is a $\mathbb{P}^1$-fibration, moreover, $\psi^*(D)_{\mathrm{red.}}$ is the union of a section and all singular fibers of $\bar{\varphi}$. Thus, if D has a cycle, then some singular fibers of $\bar{\varphi}$ also have a cycle. However, it is impossible. Indeed, it is known that any singular fiber of $\mathbb{P}^1$-fibration from a smooth projective surface does not have a cycle (see, e.g., [54, Lemma 12.5]). This completes the proof. $\square$

Now, we shall prepare the *variant of Corti's inequality*. It seems to be a useful tool for proving the absence of cylinders in smooth surfaces over algebraically non-closed fields.

Lemma 2.5.4 (The variant of Corti's inequality)**.** Let V be a smooth projective surface defined over $\overline{k}$, let $\mathscr{L}$ be a mobile linear system on V, let p be a closed point on V and let C_1 and C_2 be two curves on V such that these curves meet transversely at p. Assume that $(V, (1-a_1)C_1 + (1-a_2)C_2 + \mu\mathscr{L})$ is not log canonical at p for some $a_1, a_2 \in \mathbb{Q}_{\geq 0}$ and $\mu \in \mathbb{Q}_{>0}$.

(1) If either $a_1 \leq 1$ or $a_2 \leq 1$, then the following inequality holds:

$$i(L_1, L_2; p) > 4a_1a_2\mu^2,$$

where L_1 and L_2 are general members of $\mathscr{L}$.

(2) If both $a_1 > 1$ and $a_2 > 1$, then the following inequality holds:

$$i(L_1, L_2; p) > 4(a_1 + a_2 - 1)\mu^2,$$

where L_1 and L_2 are general members of $\mathscr{L}$.

Proof. See [16, Theorem 3.1]. □

From now on, we present two lemmas, which are obtained by using Lemma 2.5.4, on cylinders. The first lemma is a result of generalizing [65, Lemma 4.7], which is a key lemma for the proof of Theorem 1.3.4:

Lemma 2.5.5. Let V be a smooth geometrically rational projective surface with $\rho_k(V) \geq 2$ and $(-K_V)^2 \leq 4$ over k, which endowed with a structure of $\mathbb{P}^1$-fibration $\pi : V \to \mathbb{P}^1_k$. Let $\mathscr{L}$ be a linear system on V such that $\mathrm{Bs}(\mathscr{L})$ consists of exactly one k-rational point p. Assume that a general member L of $\mathscr{L}$ satisfies $L\backslash\{p\} \simeq \mathbb{A}^1_k$ and is $\mathbb{Q}$-linearly equivalent to $a(-K_V) + bF$ for some $a, b \in \mathbb{Q}$, where F is the closed fiber of $\pi : V \to \mathbb{P}^1_k$ passing through p. Then $a > 0$ and $b < 0$.

Proof. The assertion $a > 0$ can be easily seen by $0 \leq (\mathscr{L} \cdot F) = 2a$ and $0 < (\mathscr{L})^2 = a(da+4b)$. Suppose $b \geq 0$. Let $\Phi_{\mathscr{L}} : V \dashrightarrow \mathbb{P}^1_k$ be the rational map associate to $\mathscr{L}$, and let $\psi : \bar{V} \to V$ be the shortest succession of blow-ups the point $p \in \mathrm{Bs}(\mathscr{L})$ and its infinitely near points such that the proper transform $\bar{\mathscr{L}} := \psi^{-1}_*(\mathscr{L})$ of $\mathscr{L}$ is free of base points to give rise to a morphism $\bar{\varphi} := \Phi_{\bar{\mathscr{L}}} \circ \psi$ (see the following diagram):

$$\begin{tikzcd} V \arrow[r, dotted, "\Phi_{\mathscr{L}}"] & \mathbb{P}^1_k \\ \bar{V} \arrow[u, "\psi"] \arrow[ur, "\bar{\varphi}"'] & \end{tikzcd}$$

Notice that ψ is defined over k by construction. Letting $\{\bar{E}_i\}_{1\leq i\leq n}$ be the exceptional divisors of ψ with $\bar{E}_n$ the last exceptional one, which is a section of $\bar{\varphi}$, we have:

$$(\bar{\mathscr{L}} \cdot \bar{E}_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq n-1 \\ 1 & \text{if } i = n \end{cases} \tag{2.5.1}$$

and

$$K_{\bar{V}} - \frac{b}{a}\psi^*(F) + \frac{1}{a}\bar{\mathscr{L}} = \psi^*\left(K_V - \frac{b}{a}F + \frac{1}{a}\mathscr{L}\right) + \sum_{i=1}^{n} c_i\bar{E}_i \tag{2.5.2}$$

for some rational numbers $c_1, \dots, c_n$. As $a > 0$, $b \geq 0$ and $(\bar{\mathscr{L}})^2 = 0$, we have:

$$\begin{aligned} -2 &= (\bar{\mathscr{L}} \cdot K_{\bar{V}}) \\ &= \left(\bar{\mathscr{L}} \cdot K_{\bar{V}} + \frac{1}{a}\bar{\mathscr{L}}\right) \\ &\geq \left(\bar{\mathscr{L}} \cdot K_{\bar{V}} - \frac{b}{a}\psi^*(F) + \frac{1}{a}\bar{\mathscr{L}}\right) \\ &\underset{(2.5.2)}{=} \left(\bar{\mathscr{L}} \cdot \psi^*\left(K_V - \frac{b}{a}F + \frac{1}{a}\mathscr{L}\right)\right) + \sum_{i=1}^{n} c_i(\bar{\mathscr{L}} \cdot \bar{E}_i) \\ &\underset{(2.5.1)}{=} \left(\bar{\mathscr{L}} \cdot \psi^*\left(K_V - \frac{b}{a}F + \frac{1}{a}\mathscr{L}\right)\right) + c_n. \end{aligned}$$

Since $K_V - \frac{b}{a}F + \frac{1}{a}\mathscr{L} \sim_{\mathbb{Q}} 0$, we have $c_n \le -2$. This implies that $(V, -\frac{b}{a}F + \frac{1}{a}\mathscr{L})$ is not log canonical (see Definition 2.3.2, for this definition). We will consider whether $p \in F$ is smooth or not in what follows.

In the case that $p \in F$ is smooth: By Lemma 2.5.4 (1), we have:

$$i(L_1, L_2; p) > 4\left(1 + \frac{b}{a}\right)a^2 = 4a(a+b), \tag{2.5.3}$$

where L_1 and L_2 are general members of $\mathscr{L}$. Meanwhile, since L_1 and L_2 meet at only p, the left hand side of (2.5.3) can be written as:

$$i(L_1, L_2; x) = (\mathscr{L})^2 = (-K_V)^2a^2 + 4ab \le 4a(a+b),$$

where we recall that $(-K_V)^2 \le 4$. It is a contradiction to (2.5.3).

In the case that $p \in F$ is not smooth: We then know that $F_{\overline{k}}$ is a singular fiber of $\pi_{\overline{k}}$, hence, there exists exactly two irreducible components F_1 and F_2 on F meeting transversely at p (see, e.g., [55, Lemma 2.11.2]). Hence, $(V, -\frac{bm_1}{a}F_1 - \frac{bm_2}{a}F_2 + \frac{1}{a}\mathscr{L})$ is not log canonical at p for some positive integers m_1 and m_2. By Lemma 2.5.4, we have:

$$i(L_1, L_2; p) > 4\left\{\left(1 + \frac{bm_1}{a}\right) + \left(1 + \frac{bm_2}{a}\right) - 1\right\}a^2 = 4a\{a + (m_1+m_2)b\}, \tag{2.5.4}$$

where L_1 and L_2 are general members of $\mathscr{L}$. By the similar argument as above, we see:

$$i(L_1, L_2; p) \le 4a(a+b) \le 4a\{a + (m_1+m_2)b\},$$

which is a contradiction to (2.5.4). $\square$

The other lemma, which is a generalization of [19, Proposition 9] in the sense of singularities, will play important role in Chapter 4:

Lemma 2.5.6. Let S be a Du Val del Pezzo surface of rank one and of degree d defined over k. Assume that S contain a cylinder $U \simeq \mathbb{A}^1_k \times Z$. The closures in S of fibers of the projection $pr_Z : U \simeq \mathbb{A}^1_k \times Z \to Z$ yields a linear system on S, say $\mathscr{L}$. Then the following assertions hold:

(1) $\mathrm{Bs}(\mathscr{L})$ consists of exactly one k-rational point, say p.

(2) If $d \le 4$, then p is a singular point on $S_{\overline{k}}$.

Proof. In (1), since S is of rank one, $\mathrm{Cl}(S)_{\mathbb{Q}}$ is generated by only $-K_S$. Hence, two general members L_1 and L_2 on $\mathscr{L}$ meet at a point because of $(-K_S)^2 > 0$. This implies that $\mathrm{Bs}(\mathscr{L}) \ne \emptyset$. Hence, $\mathrm{Bs}(\mathscr{L})$ consists of exactly one k-rational point by construction of $\mathscr{L}$.

In (2), suppose that $d \le 4$ and p is a smooth point on $S_{\overline{k}}$. Let $\sigma : \widetilde{S} \to S$ be the minimal resolution over k. Then $\widetilde{S}$ also contains a cylinder $\widetilde{U} := \sigma^{-1}(U) \simeq U$. The closures in $\widetilde{S}$ of fibers of the projection $pr_Z : \widetilde{U} \simeq \mathbb{A}^1_k \times Z \to Z$ yields a linear system, say $\widetilde{\mathscr{L}}$, on $\widetilde{S}$. By assumption, $\sigma^{-1}(p)$ consists of only one k-rational point, say $\widetilde{p}$. Hence, $\mathrm{Bs}(\widetilde{\mathscr{L}}) = \{\widetilde{p}\}$. On the other hand, notice that we can write $\mathscr{L} \sim_{\mathbb{Q}} a(-K_S)$ for some $a \in \mathbb{Q}_{>0}$ Then since p is smooth on $S_{\overline{k}}$ and $S_{\overline{k}}$ has at most Du Val singularities, we have:

$$\widetilde{\mathscr{L}} = \sigma_*^{-1}(\mathscr{L}) = \sigma^*(\mathscr{L}) \sim_{\mathbb{Q}} a\sigma^*(-K_S) \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}).$$

Thus, we can obtain a contradiction by the argument similar to Lemma 2.5.5. Therefore, p must be a singular point on $S_{\overline{k}}$ provided that $d \le 4$. $\square$

Chapter 3

Cylinders in weak del Pezzo fibrations

The purpose of this chapter is to prove Theorems 1.3.3 and 1.3.4. Throughout this chapter, let k be a field of characteristic zero, let $\widetilde{S}$ be a weak del Pezzo surface defined over k and let d be the degree of $\widetilde{S}$, i.e., $d := (-K_{\widetilde{S}})^2$.

3.1 Properties of Mori conic bundles from minimal weak del Pezzo surfaces

Let the notation be the same as at the beginning of Chapter 3 and assume further that $\rho_k(\widetilde{S}) > 1$ and $\widetilde{S}$ is minimal over k. By Proposition 2.2.4, we then obtain that $\rho_k(\widetilde{S}) = 2$ and $\widetilde{S}$ is endowed with a structure of Mori conic bundle defined over k. In this section, we shall prepare the basic properties of this Mori conic bundle for later use.

Lemma 3.1.1. With the notation and the assumptions as above, let $\pi : \widetilde{S} \to B$ be a Mori conic bundle over k. Then:

(1) $B_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}}$.

(2) $\pi_{\overline{k}} : \widetilde{S}_{\overline{k}} \to B_{\overline{k}}$ is a $\mathbb{P}^1$-bundle if and only if $d = 8$.

(3) If $d < 8$, then π does not admit any section defined over k.

Proof. In (1) and (2), see [50, Exercise 3.13]. We shall show (3). By (1), we have $B_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}}$. Note that the base extension of π to the algebraic closure $\pi_{\overline{k}} : \widetilde{S}_{\overline{k}} \to B_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}}$ admits always a section defined over $\overline{k}$, by the Tsen's theorem. Let Γ be a section of $\pi_{\overline{k}}$. By the assumption that $d < 8$ and (2), $\pi_{\overline{k}}$ admits a singular fiber F. We can easily see by the minimality of $\widetilde{S}$ that F is the union $E + E'$ of (-1)-curves E and E' on $\widetilde{S}_{\overline{k}}$ meeting transversally at a point, say p, in such a way that E and E' lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Since Γ is a section of $\pi_{\overline{k}}$, Γ does not pass through p. Hence, we may assume that there exists a closed point $q \in E\backslash\{p\}$ such that Γ passes through q. Since E and E' lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, there exists a closed point $q' \in E'\backslash\{p\}$ such that q and q' are contained in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. This implies that Γ is not defined over k. □

The following two lemmas will play important roles in Subsection 3.3.2:

Lemma 3.1.2. With the notation and the assumptions as above, any $\mathbb{P}^1$-fibration $\pi : \widetilde{S} \to B$ over a geometrically rational curve B defined over k is a Mori conic bundle.

Proof. Assume that $\pi_{\overline{k}}$ admits a singular fiber F. Since $\widetilde{S}$ is minimal over k, we know that F does not contain any (-2)-curve by [70, Lemma 1.5]. Moreover, F is the union $E_1 + E_2$ of two (-1)-curves E_1 and E_2 on $\widetilde{S}_{\overline{k}}$ meeting transversally at a point in such a way that E_1 and E_2 lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. This implies that π is a Mori conic bundle. □

The following lemma can be found in [32]. However, we give the proof for the reader's convenience since this is an important result.

Lemma 3.1.3. With the notation and the assumptions as above, assume further that $\widetilde{S}(k) \neq \emptyset$, $-K_{\widetilde{S}}$ is ample, and d is equal to 1, 2 or 4. Then $\widetilde{S}$ is endowed with two distinct structures of Mori conic bundles $\pi_i : \widetilde{S} \to \mathbb{P}^1_k$ defined over k for $i = 1, 2$ such that $F_1 + F_2 \sim \frac{4}{d}(-K_{\widetilde{S}})$, where F_i is a general fiber of $\pi_{i,\overline{k}}$, which is defined over k, for $i = 1, 2$.

Proof. For any Mori conic bundle $\pi : \widetilde{S} \to B$ over k, note that $B \simeq \mathbb{P}^1_k$, in particular, there exists a general fiber of $\pi_{\overline{k}}$ defind over k. Indeed, since $\widetilde{S}$ has a k-rational point, so it its image via π by Lemmas 2.2.2 and 3.1.1 (1).

By Proposition 2.2.4, we see that $\rho_k(\widetilde{S}) = 2$ and $\widetilde{S}$ is endowed with a structure of Mori conic bundle $\pi_1 : S \to \mathbb{P}^1_k$ defined over k. In particular, there exists a general fiber F_1 of π_1, which is geometrically irreducible. By $\rho_k(\widetilde{S}) = 2$, the Mori cone $\overline{\mathrm{NE}}(\widetilde{S})$ contains exactly two extremal rays, say R_1 and R_2 (cf. [49, §1.3]). Moreover, we can assume $R_1 = \mathbb{R}_{\geq 0}[F_1]$ and we write $R_2 = \mathbb{R}_{\geq 0}[\ell]$ for some curve ℓ on $\widetilde{S}$. Noticing that $\frac{4}{d}$ is an integer by $d \in \{1, 2, 4\}$, let D be the divisor on $\widetilde{S}$ defined by $D := \frac{4}{d}(-K_{\widetilde{S}}) - F_1$. By the Riemann-Roch theorem combined with $(D)^2 = 0$ and $(-K_{\widetilde{S}} \cdot D) = 2$, we have $\chi(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}(D)) = 1 + \chi(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}})$. Moreover, by the Serre duality theorem combined with $(K_{\widetilde{S}_{\overline{k}}} - D \cdot F_1) = -2(1 + \frac{4}{d}) < 0$, we have $h^2(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}(D)) = h^0(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}(K_{\widetilde{S}_{\overline{k}}} - D)) = 0$. Thus, we have $\dim |D| = h^0(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}(D)) - 1 \geq \chi(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}(D))$. On the other hand, since $\widetilde{S}_{\overline{k}}$ is a rational surface by Lemma 2.1.3, we see $\chi(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}) = 1$. Therefore, we have $\dim |D| \geq 1$. In particular, $D_{\overline{k}}$ is linearly equivalent to a union $\sum_{i=1}^{r} C_i$ of some irreducible curves $\{C_i\}_{1 \leq i \leq r}$ on $\widetilde{S}_{\overline{k}}$. Since $-K_{\widetilde{S}}$ is ample, we have $r \leq 2$ by $(-K_{\widetilde{S}} \cdot D) = 2$, moreover, there are at most finitely many unions $C_1 + C_2$ of two irreducible curves C_1 and C_2 on $\widetilde{S}_{\overline{k}}$ with $C_1 + C_2 \sim D_{\overline{k}}$ because these unions consist of two (-1)-curves on $\widetilde{S}_{\overline{k}}$. Hence, there exists an irreducible curve Γ on $\widetilde{S}_{\overline{k}}$ such that $D_{\overline{k}} \sim \Gamma$. Let Γ' be a $\mathrm{Gal}(\overline{k}/k)$-orbit of Γ. Thus, we can write $[\Gamma'] = a_1[F_1] + a_2[\ell]$ in $\overline{\mathrm{NE}}(\widetilde{S})$ for some non-negative real numbers a_1, a_2. By $(\Gamma')^2 = 0$ and $(F_1 \cdot \Gamma') > 0$, we obtain $a_1 = 0$. Namely, $\Gamma' \in R_2$. This implies that there exists a Mori conic bundle $\pi_2 : \widetilde{S} \to \mathbb{P}^1_k$, which is different from π_1, such that a general fiber of $\pi_{2,\overline{k}}$ is linearly equivalent to Γ on $\widetilde{S}_{\overline{k}}$. Furthermore, there exists a general fiber F_2 of $\pi_{2,\overline{k}}$, which is defined over k. By construction of π_2, we know $F_1 + F_2 \sim \frac{4}{d}(-K_{\widetilde{S}})$. □

Remark 3.1.4. Assuming that $-K_{\widetilde{S}}$ is not ample, then we have either $(\ell_1)^2 \neq 0$ or $(\ell_2)^2 \neq 0$ for two curves ℓ_1 and ℓ_2 on $\widetilde{S}$ such that $\overline{\mathrm{NE}}(\widetilde{S}) = \mathbb{R}_{\geq 0}[\ell_1] + \mathbb{R}_{\geq 0}[\ell_2]$. Otherwise, we obtain $(\ell_1 \cdot \ell_2) > 0$ by virtue of $(-K_{\widetilde{S}})^2 > 0$, however, this contradicts $(-K_{\widetilde{S}} \cdot M) = 0$, where M is a $\mathrm{Gal}(\overline{k}/k)$-orbit of a (-2)-curve on $\widetilde{S}_{\overline{k}}$. Hence, the assertion of Lemma 3.1.3 is not true unless $-K_{\widetilde{S}}$ is ample.

3.2 Proof of Theorem 1.3.3

Let the notation be the same as at the beginning of Chapter 3 and assume further that $-K_{\widetilde{S}}$ is not ample. In this section, we will prove the proof of Theorem 1.3.3. In other words, we shall classify minimal weak del Pezzo surfaces, whose anti-canonical divisor is not ample, over a field of characteristic zero. Note that minimal weak del Pezzo surfaces of degree ≥ 4 with anti-canonical divisor not ample over a perfect field are already classified by [15]. Our result thus gives a generalization to any degree of their work in the characteristic zero case.

3.2.1 Quasi-minimal weak del Pezzo surfaces

The purpose of this section is that we shall give to classify of minimal weak del Pezzo surfaces with anti-canonical divisor not ample. In order to state this classification, we shall introduce a weaker version of being minimal, which depends only on degree and type, the so-called being *quasi-minimal*.

Lemma 3.2.1. With the notation and the assumptions as above, assume further that $\rho_k(\widetilde{S}) = 2$. Then the type of $\widetilde{S}$ is either mA_1-type or mA_2-type for some $m \in \mathbb{Z}_{>0}$. In particular, the type of $\widetilde{S}$ is one of the following:

- $d = 7$ or 8 and A_1-type.
- $d = 6$ and A_2, $2A_1$, $(A_1)_<$ or $(A_1)_>$-type.
- $d = 5$ and A_2, $2A_1$ or A_1-type.
- $d = 4$ and $4A_1$, $3A_1$, A_2, $(2A_1)_<$, $(2A_1)_>$ or A_1-type.
- $d = 3$ and $3A_2$, $2A_2$, $4A_1$, $3A_1$, A_2, $2A_1$ or A_1-type.
- $d = 2$ and $3A_2$, $6A_1$, $5A_1$, $2A_2$, $(4A_1)_<$, $(4A_1)_>$, $(3A_1)_<$, $(3A_1)_>$, A_2, $2A_1$ or A_1-type.
- $d = 1$ and $4A_2$, $3A_2$, $6A_1$, $5A_1$, $2A_2$, $(4A_1)_<$, $(4A_1)_>$, $3A_1$, A_2, $2A_1$ or A_1-type.

Proof. At first, we show that the type of $\widetilde{S}$ is either mA_1-type or mA_2-type for some positive integer m. Let $\sigma : \widetilde{S} \to S$ be the contraction of all (-2)-curves on $\widetilde{S}_{\overline{k}}$, where σ is defined over k (see Section 2.4). By virtue of $1 \leq \rho_k(S) < \rho_k(\widetilde{S}) = 2$, it follows that S is a Du Val del Pezzo surface of rank one. Hence, we obtain $\rho_k(\widetilde{S}) - \rho_k(S) = 1$. This implies that all (-2)-curves on $\widetilde{S}_{\overline{k}}$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Thus, it must be that $\widetilde{S}$ is of mA_1-type or mA_2-type for some $m \in \mathbb{Z}_{>0}$ and all singularities on $S_{\overline{k}}$ are transformed to each other by means of the action of $\mathrm{Gal}(\overline{k}/k)$. Otherwise, by the dual graph of the union of all (-2)-curves on $\widetilde{S}_{\overline{k}}$, we can easily see $\rho_k(\widetilde{S}) - \rho_k(S) > 1$, which is a contradiction. Moreover, the remaining assertion follows from the above argument by combined with the classification of weak del Pezzo surfaces over algebraically closed fields of characteristic zero (see Appendix A.1). □

By Proposition 2.2.4 and Lemma 3.2.1, the type of any minimal weak del Pezzo surface, whose anti-canonical divisor is not ample, is one of those in the list of Lemma 3.2.1.

Now, let us consider an example of minimal weak del Pezzo surfaces. We say that a singular intersection of two quadrics $S \subseteq \mathbb{P}^4_k$ is an *Iskovskih surface* if its minimal resolution is a weak del Pezzo surface of degree 4 and of $(2A_1)_>$-type, and two Du Val singular points of type A_1 on $S_{\overline{k}}$ are exchanged by the $\mathrm{Gal}(\overline{k}/k)$-action. It is known that a weak del Pezzo surface of degree 4 is minimal if and only if it is the minimal resolution of an Iskovskih surface ([15, Theorem 7.2]). The following is an example of an Iskovskih surface studied by [31]:

Example 3.2.2. Let S be the singular intersection of two quadrics in $\mathbb{P}^4_{\mathbb{Q}}$ defined by:

$$S := (x^2 + y^2 + z^2 + zv = zv - w^2 + 3v^2 = 0) \subseteq \mathbb{P}^4_{\mathbb{Q}} = \mathrm{Proj}(\mathbb{Q}[x, y, z, v, w]).$$

Then S is an Iskovskih surface such that $S_{\overline{\mathbb{Q}}}$ has two Du Val singular points $[1 : \pm\sqrt{-1} : 0 : 0 : 0]$ of type A_1. Let $\widetilde{\tau} : \widetilde{S} \to S$ be the minimal resolution over $\mathbb{Q}$, so that S is a minimal weak del Pezzo surface of degree 4 and of $(2A_1)_<$-type over $\mathbb{Q}$. In particular, there are exactly eight (-1)-curves on $\widetilde{S}_{\overline{\mathbb{Q}}}$, which are proper transforms the following defining equations in $\mathbb{P}^4_{\overline{\mathbb{Q}}}$:

$$x \pm \sqrt{-1}y = z = \sqrt{3}w + u = 0,\ x \pm \sqrt{-1}y = z + w = \sqrt{2}w + u = 0,$$
$$x \pm \sqrt{-1}y = z = \sqrt{3}w - u = 0,\ x \pm \sqrt{-1}y = z + w = \sqrt{2}w - u = 0.$$

Thus, we know that $\widetilde{S}$ is minimal over $\mathbb{Q}$.

Remark 3.2.3. Note that the minimality of weak del Pezzo surfaces can not be detected by the type only. For instance, if we change the defining equation of S in Example 3.2.2 to $x^2 - y^2 + z^2 + zv,\ zv - w^2 + 3v^2 \in \mathbb{Q}[x, y, z, w, u]$, then $\widetilde{S}$ is also a weak del Pezzo surface of degree 4 and of $(2A_1)_<$-type but not minimal over $\mathbb{Q}$.

Now, letting E be any (-1)-curve on $\widetilde{S}_{\overline{k}}$, if $\widetilde{S}_{\overline{k}}$ is minimal, then there exists a (-1)-curve E' on $\widetilde{S}_{\overline{k}}$ such that $(E \cdot E') > 0$ and $|\mathscr{M}_E(i, j)| = |\mathscr{M}_{E'}(i, j)|$ for $i = 1, 2$ and $j = 1, 2$, where $\mathscr{M}_C(i, j)$ is the set defined by:

$$\mathscr{M}_C(i, j) := \{M \mid M : (-i)\text{-curve on } \widetilde{S}_{\overline{k}},\ (C \cdot M) = j\}$$

for $i = 1, 2,\ j = 1, 2$ and a projective curve C on $\widetilde{S}_{\overline{k}}$. By noticing this observation, we shall define a weaker version of minimality as follows:

Definition 3.2.4. Let the notation and the assumptions be the same as above. Then $\widetilde{S}$ is *quasi-minimal* if the following two conditions hold:

- $\widetilde{S}$ is either of mA_1-type or mA_2-type for some positive integer m.
- For any (-1)-curve E on $\widetilde{S}_{\overline{k}}$, there exists a (-1)-curve E' on $\widetilde{S}_{\overline{k}}$ such that $(E \cdot E') > 0$ and $|\mathscr{M}_E(i, j)| = |\mathscr{M}_{E'}(i, j)|$ for $i = 1, 2$ and $j = 1, 2$.

By definition, if $\widetilde{S}$ is minimal, then $\widetilde{S}$ is quasi-minimal. Furthermore, we actually see that quasi-minimality depends only on the type by the classification of weak del Pezzo surfaces over algebraically closed fields of characteristic zero (see also Definition 2.4.1).

Theorem 1.3.3 is a consequence of the following proposition:

Proposition 3.2.5. With the notation and the assumptions as above, the following three conditions are equivalent:

(1) $\widetilde{S}$ is minimal.

(2) $\rho_k(\widetilde{S}) = 2$ and $\widetilde{S}$ is quasi-minimal.

(3) $\rho_k(\widetilde{S}) = 2$ and the type of $\widetilde{S}$ is one of those in the list of Theorem 1.3.3.

Remark 3.2.6. Assume that Proposition 3.2.5 is true and there exists a weak del Pezzo surface $\widetilde{S}'$ with $\rho_k(\widetilde{S}') = 2$ such that $\widetilde{S}$ and $\widetilde{S}'$ have the same type. Then we see that $\widetilde{S}$ is quasi-minimal if and only if the type of $\widetilde{S}$ is one of those in the list of Theorem 1.3.3. Indeed, by Proposition 3.2.5, $\widetilde{S}'$ is quasi-minimal if and only if $\widetilde{S}'$ is one of those in the list of Theorem 1.3.3, moreover, since quasi-minimality depends on the type, $\widetilde{S}'$ is quasi-minimal if and only if $\widetilde{S}$ is quasi-minimal.

Let us prove Proposition 3.2.5. It is clear that (1) implies (2) in Proposition 3.2.5. Let us show that (2) implies (3) and (3) implies (1) in Proposition 3.2.5. In the case of $d = 8$, it can be easily seen that these two implications hold, indeed, $\widetilde{S}$ is always minimal since $\widetilde{S}$ is a k-form of the Hirzebruch surface $\mathbb{F}_2$ of degree two, i.e., $\widetilde{S}_{\overline{k}} \simeq \mathbb{F}_2$. However, in the case of $d < 8$, the proofs of these two implications are a bit long. Thus, we will give the proof for the case of $d < 8$ in Subsection 3.2.2.

3.2.2 Proof of Proposition 3.2.5

In this subsection, assume further $d \leq 7$. In order to prove Proposition 3.2.5, we prepare some notation.

We shall consider the following composition of blowing-ups over $\overline{k}$ from the projective plane $\mathbb{P}^2_{\overline{k}}$ to a weak del Pezzo surface $\widetilde{S}_d$ of degree d:

$$\widetilde{\tau} : \widetilde{S}_d \overset{\widetilde{\tau}_{9-d}}{\to} \widetilde{S}_{d+1} \overset{\widetilde{\tau}_{8-d}}{\to} \dots \overset{\widetilde{\tau}_2}{\to} \widetilde{S}_8 \overset{\widetilde{\tau}_1}{\to} \widetilde{S}_9 = \mathbb{P}^2_{\overline{k}} \tag{3.2.1}$$

such that $\widetilde{S}_d$ and $\widetilde{S}_{\overline{k}}$ have the same type as $\widetilde{S}_{\overline{k}}$ (see Definition 2.4.1), where $\widetilde{\tau}_i$ is a blow-up at a closed point for $i = 1, \dots, 9-d$. Notice that there exists such a birational morphism $\widetilde{\tau} : \widetilde{S}_d \to \mathbb{P}^2_{\overline{k}}$ by Lemma 2.1.3 and by the assumption $d \leq 7$. In what follows, we shall take a composite of blowing-ups (3.2.1).

Let e_0 be the proper transform on $\widetilde{S}_d$ of a general line on $\mathbb{P}^2_{\overline{k}}$ and let e_i be the total transform on $\widetilde{S}_d$ of the exceptional divisor of $\widetilde{\tau}_i$ for $i = 1, \dots, 9-d$. Then $\mathrm{Pic}(\widetilde{S}_d)$ can be expressed as the free $\mathbb{Z}$-module $I_d := \bigoplus_{i=0}^{9-d} \mathbb{Z}e_i$ with a bilinear form generated by $(e_0)^2 = 1$, $(e_i)^2 = -1$ for $i > 0$ and $(e_i \cdot e_j) = 0$ for $i, j \geq 0$ with $i \neq j$.

Letting $M_1, \dots, M_r$ be all (-2)-curves on $\widetilde{S}_d$, we note that each (-2)-curve corresponds to one of the following element in I_d (see [18, Proposition 8.2.7]):

$$\begin{array}{ll}
m^0_{i,j} := e_i - e_j & (0 < i < j \leq 9-d,\ d \leq 7); \\
m^1_{i_1,i_2,i_3} := e_0 - (e_{i_1} + e_{i_2} + e_{i_3}) & (0 < i_1 < i_2 < i_3 \leq 9-d,\ d \leq 6); \\
m^2 := 2e_0 - (e_1 + \cdots + e_6) & (d = 3); \\
m^2_{i_1,\dots,i_{3-d}} := 2e_0 - (e_{i_1} + \cdots + e_{i_{9-d}}) & (0 < i_1 < \cdots < i_{3-d} \leq 9-d,\ d \leq 2); \\
m^3_i := 3e_0 - (e_1 + \cdots + e_8) - e_i & (0 < i \leq 9-d,\ d = 1).
\end{array} \tag{3.2.2}$$

Letting $k_d := -3e_0 + e_1 + \cdots + e_{9-d} \in I_d$, which corresponds to the canonical divisor on $\widetilde{S}_d$, we also note that any $e \in I_d$ satisfying $(e)^2 = (e \cdot k_d) = -1$ is expressed as one of the following (see [18, Proposition 8.2.19]):

$$\begin{array}{ll}
e_i & (0 < i \leq 9-d,\ d \leq 7); \\
\ell_{i,j} := e_0 - (e_i + e_j) & (0 < i < j \leq 9-d,\ d \leq 7); \\
2e_0 - (e_{i_1} + \cdots + e_{i_5}) & (0 < i_1 < \cdots < i_5 \leq 9-d,\ d \leq 5); \\
-k_2 - e_i & (0 < i \leq 7,\ d = 2); \\
-k_1 - e_i + e_j & (0 < i, j \leq 8,\ i \neq j,\ d = 1); \\
-k_1 + e_0 - (e_{i_1} + e_{i_2} + e_{i_3}) & (0 < i_1 < i_2 < i_3 \leq 8,\ d = 1); \\
-k_1 + 2e_0 - (e_{i_1} + \cdots + e_{i_6}) & (0 < i_1 < \cdots < i_6 \leq 8,\ d = 1); \\
-2k_1 - e_i & (0 < i \leq 8,\ d = 1).
\end{array} \tag{3.2.3}$$

Table 3.1: Configuration of all (-2)-curves.

d	Type	all (-2)-curves
4	$(2A_1)_<$	$m^0_{1,2}$, $m^1_{3,4,5}$
2	A_1	m^2_1
2	A_2	$m^1_{2,3,4}$, $m^1_{5,6,7}$
2	$(4A_1)_>$	$m^1_{2,3,4}$, $m^1_{2,5,6}$, $m^1_{3,5,7}$, $m^1_{4,6,7}$
1	$2A_1$	$m^2_{1,2}$, m^3_2
1	$2A_2$	$m^1_{2,3,7}$, $m^1_{4,5,8}$, $m^1_{6,7,8}$, $m^2_{7,8}$

By Lemma 2.1.4, the set of all (-1)-curves on $\widetilde{S}_d$ has one-to-one correspondence to the set of all elements in (3.2.3) which have non-negative intersection number with elements in I_d corresponding to all (-2)-curves on $\widetilde{S}_d$. Thus, we are able to see the intersection form of all (-1)-curves and (-2)-curves on $\widetilde{S}_{\overline{k}}$ as the surfaces $\widetilde{S}_{\overline{k}}$ and $\widetilde{S}_d$ have the same type. In what follows, we will determine the quasi-minimality of $\widetilde{S}$ by studying elements as in (3.2.3) and (3.2.2) according to the type of $\widetilde{S}$.

Remark 3.2.7. A (-2)-curve M, which corresponds to an element $m^1_{i_1,i_2,i_3}$ (resp. m^2 or $m^2_{i_1,\dots,i_{3-d}}$, $m^3_{i_1}$) in I_d, is a proper transform of a line (resp. an irreducible conic, an irreducible cubic with a singular point) by a blow-up at some points on $\mathbb{P}^2_{\overline{k}}$, which may include infinitely near points. For instance, assuming that M corresponds to $m^1_{i_1,i_2,i_3}$, this blow-up includes infinitely near points if and only if there exists a (-2)-curve on $\widetilde{S}_d$ corresponding to $m^0_{i_1,i_2}$, $m^0_{i_1,i_3}$ or $m^0_{i_2,i_3}$in I_d.

Proof of (3) $\Longrightarrow$ (1) in Proposition 3.2.5

Let us prove that (3) implies (1) in Proposition 3.2.5. Assume that $\rho_k(\widetilde{S}) = 2$ and the type of $\widetilde{S}$ is one of those in the list of Theorem 1.3.3 other than the type of $d = 8$.

We shall take a composite of blowing-ups (3.2.1) in such a way that elements $m_1, \dots, m_r \in I_d$ corresponding to all (-2)-curves on $\widetilde{S}_d$ are as in Table 3.1 according to the type of $\widetilde{S}_d$ (for the notation of their elements, see (3.2.2)), where "all (-2)-curves" in Table 3.1 mean all elements in I_d corresponding to all (-2)-curves on $\widetilde{S}_d$, respectively. By construction, we see $\mathrm{Pic}(\widetilde{S}_{\overline{k}}) \simeq \mathrm{Pic}(\widetilde{S}_d) \simeq I_d$ preserving the intersection form. Then we obtain the following claim:

Claim 3.2.8. The following three assertions hold:

(1) For an arbitrary integer i with $2 \le i \le 9-d$, there exist two (-1)-curves $E_{i,+}$ and $E_{i,-}$ on $\widetilde{S}_d$ corresponding to elements e_i and $\ell_{1,i}$ in I_d, respectively.

(2) If $d \ge 2$, then all (-1)-curves meeting at least one (-2)-curve on $\widetilde{S}_d$ are only $E_{i,+}$ and $E_{i,-}$ for $2 \le i \le 9-d$.

(3) If $d = 1$, then all (-1)-curves meeting at least two (-2)-curves on $\widetilde{S}_d$ are only $E_{i,+}$ and $E_{i,-}$ for $2 \le i \le 9-d$.

Proof. In (1), we shall check that intersection numbers $(e_i \cdot m_j)$ and $(\ell_{1,i} \cdot m_j)$ are non-negative for $2 \le i \le 9-d$ and $1 \le j \le r$, however, it is left to the reader since it can be easily shown by explicit computing.

In (2) and (3), let E be a (-1)-curve on $\widetilde{S}_d$, let e be an element in I_d corresponding to E and set $m := m_1 + \cdots + m_r \in I_d$. Noting that e is one of those in the list of (3.2.3), we shall calculate the intersection number $(e \cdot m)$ according to degree d:

If $d = 4$, then $m = e_0 + e_1 - (e_2 + \cdots + e_5)$, so that we have:

$$(e \cdot m) = \begin{cases} 1 & \text{if } e = e_i \text{ or } \ell_{1,i} \\ -1 & \text{otherwise} \end{cases} \quad (2 \leq i \leq 5). \tag{3.2.4}$$

If $d = 2$ and $\widetilde{S}$ is of A_1-type or A_2-type (resp. $(4A_1)_>$-type), then $m = 2e_0 - (e_2 + \cdots + e_7)$ (resp. $m = 2\{2e_0 - (e_2 + \cdots + e_7)\}$), so that we have:

$$(e \cdot m) = \begin{cases} 1 \text{ (resp. } 2) & \text{if } e = e_i \text{ or } \ell_{1,i} \\ -1 \text{ (resp. } -2) & \text{if } e = -k_2 - e_i \text{ or } -k_2 - \ell_{1,i} \quad (2 \leq i \leq 7), \\ 0 & \text{otherwise} \end{cases} \tag{3.2.5}$$

where we note $-k_2 - \ell_{1,i} = 2e_0 - (e_2 + \cdots + e_7) + e_i$ for $2 \leq i \leq 7$.

If $d = 1$, then $m = 5e_0 - e_1 - 2(e_2 + \cdots + e_8) = -2k_1 - (e_0 - e_1)$, so that we have:

$$\begin{aligned} (e_i \cdot m) &= \begin{cases} 2 & \text{if } i > 1 \\ 1 & \text{if } i = 1 \end{cases} \quad (1 \leq i \leq 8); \\ (\ell_{i,j} \cdot m) &= \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i > 1 \end{cases} \quad (1 \leq i < j \leq 8); \\ (e \cdot m) &< 2 \quad \text{if } (e \cdot e_0) \geq 2, \end{aligned} \tag{3.2.6}$$

where we note $(e \cdot m) = (e \cdot -2k_1) - (e \cdot e_0 - e_1)$ and $(e \cdot e_0 - e_1) > 0$ by (3.2.3) if $(e \cdot e_0) \geq 2$.

Therefore, we obtain the assertions (2) and (3) of the claim. Indeed, if $d \geq 2$ (resp. $d = 1$) and E meets at least one (-2)-curve (resp. at least two (-2)-curves) on $\widetilde{S}_d$, then E is $E_{i,+}$ or $E_{i,-}$ for some $2 \leq i \leq 9 - d$ by virtue of (3.2.4) and (3.2.5) (resp. (3.2.6)). □

Now, we shall prove that (3) implies (1) in Proposition 3.2.5. Let $\widetilde{S}_d$ be the same as above. Let D be the union of (-1)-curves on $\widetilde{S}_{\overline{k}}$ corresponding to elements e_i and $\ell_{1,i}$ in I_d for $2 \leq i \leq 9 - d$ in I_d. By Claim 3.2.8, we see that D is defined over k. Moreover, we have $(D)^2 = \left(\sum_{i=2}^{9-d}(e_i + \ell_{1,i})\right)^2 = (8 - d)^2\{(e_0)^2 - (e_1)^2\} = 0$.

Suppose on the contrary that there exists a birational morphism $\tau : S \to V$ to smooth projective surface V with $\rho_k(V) < \rho_k(\widetilde{S})$ defined over k. Then V is a smooth del Pezzo surface of $\rho_k(V) = 1$ by virtue of $\rho_k(\widetilde{S}) = 2$ (see also Lemma 2.2.4). Hence, there exists a (-1)-curve E meeting at least one (-2)-curve on $\widetilde{S}_{\overline{k}}$ such that $\tau_{\overline{k}}$ is a contraction of the $\mathrm{Gal}(\overline{k}/k)$-orbit of E. Notice that E is not any irreducible component of D. Otherwise, we have $\tau_*(D) \neq 0$ and $(\tau_*(D))^2 = 0$ by Claim 3.2.8(1). This is a contradiction to the fact $\rho_k(V) = 1$. Hence, we see that $d = 1$ and E meets only one (-2)-curve on $\widetilde{S}_{\overline{k}}$ by Claim 3.2.8(2) and (3). Let $M_1, \ldots, M_r$ be all (-2)-curves on $\widetilde{S}_{\overline{k}}$, where $r = 2$ (resp. $r = 4$) if $\widetilde{S}$ is of $2A_1$-type (resp. $2A_2$-type). Furthermore, let s be the number of (-1)-curves on $\widetilde{S}_{\overline{k}}$, which meet a (-2)-curve M_1 and are contracted by $\tau_{\overline{k}}$, where we note that $\widetilde{S}$ is constant not depending on the way to take a (-2)-curve M_1 on $\widetilde{S}_{\overline{k}}$. Indeed, all (-2)-curves on $\widetilde{S}_{\overline{k}}$ lie on the same $\mathrm{Gal}(\overline{k}/k)$-orbit since $V_{\overline{k}}$ does not contain any (-2)-curve. If $\widetilde{S}$ is of $2A_1$-type (resp. $2A_2$-type), then the degree of $V_{\overline{k}}$ is equal to $2s + 1$ (resp. $4s + 1$), which is not equal to 7 and is at most 9, and we obtain $0 < (\tau_*(M_1 + \cdots + M_r))^2 = -4 + 2s$ (resp. $-4 + 4s$) by virtue of $\rho_k(V) = 1$. Thus, $V_{\overline{k}}$ is of degree 9, namely, $V_{\overline{k}} \simeq \mathbb{P}^2_{\overline{k}}$. In particular, the self-intersection number of any irreducible curve on $V_{\overline{k}}$ is a positive square number, however, $(\tau_{\overline{k},*}(M_1))^2$ is equal to 2 (resp. 0) if $\widetilde{S}$ is of $2A_1$-type (resp. $2A_2$-type). It is a contradiction.

Therefore, we see that $\widetilde{S}$ must be k-minimal.

Remark 3.2.9. Assuming $\rho_k(\widetilde{S}) = 2$, we see that $\widetilde{S}$ is minimal by the above argument. Letting $\widetilde{S}_d$ be the same as above, for any $2 \le i \le 9-d$, two (-1)-curves on $\widetilde{S}_{\overline{k}}$, which correspond to e_i and $\ell_{1,i}$ respectively, lie on the same $\mathrm{Gal}(\overline{k}/k)$-orbit.

Proof of (2) $\Longrightarrow$ (3) in Proposition 3.2.5

In order to prove that (2) implies (3) in Proposition 3.2.5, assume that the type of $\widetilde{S}$ is one of those in the list of Lemma 3.2.1 such that it does not appear in the list of Theorem 1.3.3. Then we shall show that $\widetilde{S}$ is not quasi-minimal.

At first, we deal with the case in which $\widetilde{S}$ is of degree $d = 1$ and of A_1-type. We can choose a composite of blowing-ups (3.2.1) in such a way that $\widetilde{S}_d = \widetilde{S}_1$ contains only one (-2)-curve corresponding to $m_1^3 \in I_1$ (for the notation of m_1^3, see (3.2.2)). Letting e be one of those in the list of (3.2.3), we obtain that $(e \cdot m_1^3) = 2$ if and only if $e = e_1$, indeed, $(e \cdot m_1^3) = (e \cdot -k_1) - (e \cdot e_1) = 1 - (e \cdot e_1)$. Since $\widetilde{S}_1$ and $\widetilde{S}_{\overline{k}}$ have the same type, there exists a unique (-1)-curve E satisfying $(E \cdot M) = 2$ on $\widetilde{S}_{\overline{k}}$, where M is the unique (-2)-curve on $\widetilde{S}_{\overline{k}}$. This means that $|\mathscr{M}_E(2,2)| = 1$ and there is no (-1)-curve E' meeting E on $\widetilde{S}_{\overline{k}}$ such that $|\mathscr{M}_{E'}(2,2)| = 1$. Hence $\widetilde{S}$ is not quasi-minimal.

In what follows, we deal with the remaining cases. As an example, we shall explain the case in which $\widetilde{S}$ is of degree $d = 2$ and of $(3A_1)_>$-type. Then $\widetilde{S}_{\overline{k}}$ contains exactly three (-2)-curves. Let us put $\alpha := 2$, where we notice that α is smaller than or equal to the number of (-2)-curves on $\widetilde{S}_{\overline{k}}$. Let β be the number of (-1)-curves on $\widetilde{S}_{\overline{k}}$ meeting exactly α-times of (-2)-curves on $\widetilde{S}_{\overline{k}}$. In order to determine the value of β, we shall take a composite of blowing-ups (3.2.1) in such a way that $\widetilde{S}_d = \widetilde{S}_2$ contains exactly three (-2)-curves corresponding to $m_{1,2,3}^1$, $m_{1,3,4}^1$, $m_1^2 \in I_2$ (see (3.2.2)). Then we see that elements in I_2 corresponding to all (-1)-curves meeting exactly α-times of (-2)-curves on $\widetilde{S}_2$ are only $e_1, \dots, e_5$ and $\ell_{6,7}$ (see Example 3.2.11). Hence, we obtain $\beta = 6$. Moreover, the union of β-times of (-1)-curves on $\widetilde{S}_2$, which correspond to $e_1, \dots, e_5$ and $\ell_{6,7}$ in I_2, is disjoint. Since $\widetilde{S}_2$ and $\widetilde{S}_{\overline{k}}$ have the same type, letting E be a (-1)-curve on $\widetilde{S}_{\overline{k}}$ corresponding to one of $e_1, \dots, e_5$ or $\ell_{6,7}$ in I_2, we see that $|\mathscr{M}_E(2,1)| = \alpha$ and there is no (-1)-curve E' meeting E on $\widetilde{S}_{\overline{k}}$ such that $|\mathscr{M}_{E'}(2,1)| = \alpha$. Thus, $\widetilde{S}$ is not quasi-minimal.

The other cases can be shown by a similar argument, by changing the value of α and elements in I_d, which correspond to all (-2)-curves on $\widetilde{S}_d$, according to the type of $\widetilde{S}$. We will now explain how to do this. Let α be this as in Table 3.2 according to the type of $\widetilde{S}$, and let us take a composite of blowing-ups (3.2.1) in such a way that all (-2)-curves on $\widetilde{S}_d$ corresponding to elements in I_d, which are these as in "all (-2)-curves" in Table 3.2 according to the type of $\widetilde{S}$. Then we see that elements in I_2, which correspond to all (-1)-curves meeting exactly α-times of (-2)-curves on $\widetilde{S}_d$, are only these as in "β-times of (-1)-curves" in Table 3.2 according to the type of $\widetilde{S}$ (see Examples 3.2.11 and 3.2.12 for how to find all elements in I_d). For instance, if $d = 2$ and the type of $\widetilde{S}$ is $(3A_1)_>$-type, then such these elements yield $e_1, \dots, e_5, \ell_{6,7} \in I_2$ as demonstrated above. Hence, β is this as in Table 3.2 according to the type of $\widetilde{S}$. Moreover, we see that the union of β-times of (-1)-curves, which meet exactly α-times of (-2)-curves on $\widetilde{S}_d$, is disjoint. Since $\widetilde{S}_d$ and $\widetilde{S}_{\overline{k}}$ have the same type, letting E be a (-1)-curve on $\widetilde{S}_{\overline{k}}$ corresponding to one of β-times of (-1)-curves meeting exactly α-times of

(-2)-curves on $\widetilde{S}_d$, we see that $|\mathscr{M}_E(2,1)| = \alpha$ and there is no (-1)-curve E' meeting E on $\widetilde{S}_{\overline{k}}$ such that $|\mathscr{M}_{E'}(2,1)| = \alpha$. Thus, $\widetilde{S}$ is not quasi-minimal.

In summary, we show that $\widetilde{S}$ is quasi-minimal if the type of $\widetilde{S}$ is one of those in the list of Theorem 1.3.3.

Remark 3.2.10. In the above argument, we do not actually use the assumption $\rho_k(\widetilde{S}) = 2$.

Example 3.2.11. Assume that $\widetilde{S}$ is of degree $d = 2$ and of $(3A_1)_>$-type. Then $\widetilde{S}_{\overline{k}}$ contains exactly three (-2)-curves. Let us put $\alpha := 2$ and let us choose a composite of blowing-ups (3.2.1) in such a way that $\widetilde{S}_d = \widetilde{S}_2$ contains exactly three (-2)-curves corresponding to $m^1_{1,2,3}$, $m^1_{1,4,5}$, $m^2_1 \in I_2$ (see Table 3.2). Then we shall determine all elements in I_2 corresponding to all (-1)-curves meeting exactly two (-2)-curves on $\widetilde{S}_{\overline{k}}$. At first, we can easily check that intersection numbers $(e \cdot m^1_{1,2,3})$, $(e \cdot m^1_{1,4,5})$ and $(e \cdot m^2_1)$ are equal to 0 or 1 for any $e = e_1, \dots, e_5$, $\ell_{6,7}$. Next, we put $m := m^1_{1,2,3} + m^1_{1,4,5} + m^2_1$ and determine any element $e \in I_2$ as in (3.2.3) satisfying $(e \cdot m) = \alpha (= 2)$. In consideration of $m = 4e_0 - 2(e_1 + \dots + e_5) - (e_6 + e_7)$, we can calculate as follows:

- If $e = e_i$ $(1 \leq i \leq 7)$, then $(e \cdot m) = 2$ if and only if $1 \leq i \leq 5$.
- If $e = \ell_{i,j}$ $(1 \leq i < j \leq 7)$, then $(e \cdot m) = 2$ if and only if $(i,j) = (6,7)$.
- If $e = 2e_0 - (e_{i_1} + \dots + e_{i_5})$ $(1 \leq i_1 < \dots < i_5 \leq 7)$, then $(e \cdot m) \leq 8 - 2 \cdot 1 - 3 \cdot 2 = 0 < 2$.
- If $e = -k_1 - e_i$ $(1 \leq i \leq 7)$, then $(e \cdot m) \leq -1 < 2$.

Thus, we certainly see that all elements in I_2, which correspond to all (-1)-curves meeting exactly $\alpha (= 2)$-times of (-2)-curves on $\widetilde{S}_{\overline{k}}$, are exhausted by $e_1, \dots, e_5$ and $\ell_{6,7}$. For the other cases with $d \geq 2$, we can calculate in a similar way.

The following deals with all cases of $d = 1$:

Example 3.2.12. Assume that $\widetilde{S}$ is of degree $d = 1$. We shall take a composite of blowing-ups (3.2.1) in such a way that elements $m_1, \dots, m_r \in I_1$ corresponding to all (-2)-curves on $\widetilde{S}_1$ are as in Table 3.2, according to the type of $\widetilde{S}$. Then the element $m := m_1 + \dots + m_r$ is expressed as follows depending on the types of $\widetilde{S}$:

- $4A_2$ or $(4A_1)_<$: $(3\alpha - 1)e_0 - \alpha(e_1 + \dots + e_8)$, where $\alpha = 3$, $\beta = 8$;
- $5A_1$: $(3\alpha - 5)e_0 - \alpha e_1 - (\alpha - 1)(e_2 + e_3 + e_4) - (\alpha - 2)(e_5 + \dots + e_8)$, where $\alpha = 4$, $\beta = 1$;
- Otherwise: $3\alpha' e_0 - \alpha(e_1 + \dots + e_\beta) - \alpha'(e_{\beta+1} + \dots + e_8)$, where $\alpha' < \alpha$.

Hence, letting e be one of those in the list of (3.2.3), if $(e \cdot m) = \alpha$, then we see that $(e \cdot e_0) = 0$, i.e., $e = e_i$ for some $1 \leq i \leq 8$. Indeed, assuming $(e \cdot e_0) > 0$, we have $(e \cdot m) < (e \cdot -\alpha k_1) = \alpha$ by noting $(e \cdot e_i) \geq 0$ $(1 \leq i \leq 8)$. Moreover, we see that $(e_i \cdot m) = \alpha$ if and only if $1 \leq i \leq \beta$. Obviously we obtain $(e_i \cdot m_j) \geq 0$ for $1 \leq i \leq \beta$ and $1 \leq j \leq r$.

3.3 Proof of Theorem 1.3.4

Let the notation be the same as at the beginning of Chapter 3 and assume further that $\widetilde{S}$ is minimal over k and $\rho_k(\widetilde{S}) = 2$. In this section, we shall prove Theorem 1.3.4. Notice that Theorem 1.3.4 is a consequence of the following proposition:

Proposition 3.3.1. With the notation and the assumptions as above, the following three assertions hold true:

Table 3.2: The value of β and configuration of β-times of (-1)-curves.

d	Type	α	all (-2)-curves	β	β-times of (-1)-curves
7	A_1	1	$m^0_{1,2}$	1	e_2
6	A_2	1	$m^0_{1,2},\quad m^0_{2,3}$	2	$e_3,\ \ell_{1,2}$
6	$2A_1$	2	$m^0_{1,2},\quad m^1_{1,2,3}$	1	e_2
6	$(A_1)_<$	1	$m^1_{1,2,3}$	3	$e_1,\ e_2,\ e_3$
6	$(A_1)_>$	1	$m^0_{1,2}$	2	$e_2,\ \ell_{1,3}$
5	A_2	1	$m^0_{1,2},\quad m^1_{1,3,4}$	3	$e_2,\ e_3,\ e_4$
5	$2A_1$	2	$m^0_{1,2},\quad m^1_{1,2,3}$	1	e_2
5	A_1	1	$m^1_{1,2,3}$	3	$e_1,\ e_2,\ e_3$
4	$4A_1$	2	$m^0_{1,2},\quad m^0_{3,4},\quad m^1_{1,2,5},\ m^1_{3,4,5}$	4	$e_2,\ e_4,\ e_5,\ \ell_{1,3}$
4	$3A_1$	2	$m^0_{1,2},\quad m^0_{3,4},\quad m^1_{1,2,5}$	2	$e_2,\ \ell_{1,3}$
4	A_2	1	$m^1_{1,2,3},\ m^0_{4,5}$	4	$e_1,\ e_2,\ e_3,\ \ell_{4,5}$
4	$(2A_1)_>$	2	$m^1_{1,2,3},\ m^1_{1,4,5}$	1	e_1
4	A_1	1	$m^1_{1,2,3}$	4	$e_1,\ e_2,\ e_3,\ \ell_{4,5}$
3	$3A_2$	2	$m^0_{1,2},\quad m^1_{1,3,4},\ m^0_{3,4},\quad m^1_{3,5,6},$ $m^0_{5,6},\quad m^1_{1,2,5}$	3	$e_2,\ e_4,\ e_6$
3	$2A_2$	2	$m^0_{1,2},\quad m^1_{1,5,6},\ m^0_{3,4},\quad m^1_{1,2,3}$	1	e_2
3	$4A_1$	2	$m^1_{1,2,3},\ m^1_{1,4,5},\ m^1_{2,4,6},\ m^1_{3,5,6}$	6	$e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6$
3	$3A_1$	2	$m^1_{1,2,4},\ m^1_{1,3,5},\ m^1_{2,3,6}$	3	$e_1,\ e_2,\ e_3$
3	A_2	1	$m^1_{1,2,3},\ m^1_{4,5,6}$	6	$e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6$
3	$2A_1$	2	$m^1_{1,2,3},\ m^1_{1,4,5}$	1	e_1
3	A_1	1	m^2	6	$e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6$
2	$3A_2$	2	$m^0_{1,2},\quad m^1_{1,3,4},\ m^0_{3,4},\quad m^1_{3,5,6},$ $m^0_{5,6},\quad m^1_{1,2,5}$	6	$e_2,\ e_4,\ e_6,\ \ell_{1,7},\ \ell_{3,7},\ \ell_{5,7}$
2	$6A_1$	3	$m^1_{1,2,5},\ m^1_{1,3,6},\ m^1_{1,4,7},\ m^1_{2,3,7},$ $m^1_{2,4,6},\ m^1_{3,4,5}$	4	$e_1,\ e_2,\ e_3,\ e_4$
2	$5A_1$	3	$m^1_{1,2,3},\ m^1_{1,4,5},\ m^1_{1,6,7},\ m^1_{2,4,6},\ m^1_{2,5,7}$	2	$e_1,\ e_2$
2	$2A_2$	2	$m^0_{1,2},\quad m^1_{1,3,7},\ m^1_{1,2,6},\ m^1_{3,4,5}$	2	$e_2,\ e_3$
2	$(4A_1)_<$	3	$m^1_{1,2,3},\ m^1_{1,4,5},\ m^1_{1,6,7},\ m^1_{2,4,6}$	1	e_1
2	$(3A_1)_<$	3	$m^1_{1,2,3},\ m^1_{1,4,5},\ m^1_{1,6,7}$	1	e_1
2	$(3A_1)_>$	2	$m^1_{1,2,3},\ m^1_{1,4,5},\ m^2_1$	6	$e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ \ell_{6,7}$
2	$2A_1$	2	$m^1_{1,2,3},\ m^2_3$	2	$e_1,\ e_2$
1	$4A_2$	3	$m^1_{1,3,4},\ m^1_{2,5,8},\ m^1_{1,5,6},\ m^1_{2,4,7},$ $m^1_{1,7,8},\ m^1_{2,3,6}\quad m^1_{3,5,7},\ m^1_{4,6,8}$	8	$e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7,\ e_8$
1	$3A_2$	3	$m^1_{1,3,4},\ m^1_{2,5,8},\ m^1_{1,5,6},\ m^1_{2,4,7},$ $m^1_{1,7,8},\ m^1_{2,3,6}$	2	$e_1,\ e_2$
1	$6A_1$	4	$m^1_{1,2,4},\ m^1_{1,3,5},\ m^1_{2,3,6},\ m^2_{1,6},$ $m^2_{2,5},\quad m^2_{3,4}$	3	$e_1,\ e_2,\ e_3$
1	$5A_1$	4	$m^1_{1,3,6},\ m^1_{1,4,5},\ m^1_{2,3,4},\ m^2_{3,5},\ m^2_{4,6}$	1	e_1
1	$(4A_1)_<$	3	$m^2_{1,2},\quad m^2_{3,4},\quad m^2_{5,6},\quad m^2_{7,8}$	8	$e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7,\ e_8$
1	$(4A_1)_>$	4	$m^1_{1,2,3},\ m^1_{1,4,5},\ m^1_{1,6,7},\ m^3_8$	1	e_1
1	$3A_1$	3	$m^1_{1,2,3},\ m^1_{1,4,5},\ m^1_{1,6,7}$	1	e_1
1	A_2	2	$m^1_{1,2,3},\ m^2_{2,3}$	1	e_1

(1) If $d = 8$, then $\widetilde{S}$ contains an $\mathbb{A}^1_k$-cylinder if and only if there exists a conic bundle $\pi : \widetilde{S} \to B$, which admits a section defined over k.

(2) If $d = 8$, then $\widetilde{S}$ contains the affine plane $\mathbb{A}^2_k$ if and only if $\widetilde{S}(k) \neq \emptyset$.

(3) If $d < 8$, then $\widetilde{S}$ does not contain any $\mathbb{A}^1_k$-cylinder.

We will prove Proposition 3.3.1 according to the degree d of $\widetilde{S}$. More precisely, Proposition 3.3.1 (1) and (2) will be shown in Subsection 3.3.1 and Proposition 3.3.1 (3) will be shown in Subsection 3.3.2.

3.3.1 Case of degree 8

In this subsection, we shall show Proposition 3.3.1 (1) and (2). Let us assume $d = 8$. Then $\widetilde{S}$ is a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ or the Hirzebruch surface $\mathbb{F}_2$ of degree two, i.e., $\widetilde{S}_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ or $\widetilde{S}_{\overline{k}} \simeq \mathbb{F}_2$. Moreover, $\widetilde{S}$ is endowed with a structure of Mori conic bundle $\pi : \widetilde{S} \to B$ such that the base extension of π to the algebraic closure $\pi_{\overline{k}} : \widetilde{S}_{\overline{k}} \to B_{\overline{k}}$ is a $\mathbb{P}^1$-bundle over $B_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}}$ by Lemma 3.1.1.

We shall consider the following three conditions:

(A) $\widetilde{S}$ contains an $\mathbb{A}^1_k$-cylinder.

(B) There exists a Mori conic bundle $\pi : \widetilde{S} \to B$, which admits a section defined over k.

(C) $\widetilde{S}(k) \neq \emptyset$.

Then the following three lemmas hold:

Lemma 3.3.2. (C) implies (B).

Proof. Noting $\widetilde{S}(k) \neq \emptyset$ and $\rho_k(\widetilde{S}) = 2$, we see that $\widetilde{S} \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$ or $\widetilde{S}$ is the Hirzebruch surface of degree two defined over k (i.e., $\widetilde{S} \simeq \mathbb{P}(\mathscr{O}_{\mathbb{P}^1_k} \oplus \mathscr{O}_{\mathbb{P}^1_k}(2))$) by using Lemma 2.2.2. In particular, there exists a $\mathbb{P}^1$-bundle $\widetilde{S} \to \mathbb{P}^1_k$ over k, which admits a section defined over k. □

Lemma 3.3.3. (A) implies (B).

Proof. Suppose that $\widetilde{S}$ contains an $\mathbb{A}^1_k$-cylinder, say $U \simeq \mathbb{A}^1_k \times Z$, and there is no Mori conic bundle, which admits a section defined over k. The closures in $\widetilde{S}$ of fibers of the projection $pr_Z : U \simeq \mathbb{A}^1_k \times Z \to Z$ yields a linear system, say $\widetilde{\mathscr{L}}$, on $\widetilde{S}$. Hence, we obtain the rational map $\Phi_{\widetilde{\mathscr{L}}} : \widetilde{S} \dashrightarrow \overline{Z}$ associated with $\widetilde{\mathscr{L}}$, where $\overline{Z}$ is the smooth projective model of Z. If $\Phi_{\widetilde{\mathscr{L}}}$ is a morphism, then $\Phi_{\widetilde{\mathscr{L}}}$ is a Mori conic bundle, which admits a section defined over k and is contained in $\widetilde{S} \backslash U$, by Lemma 3.1.2. It is a contradiction to the assumption. Hence, $\widetilde{\mathscr{L}}$ is not base point-free. Then the base extension of $\widetilde{\mathscr{L}}$, say $\widetilde{\mathscr{L}_{\overline{k}}}$, is not also base point-free. Since fibers of the base extension $pr_{Z_{\overline{k}}} : U_{\overline{k}} \simeq \mathbb{A}^1_{\overline{k}} \times Z_{\overline{k}} \to Z_{\overline{k}}$ are isomorphic to the affine line, in particular, having only one-place at infinity, $\mathrm{Bs}(\widetilde{\mathscr{L}_{\overline{k}}})$ is composed of one point. Furthermore, this point is defined over k. Thus, $\mathrm{Bs}(\widetilde{\mathscr{L}})$ consists of only one k-rational point, which contradicts Lemma 3.3.2. □

Lemma 3.3.4. (B) implies (A).

Proof. By the assumption, we can take a Mori conic bundle $\pi : \widetilde{S} \to B$, which admits a section defined over k, and let Γ be a section of π defined over k. As π itself is defined over k, the base curve $B_{\overline{k}}$ is also equipped with an action of $\mathrm{Gal}(\overline{k}/k)$ induced from that on $\widetilde{S}_{\overline{k}}$. The complement, say U', of a divisor composed of Γ and the pull-back by $\pi_{\overline{k}}$ of a $\mathrm{Gal}(\overline{k}/k)$-orbit on $B_{\overline{k}}$ is then a smooth affine surface defined over k. The restriction $\varphi := \pi|_{U'}$ of π to U' yields a morphism over an affine curve $Z' \subseteq B$. By construction, the base extension $\varphi_{\overline{k}}$ is an $\mathbb{A}^1$-bundle to conclude that so is φ by [34, Theorem 1], which implies that there exists an open subset $Z \subseteq Z'$ such that $\varphi^{-1}(Z) \simeq \mathbb{A}^1_k \times Z$. This completes the proof. □

Proposition 3.3.1 (1) follows from Lemmas 3.3.3 and 3.3.4. Next, we will show Proposition 3.3.1 (2) as follows:

Proof of Proposition 3.3.1 (2). Assume that $\widetilde{S}$ admits a k-rational point. Let $\pi : \widetilde{S} \to B$ be a Mori conic bundle. Then the base B is a geometrically rational curve admitting a k-rational point to conclude that B is isomorphic to $\mathbb{P}^1_k$ by Lemma 2.2.2. Thus, $\widetilde{S}$ contains the affine plane $\mathbb{A}^2_k$. The converse direction is obvious. □

3.3.2 Case of degree less than 8

In this subsection, let us assume $d < 8$. The purpose of this subsection is to prove Proposition 3.3.1 (3). In other words, we shall show that $\widetilde{S}$ does not contain any $\mathbb{A}^1_k$-cylinder by using Lemma 2.5.5. By Lemma 3.1.1, $\widetilde{S}$ is endowed with a structure of Mori conic bundle $\pi : \widetilde{S} \to B$ such that $\pi_{\overline{k}}$ admits a singular fiber. Notice that B is isomorphic to $\mathbb{P}^1_k$ provided that $\widetilde{S}$ admits a k-rational point.

Lemma 3.3.5. With a notation and the assumptions as above, then $d \leq 4$.

Proof. If $-K_{\widetilde{S}}$ is not ample, then it follows from Theorem 1.3.3. Hence, we may assume that $-K_{\widetilde{S}}$ is ample in what follows. Then $\widetilde{S}$ is a smooth minimal del Pezzo surface of rank two. Noting $d \neq 7, 9$, suppose that $d = 5$ or 6. By Proposition 2.2.4, $\widetilde{S}$ is endowed with a structure of Mori conic bundle defined over k, say $\pi : \widetilde{S} \to B$. Any (-1)-curve on $S_{\overline{k}}$, which is not an irreducible component of any singular fiber of $\pi_{\overline{k}}$, meets all singular fibers of $\pi_{\overline{k}}$. Notice that $\widetilde{S}_{\overline{k}}$ contains exactly $(8-d)$-times of singular fibers of $\pi_{\overline{k}}$ such that each one consists of two (-1)-curves, which lying the same $\mathrm{Gal}(\overline{k}/k)$-orbit, on $\widetilde{S}_{\overline{k}}$ meeting transversally at a point. By the hypothesis, it can be easily seen that any (-1)-curve on $\widetilde{S}_{\overline{k}}$ meets transversally exactly $(8-d)$-times of (-1)-curves on $S_{\overline{k}}$ since $d \geq 5$ and there exists a birational morphism to $\mathbb{P}^2_{\overline{k}}$, which is a composite of $(9-d)$-times blow-up. Thus, the union of all (-1)-curves on $\widetilde{S}_{\overline{k}}$, none of which is an irreducible component of any singular fiber of $\pi_{\overline{k}}$, is defined over k and is disjoint. It is a contradiction to the minimality of $\widetilde{S}$. □

Remark 3.3.6. If $-K_{\widetilde{S}}$ is not ample, then we further see $d \neq 3$ by Theorem 1.3.3. Moreover, $d \neq 3$ even if $-K_{\widetilde{S}}$ is ample (see [52, Theorem 28.1]).

Suppose on the contrary that $\widetilde{S}$ contains an $\mathbb{A}^1_k$-cylinder, say $U \simeq \mathbb{A}^1_k \times Z$, where Z is a smooth affine curve defined over k. The closures in $\widetilde{S}$ of fibers of the projection $pr_Z : U \simeq Z \times \mathbb{A}^1_k \to Z$ yields a linear system, say $\widetilde{\mathscr{L}}$, on $\widetilde{S}$.

Claim 3.3.7. The base locus $\mathrm{Bs}(\widetilde{\mathscr{L}})$ consists of only one point, which is k-rational.

Proof. Let $\Phi_{\widetilde{\mathscr{L}}} : \widetilde{S} \dashrightarrow \overline{Z}$ be the rational map associated with $\widetilde{\mathscr{L}}$, where $\overline{Z}$ is the smooth projective model of Z. Assume that $\mathrm{Bs}(\widetilde{\mathscr{L}})$ is base point-free. Then $\Phi_{\widetilde{\mathscr{L}}}$ is a morphism, in particular, it is a Mori conic bundle, which admits a section defined over k and is contained in $\widetilde{S} \backslash U$, by Lemma 3.1.2. However, this is a contradiction to Lemma 3.1.1 (3). Thus, $\mathrm{Bs}(\widetilde{\mathscr{L}})$ is not base point-free. By the similar argument as Lemma 3.3.3, we see that $\mathrm{Bs}(\widetilde{\mathscr{L}})$ consists of only one k-rational point. $\square$

Let us denote by p the base point of the linear system $\widetilde{\mathscr{L}}$. Recall that $\widetilde{S}$ is endowed with a structure of a Mori conic bundle $\pi : \widetilde{S} \to B$ over a geometrically rational curve B defined over k. Since p is k-rational by Claim 3.3.7, so is its image via π, in particular, $B \simeq \mathbb{P}^1_k$ by Lemma 2.2.2. Since Z is contained in a projective line $\mathbb{P}^1_k$ on k by the similar argument, $\widetilde{\mathscr{L}}$ is a linear pencil on $\widetilde{S}$. Moreover, we can easily to see $\mathrm{Pic}(\widetilde{S})_{\mathbb{Q}} = \mathbb{Q}[-K_{\widetilde{S}}] \oplus \mathbb{Q}[F]$, where F is a general fiber of π, which passes through p. In particular, $\widetilde{\mathscr{L}}$ is $\mathbb{Q}$-linearly equivalent to $a(-K_{\widetilde{S}}) + bF$ for some rational numbers a, b.

Proof of Proposition 3.3.1 (3). With the notation and the assumptions as above, we notice $d \leq 4$ by Lemma 3.3.5. In this proof, we will consider whether $-K_{\widetilde{S}}$ is ample or not as follows.

At first, we shall consider the case that $-K_{\widetilde{S}}$ is not ample. By the assumption, there exists a $\mathrm{Gal}(\overline{k}/k)$-orbit of a (-2)-curve on $\widetilde{S}$, say M. Then we have $(M \cdot -K_{\widetilde{S}}) = 0$. Moreover, we notice $(M \cdot F) > 0$ since every (-2)-curve on $\widetilde{S}_{\overline{k}}$ is not included in any singular fiber of $\pi_{\overline{k}}$. Thus, we have $b \geq 0$ by virtue of $0 \leq (M \cdot \widetilde{\mathscr{L}}) = b(M \cdot F)$. However, it is a contradiction to Lemma 2.5.5.

Next, we shall consider the case that $-K_{\widetilde{S}}$ is ample. By Lemma 2.5.5, we obtain $a > 0$ and $b < 0$. By Lemma 3.1.3, there exists a Mori conic bundle $\pi_2 : \widetilde{S} \to \mathbb{P}^1_k$ such that a fiber F_2 of π_2 passing through p is linearly equivalent to $\frac{4}{d}(-K_{\widetilde{S}}) - F$. Thus, we can write $\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} (a + \frac{4}{d}b)(-K_{\widetilde{S}}) - bF_2$. Then we obtain $-b > 0$ by Lemma 2.5.5 again. However, it is a contradiction to $b > 0$.

Therefore, $\widetilde{S}$ never contains an $\mathbb{A}^1_k$-cylinder for both cases. $\square$

Chapter 4

Cylinders in canonical del Pezzo fibrations

The purpose of this chapter is to prove Theorem 1.3.9. Throughout this chapter, let k be a field of characteristic zero, let S be a Du Val del Pezzo surface defined over k such that $\mathrm{Sing}(S_{\overline{k}}) \neq \emptyset$. Here, if S is smooth, then S clearly satisfies all assertions in Theorem 1.3.9 by virtue of Theorem 1.2.4. Hence, the assumption $\mathrm{Sing}(S_{\overline{k}}) \neq \emptyset$ is reasonable. Let d be the degree of S, i.e., $d := (-K_S)^2$, and let $\sigma : \widetilde{S} \to S$ be the minimal resolution over k. Notice that $\widetilde{S}$ is a weak del Pezzo surface with $(-K_{\widetilde{S}})^2 = d$.

4.1 Du Val singularities over non-closed fields

In this section, in order to state and prove Theorem 1.3.9, we prepare the notation about Du Val singularities over algebraically non-closed fields.

4.1.1 Du Val singularities over algebraically non-closed fields

Let V be a normal algebraic surface over k and let p be a Du Val singular point on $V_{\overline{k}}$, which is k-rational. Notice that the exceptional set of the minimal resolution at $p \in V_{\overline{k}}$ is invariant under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$. Thus, depending on a fashion of the $\mathrm{Gal}(\overline{k}/k)$-action on the exceptional set, we shall divide the type of Du Val singularities in a more refined way as follows:

Definition 4.1.1. Let V be a normal algebraic surface over k, let p be a Du Val singular point on $V_{\overline{k}}$, which is k-rational, let $\sigma : \widetilde{V} \to V$ be the minimal resolution of p over k and let Δ be the exceptional set of σ on $\widetilde{V}$.

(1) In the case that p is of typeA_1 on $V_{\overline{k}}$, then:

  (i) p is of *type* A_1^+ over k if $\Delta(k) \neq \emptyset$.

  (ii) p is of *type* A_1^{++} over k if $\Delta(k) = \emptyset$.

(2) In the case that p is of type A_n for $n \geq 2$ on $V_{\overline{k}}$, then:

  (i) p is of *type* A_n^- over k if $\rho_k(\widetilde{V}) - \rho_k(V) = n$.

  (ii) p is of *type* A_n^+ over k if $\rho_k(\widetilde{V}) - \rho_k(V) < n$ and $\Delta(k) \neq \emptyset$.

  (iii) p is of *type* A_n^{++} over k if $\rho_k(\widetilde{V}) - \rho_k(V) < n$ and $\Delta(k) = \emptyset$.

(3) In the case that p is of type X_n on $V_{\overline{k}}$, where X_n means D_n for $n \geq 4$ or E_n for $n = 6$, then:

 (i) p is of *type* X_n^- over k if $\rho_k(\widetilde{V}) - \rho_k(V) = n$.

 (ii) p is of *type* X_n^+ over k if $\rho_k(\widetilde{V}) - \rho_k(V) < n$.

Remark 4.1.2. If $k = \mathbb{R}$, then all types of Du Val singularities over k correspond to all types of *real Du Val singularities* in [48, §9] except for type A_1. Meanwhile, although [48] defines both of Du Val singularities of type A_1^+ and type A_1^-, whereas in Definition 4.1.1, we do not prepare the notation for type A_1^- intentionally in consideration of the assertion (3)(iv) in Theorem 1.3.9.

4.1.2 Du Val singularities on Du Val del Pezzo surfaces of low degree

Let the notation be the same as at beginning of Chapter 4, assume further that $d \leq 2$.

By the classification of types of weak del Pezzo surfaces, assuming that $S_{\overline{k}}$ admits at least one singular point of type A_{9-2d} or at least two singular points, one of which is of type A_{7-2d} and the other of which is of type A_1, the type of the weak del Pezzo surface $\widetilde{S}$ is not uniquely determined by only "Degree" and "Singularities" if and only if "Singularities" of $\widetilde{S}$ is one of the following:

$$\begin{cases} d = 2: & A_5 + A_1,\ A_5,\ A_3 + 2A_1 \text{ or } A_3 + A_1. \\ d = 1: & A_7 \text{ or } A_5 + A_1. \end{cases} \tag{4.1.1}$$

For the above mentioned cases (4.1.1) only, we shall adopt the notation found in [12, §§2.2] as follows to make the proof more transparent. Here, to be more precise, it seems that [69] firstly introduces their notation. We note that in (4.1.2) each of the left hand side is the notation used in [12, §§2.2], meanwhile, each of the right hand side is the one defined in Subsection 2.4. For types in (4.1.1), we will adopt the ones at the left hand side in (4.1.2):

$$\begin{cases} d = 2: & (A_5 + A_1)' = (A_5 + A_1)_<,\ (A_5 + A_1)'' = (A_5 + A_1)_>, \\ & (A_5)' = (A_5)_<,\ (A_5)'' = (A_5)_>, \\ & (A_3 + 2A_1)' = (A_3 + 2A_1)_<,\ (A_3 + 2A_1)'' = (A_3 + 2A_1)_>, \\ & (A_3 + A_1)' = (A_3 + A_1)_<,\ (A_3 + A_1)'' = (A_3 + A_1)_>. \\ d = 1: & (A_7)' = (A_7)_>,\ (A_7)'' = (A_7)_<, \\ & (A_5 + A_1)' = (A_5 + A_1)_>,\ (A_5 + A_1)'' = (A_5 + A_1)_<. \end{cases} \tag{4.1.2}$$

On the other hand, to state our main result exactly, we shall divide the types of k-rational Du Val singularity $x \in S_{\overline{k}}$ of type A_{9-2d} as follows by making use of their notation:

Definition 4.1.3 (cf. [69])**.** With the notation and the assumptions as above, let x be a Du Val singular point of type A_{9-2d} on $S_{\overline{k}}$ defined over k. Then we say that x is of type $(A_{9-2d})'$ (resp. $(A_{9-2d})''$) if there exists a (-1)-curve on $\widetilde{S}_{\overline{k}}$ meeting the (-2)-curve corresponding to the central vertex on the dual graph of the minimal resolution (resp. there does not exist such a (-1)-curve on $\widetilde{S}_{\overline{k}}$).

Remark 4.1.4. If $S_{\overline{k}}$ admits a singular point x of type A_{9-2d}, then $\widetilde{S}$ is of one of the following:

- $d = 2$: $A_5 + A_2$, $A_5 + A_1$, A_5.
- $d = 1$: $A_7 + A_1$, A_7.

In particular, if $\widetilde{S}$ is of $A_{9-2d} + A_d$-type, then the singular point x is of type $(A_{9-2d})'$.

4.2 Proof of Theorem 1.3.9 (1) and (2)

In this section, we shall prove Theorem 1.3.9 (1) and (2).

4.2.1 Configurations of Du Val del Pezzo surfaces of rank one

Let the notation be the same as at beginning of Chapter 4 and assume further that $d \geq 3$. In this subsection, we shall study the configuration of (-2)-curves and (-1)-curves on $\widetilde{S}_{\overline{k}}$, furthermore, we classify Du Val del Pezzo surfaces of rank one and of degree ≥ 3. Notice that if $d \geq 4$, then we completely know the configuration of (-2)-curves and (-1)-curves on $\widetilde{S}_{\overline{k}}$ (see [15, Propositions 6.1, 8.1, 8.3 and 8.5]). On the other hand, weak del Pezzo surfaces of degree 3 defined over an algebraically closed field is also studied (e.g., [9]) but could not find the list of the configuration of (-2)-curves and (-1)-curves for all types of weak del Pezzo surface of degree 3. Then, by using [18, §§9.2] and based on the notation in Subsection 3.2.2, we consider the configuration of (-2)-curves and (-1)-curves on weak del Pezzo surfaces of degree 3. In other words, considering a weak del Pezzo surface $\widetilde{S}_3$, which is the same type of $\widetilde{S}_{\overline{k}}$, and the following composition of blow-downs to $\mathbb{P}^2_{\overline{k}}$ over $\overline{k}$:

$$\widetilde{\tau} : \widetilde{S}_3 \overset{\widetilde{\tau}_6}{\to} \widetilde{S}_4 \overset{\widetilde{\tau}_5}{\to} \dots \overset{\widetilde{\tau}_2}{\to} \widetilde{S}_8 \overset{\widetilde{\tau}_1}{\to} \widetilde{S}_9 = \mathbb{P}^2_{\overline{k}} \tag{4.2.1}$$

Let I_3 be the free $\mathbb{Z}$-module with a bilinear form such that it is generated by the proper transform e_0 of a general line by $\widetilde{\tau}$ and total transforms $e_1, \dots, e_6$ of the exceptional divisors by $\widetilde{\tau}_1, \dots, \widetilde{\tau}_6$. Then all (-2)-curves and all (-1)-curves on $\widetilde{S}_{\overline{k}}$ correspond to elements as in (3.2.2) and (3.2.3), respectively. For simplicity, c_i denotes the element $2e_0 - (e_1 + \cdots + e_6) + e_i \in I_3$, which is included in the list (3.2.3).

At first, we shall treat the following proposition:

Proposition 4.2.1. Let the notation and the assumptions be the same as above. If $\rho_k(S) = 1$, then the type of $\widetilde{S}$ is one of the following:

- $d = 8$: A_1-type;
- $d = 6$: $A_2 + A_1$, A_2 or $(A_1)_<$-type;
- $d = 5$: A_4-type;
- $d = 4$: D_5, $A_3 + 2A_1$, D_4, $A_3 + A_1$, $A_2 + 2A_1$, $4A_1$, $(A_3)_<$, $3A_1$, A_2, $2A_1$ or A_1-type;
- $d = 3$: E_6, $A_5 + A_1$, $3A_2$, A_5, $2A_2 + A_1$, D_4, $2A_2$, $4A_1$, $3A_1$, A_2 or A_1-type.

In order to prove Proposition 4.2.1, we prepare the following two lemmas:

Lemma 4.2.2. Let the notation and the assumptions be the same as above. If there exists a (-1)-curve E on $\widetilde{S}_{\overline{k}}$, which does not meet any (-2)-curve on $\widetilde{S}_{\overline{k}}$, such that either E is defined over k or the $\mathrm{Gal}(\overline{k}/k)$-orbit of E is a disjoint union, then $\rho_k(S) > 1$.

Proof. Assume that there exists a (-1)-curve E satisfying the assumption of this lemma. Then the direct image of the $\mathrm{Gal}(\overline{k}/k)$-orbit of E via σ is contactable in S. This implies that $\rho_k(S) > 1$. □

Lemma 4.2.3. Let the notation and the assumptions be the same as above. If any (-1)-curve and (-2)-curve on $\widetilde{S}_{\overline{k}}$ are defined over k and the number of all (-2)-curves on $\widetilde{S}_{\overline{k}}$ is less than $9 - d$, then $\rho_k(S) > 1$.

Proof. By assumption we have $\rho_k(\widetilde{S}) = \rho_{\overline{k}}(\widetilde{S}_{\overline{k}}) = 10 - d$ and $\rho_k(\widetilde{S}) - \rho_k(S) < 9 - d$. Hence, we obtain $\rho_k(S) > (10 - d) - (9 - d) = 1$. $\square$

By explicitly using Lemmas 4.2.2 and 4.2.3, we obtain the following lemma:

Lemma 4.2.4. Let the notation and the assumptions be the same as in Proposition 4.2.1. If the type of $\widetilde{S}$ is one of the following, then $\rho_k(S) > 1$:

- $d = 7$ and A_1-type.
- $d = 6$ and $2A_1$ or $(A_1)_>$-type.
- $d = 5$ and A_3, $A_2 + A_1$, A_2 or A_1-type.
- $d = 4$ and A_4 or $A_2 + A_1$-type.
- $d = 3$ and D_5, $A_3 + 2A_1$, $A_4 + A_1$, A_4, $A_3 + A_1$, $A_2 + 2A_1$, A_3, $A_2 + A_1$ or $2A_1$-type.

Proof. At first, we deal with the case of $d \geq 4$. By the list of weighted dual graphs in [15, Propositions 6.1, 8.1, 8.3 and 8.5], if the type of $\widetilde{S}$ is one of the following list, then the assumption of Lemma 4.2.2 holds:

- $d = 7$ and A_1-type.
- $d = 6$ and $(A_1)_>$-type.
- $d = 5$ and $A_2 + A_1$, A_2 or A_1-type.
- $d = 4$ and A_4 or $A_2 + A_1$-type.

Similarly, if the type of $\widetilde{S}$ is one of the following list, then the assumption of Lemma 4.2.3 holds:

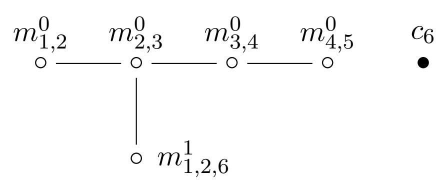
- $d = 6$ and $2A_1$-type.
- $d = 5$ and A_3-type.

Thus, this completes the proof of the case $d \geq 4$.

In what follows, we deal with the case of $d = 3$.

Then, by using mainly [18, §§9.2] and based on the notation in Subsection 3.2.2, we consider the configuration of (-2)-curves and (-1)-curves on weak del Pezzo surfaces of degree 3 according to the list in this lemma.

- *D_5-type:* By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{2,3}$, $m^0_{3,4}$, $m^0_{4,5}$ and $m^1_{1,2,6}$ in I_3 (see (3.2.2), for these notation). Then by using Lemma 2.1.4 there exists a unique (-1)-curve E on $\widetilde{S}_{\overline{k}}$, which corresponds to c_6 in I_3 (see the beginning of this subsection, for this notation), such that it does not meet any (-2)-curve on $\widetilde{S}_{\overline{k}}$. Namely, E is defined over k.



- A_3+2A_1-*type:* By [17, pp. 665–666][*1], we can choose a morphism (4.2.1) such that all (-2)-curves on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{3,4}$, $m^0_{5,6}$, $m^1_{1,2,5}$ and $m^1_{3,4,5}$ in I_3. Then by using Lemma 2.1.4 there exists a unique (-1)-curve E on $\widetilde{S}_{\overline{k}}$, which corresponds to $\ell_{5,6}$ in I_3 (see (3.2.3), for this notation), such that they do not meet any (-2)-curve on $\widetilde{S}_{\overline{k}}$. Namely, E is defined over k.

$$\begin{array}{cccccccccc} m^1_{1,2,5} & & m^0_{5,6} & & m^1_{3,4,5} & m^0_{1,2} & m^0_{3,4} & \ell_{5,6} \\ \circ & \text{———} & \circ & \text{———} & \circ & \circ & \circ & \bullet \end{array}$$

- A_4-*type:* By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{2,3}$, $m^0_{3,4}$ and $m^0_{4,5}$ in I_3. Then by using Lemma 2.1.4 there exist exactly two (-1)-curves E_1 and E_2 on $\widetilde{S}_{\overline{k}}$, which correspond to e_6 and c_5 in I_3, such that it does not meet any (-2)-curve on $\widetilde{S}_{\overline{k}}$. Namely, the union E_1+E_2 is defined over k and is disjoint.

$$\begin{array}{ccccccccc} m^0_{1,2} & & m^0_{2,3} & & m^0_{3,4} & & m^0_{4,5} & e_6 & c_6 \\ \circ & \text{———} & \circ & \text{———} & \circ & \text{———} & \circ & \bullet & \bullet \end{array}$$

- A_3+A_1-*type:* By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{2,3}$, $m^0_{3,4}$ and $m^0_{5,6}$ in I_3. Then by using Lemma 2.1.4 there exists a unique (-1)-curve E on $\widetilde{S}_{\overline{k}}$, which corresponds to $\ell_{5,6}$ in I_3, such that it does not meet any (-2)-curve on $\widetilde{S}_{\overline{k}}$. Namely, E is defined over k.

$$\begin{array}{ccccccc} m^0_{1,2} & & m^0_{2,3} & & m^0_{3,4} & m^0_{5,6} & \ell_{5,6} \\ \circ & \text{———} & \circ & \text{———} & \circ & \circ & \bullet \end{array}$$

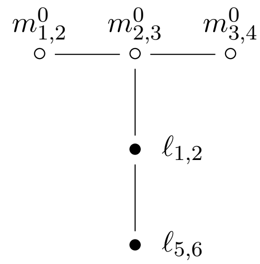
- A_2+2A_1-*type:* By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{2,3}$, $m^0_{4,5}$ and $m^1_{1,2,3}$ in I_3. Then by using Lemma 2.1.4 there exists a unique (-1)-curve E on $\widetilde{S}_{\overline{k}}$, which corresponds to e_6 in I_3, such that it does not meet any (-2)-curve on $\widetilde{S}_{\overline{k}}$. Namely, E is defined over k.

$$\begin{array}{cccccc} m^0_{1,2} & & m^0_{2,3} & m^0_{4,5} & m^1_{1,2,3} & e_6 \\ \circ & \text{———} & \circ & \circ & \circ & \bullet \end{array}$$

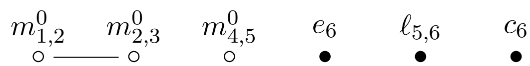
- A_3-*type:* By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{2,3}$ and $m^0_{3,4}$ in I_3. Then by using Lemma 2.1.4 there exists a unique (-1)-curve E' on $\widetilde{S}_{\overline{k}}$, which corresponds to $\ell_{1,2}$ in I_3, such that it meet the (-2)-curve corresponding to $m^0_{2,3} \in I_3$ but does not meet the others (-2)-curves on $\widetilde{S}_{\overline{k}}$. Furthermore, there exists a unique (-1)-curve E on $\widetilde{S}_{\overline{k}}$, which corresponds to $\ell_{5,6} \in I_3$, such that it meet E. Namely, E is defined over k, moreover, E does not meet any

[*1]Note that the way to construct τ using [18, p. 494] is wrong.

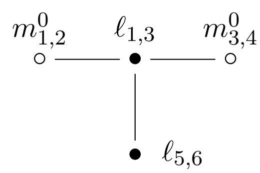
(-2)-curve on $\widetilde{S}_{\overline{k}}$.



- $A_2 + A_1$-*type:* By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{2,3}$ and $m^0_{4,5}$ in I_3. Then by using Lemma 2.1.4 there exist exactly three (-1)-curves E_1, E_2 and E_3 on $\widetilde{S}_{\overline{k}}$, which correspond to e_6, $\ell_{4,5}$ and c_6 in I_3, such that they do does not meet any (-2)-curve on $\widetilde{S}_{\overline{k}}$. Namely, the union $E_1 + E_2 + E_3$ is defined over k and is disjoint.



- $2A_1$-*type:* By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$ and $m^0_{3,4}$ in I_3. Then by using Lemma 2.1.4 there exists a unique (-1)-curve E' on $\widetilde{S}_{\overline{k}}$, which corresponds to $\ell_{1,3} \in I_3$, such that it meet two (-2)-curves on $\widetilde{S}_{\overline{k}}$. Furthermore, there exists a unique (-1)-curve E on $\widetilde{S}_{\overline{k}}$, which corresponds to $\ell_{5,6} \in I_3$, such that it meet E. Namely, E is defined over k, moreover, E does not meet any (-2)-curve on $\widetilde{S}_{\overline{k}}$.



Thus, if $\widetilde{S}$ is of D_5, $A_3 + 2A_1$, A_4, $A_3 + A_1$, $A_2 + 2A_1$, A_3, $A_2 + A_1$ or $2A_1$-type, then the assumption of Lemma 4.2.2 holds, so that $\rho_k(S) > 1$. On the other hand, assume that $\widetilde{S}$ is of $A_4 + A_1$-type. Then we know that all (-2)-curves and all (-1)-curves are defined over k by the weighted dual graph of the union of these curves (see [14, (19°)] or [17, p. 667]). That is, the assumption of Lemma 4.2.3 holds, so that $\rho_k(S) > 1$. This completes the proof of the case $d = 3$. □

Meanwhile, the following lemma holds:

Lemma 4.2.5. With the notation and the assumptions as above, assume further that the type of $\widetilde{S}$ is of one of the following:

- $d = 5$: $\widetilde{S}$ is of $2A_1$-type;
- $d = 4$: $\widetilde{S}$ is of either $(A_3)_>$ or $(2A_1)_>$-type.

Then $\rho_k(S) > 1$.

In order to prove Lemma 4.2.5, we prepare the following lemma:

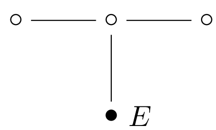
Lemma 4.2.6. With the notation as above, assume further that there exists a birational morphism $\tau' : \widetilde{S} \to \widetilde{S}'$ over k such that $\widetilde{S}'$ is smooth, $\rho_k(\widetilde{S}) - \rho_k(\widetilde{S}') = 1$ and the exceptional locus of $\tau'_{\overline{k}}$ meets only one $\mathrm{Gal}(\overline{k}/k)$-orbit of a (-2)-curve on $\widetilde{S}_{\overline{k}}$. Let $\sigma' : \widetilde{S}' \to S'$ be the contraction of all (-2)-curve, where note that σ' is defined over k. Then $\rho_k(S') = \rho_k(S)$.

Proof. By the assumption, we thus obtain $\rho_k(\widetilde{S}) - \rho_k(\widetilde{S}') = 1$ and $\rho_k(\widetilde{S}') - \rho_k(S') = \rho_k(\widetilde{S}) - \rho_k(S) - 1$. Namely, $\rho_k(S') = \rho_k(S)$. □

Proof of Lemma 4.2.5. We shall consider three cases separately:

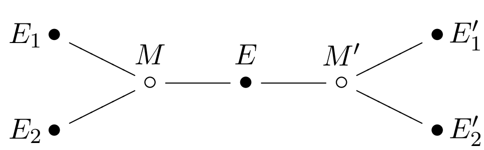
$\widetilde{S}$ *is of* $d = 5$ *and of* $2A_1$*-type:* By [15, Proposition 8.5], there exists a unique (-1)-curve E meeting two (-2)-curves M and M' on $\widetilde{S}_{\overline{k}}$. In particular, E is defined over k. If M and M' lie the same $\mathrm{Gal}(\overline{k}/k)$-orbit, letting $\tau' : \widetilde{S} \to \widetilde{S}'$ be the contraction of E over k, then $\tau_*(M + M')$ has self-intersection number 0. This implies that $\rho_k(\widetilde{S}') > 1$. Moreover, by virtue of $\rho_k(\widetilde{S}') = \rho_k(\widetilde{S}) - 1$ and $\rho_k(S) = \rho_k(\widetilde{S}) - 1$, we obtain $\rho_k(S) > 1$. In what follows, assume that M and M' are defined over k. By [15, Proposition 8.5] again, there exist exactly two (-1)-curve meeting no (-2)-curve such that they are defined over k. Hence, we see $\rho_k(S) > 1$ by Lemma 4.2.2.

$\widetilde{S}$ *is of* $d = 4$ *and of* $(A_3)_>$*-type:* By [15, Proposition 6.1], there exists a unique (-1)-curve E on $\widetilde{S}_{\overline{k}}$ meeting the (-2)-curve corresponding to the central vertex on the dual graph of all (-2)-curves on $\widetilde{S}_{\overline{k}}$ (see the following dual graph):



In particular, E is defined over k. Hence, we obtain the contraction $\tau' : \widetilde{S} \to \widetilde{S}'$ of E over k, so that $\widetilde{S}'$ is a weak del Pezzo surface of degree 5 and of $2A_1$-type. Letting $\sigma' : \widetilde{S}' \to S'$ be the contraction of the (-2)-curve, noticing that σ' is defined over k, we obtain $\rho_k(S) = \rho_k(S') > 1$ by Lemma 4.2.5 and Proposition 4.2.1.

$\widetilde{S}$ *is of* $d = 4$ *and of* $(2A_1)_>$*-type:* By [15, Proposition 6.1], there exists a unique (-1)-curve E meeting two (-2)-curves M and M' on $\widetilde{S}_{\overline{k}}$. In particular, E is defined over k. If M and M' lie the same $\mathrm{Gal}(\overline{k}/k)$-orbit, letting $\tau' : \widetilde{S} \to \widetilde{S}'$ be the contraction of E over k, then $\tau_*(M + M')$ has self-intersection number 0. This implies that $\rho_k(\widetilde{S}') > 1$. Moreover, by virtue of $\rho_k(\widetilde{S}') = \rho_k(\widetilde{S}) - 1$ and $\rho_k(S) = \rho_k(\widetilde{S}) - 1$, we obtain $\rho_k(S) > 1$. In what follows, assume that M and M' are defined over k. By [15, Proposition 6.1] again, there exist exactly two (-1)-curve E_1 and E_2 (resp. E_1' and E_2') such that they meet M (resp. M') but does not meet M' (resp. M) (see the following dual graph):



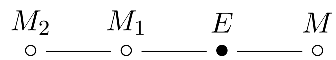
Hence, we obtain the contraction $\tau' : \widetilde{S} \to \widetilde{S}'$ of $E_1 + E_2 + E_1' + E_2'$ over k, so that $\widetilde{S}'$ is a smooth del Pezzo surface of degree 7. Noticing $\rho_k(\widetilde{S}') > 1$, we obtain $\rho_k(S) = \rho_k(\widetilde{S}') > 1$ by Lemma 4.2.5. □

By using the classification of weak del Pezzo surfaces over an algebraically closed field, Proposition 4.2.1 follows from Lemmas 4.2.4 and 4.2.5.

At the end of this subsection, for each type of a weak del Pezzo surface except for the list of Proposition 4.2.1, we will fined all Du Val del Pezzo surfaces of rank one corresponding to the above type. Moreover, assuming further $\rho_k(S) = 1$, we shall explicitly construct the birational morphism $\tau : \widetilde{S} \to V$ over a smooth k-minimal surface V according to the types of $\widetilde{S}$.

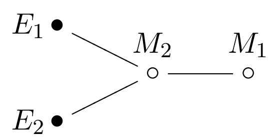
$\widetilde{S}$ is of $d = 8$ and of A_1-type: Then $\widetilde{S}$ is a k-form of the Hirzebruch surface $\mathbb{F}_2$ of degree 2. In particular, $\widetilde{S}$ is minimal over k. Hence, we set $V := \widetilde{S}$ and $\tau := id$.

$\widetilde{S}$ is of $d = 6$ and of $A_2 + A_1$-type: By [15, Proposition 8.3], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



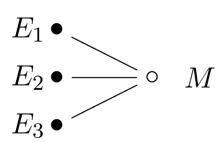
Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of E, that of the image of M_1 and that of the image of M_2. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Moreover, the direct image $\tau_*(M)$ is a line on $V \simeq \mathbb{P}^2_k$. Meanwhile, since any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 3 = 4$.

$\widetilde{S}$ is of $d = 6$ and of A_2-type: By [15, Proposition 8.3], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



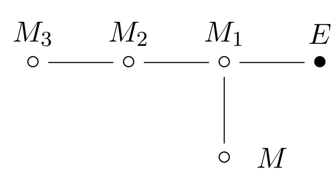
Let $\tau : \widetilde{S} \to V$ be the contraction of $E_1 + E_2$. By construction, τ is defined over k and we see $V \simeq \mathbb{F}_2$ by using Lemma 2.2.2. Moreover, the direct images $\tau_*(M_1)$ and $\tau_*(M_2)$ are the minimal section and a closed fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k, respectively. Meanwhile, since any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 2 = 3$.

$\widetilde{S}$ is of $d = 6$ and of $(A_1)_<$-type: By [15, Proposition 8.3], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



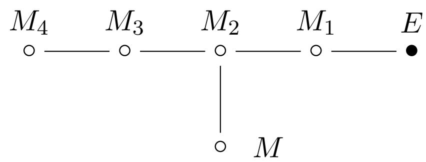
Let $\tau : \widetilde{S} \to V$ be the contraction of $E_1 + E_2 + E_3$. By construction, τ is defined over k, and V is a k-form of $\mathbb{P}^2_k$, furthermore, we know $V \simeq \mathbb{P}^2_k$. Indeed, since $\tau_{\overline{k},*}(M)$ is a line on $V_{\overline{k}} \simeq \mathbb{P}^2_{\overline{k}}$ defined over k, there exists a general line defined over k on $V_{\overline{k}} \simeq \mathbb{P}^2_{\overline{k}}$. Hence, $V_{\overline{k}}$ has a k-rational point, which is the intersection point for two general lines on $V_{\overline{k}}$, so that $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Meanwhile, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 1 = 2$.

$\widetilde{S}$ is of $d = 5$ and of A_4-type: By [15, Proposition 8.5], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



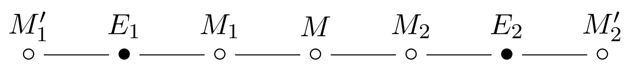
Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of E, that of the image of M_1, that of the image of M_2 and finally that of the image of M_3. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Moreover, the direct image $\tau_*(M)$ is a line on $V \simeq \mathbb{P}^2_k$. Meanwhile, since any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 4 = 5$.

$\widetilde{S}$ is of $d = 4$ and of D_5-type: By [15, Proposition 6.1], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



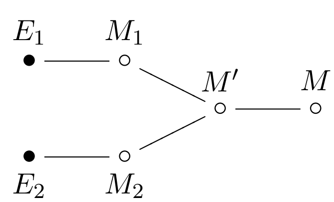
Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of E, that of the image of M_1, that of the image of M_2, that of the image of M_3 and finally that of the image of M_4. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Moreover, the direct image $\tau_*(M)$ is a line on $V \simeq \mathbb{P}^2_k$. Meanwhile, since any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 5 = 6$.

$\widetilde{S}$ is of $d = 4$ and of $A_3 + 2A_1$-type: By [15, Proposition 6.1], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



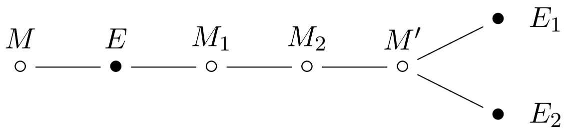
Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E_1 + E_2$ and that of the images of $M_1' + M_2'$. By construction, τ is defined over k and V is a k-form of $\mathbb{F}_2$. Moreover, the direct images $\tau_*(M)$ and $\tau_*(M_1 + M_2)$ are k-forms of the minimal section and a disjoint union of two closed fibers of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_{\overline{k}}$, respectively. Meanwhile, if since any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 5 = 6$. Otherwise, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 3 = 4$.

$\widetilde{S}$ is of $d = 4$ and of D_4-type: By [15, Proposition 6.1], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



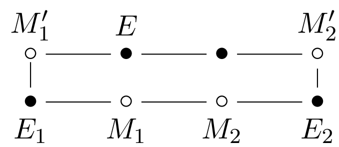
Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E_1 + E_2$ and that of the images of $M_1 + M_2$. By construction, τ is defined over k and we see $V \simeq \mathbb{F}_2$ by using Lemma 2.2.2. Moreover, the direct images $\tau_*(M)$ and $\tau_*(M')$ are the minimal section and a closed fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k, respectively. Meanwhile, note that E_1 is not defined over k. Indeed, otherwise, by Lemma 4.2.6 the contraction $\tau' : \widetilde{S} \to \widetilde{S}'$ of E_1 provides a Du Val del Pezzo surface of rank one and of degree 5 such that its minimal resolution is $\widetilde{S}'$ and of A_3-type. However, this is a contradiction to Proposition 4.2.1. Hence, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 3 = 4$.

$\widetilde{S}$ *is of* $d = 4$ *and of* $A_3 + A_1$*-type:* By [15, Proposition 6.1], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



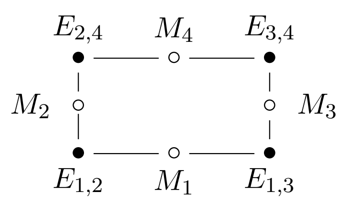
Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E_1 + E_2$, that of the image of M_1 and finally that of the image of M_2. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Moreover, the direct images $\tau_*(M)$ and $\tau_*(M')$ are two distinct lines on $V \simeq \mathbb{P}^2_k$. Meanwhile, since any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 4 = 5$.

$\widetilde{S}$ *is of* $d = 4$ *and of* $A_2 + 2A_1$*-type:* By [15, Proposition 6.1], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E_1 + E_2$ and that of the images of $M_1' + M_2'$. By construction, τ is defined over k and V is a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$. Moreover, the direct image $\tau_*(M_1 + M_2)$ is a k-form of a union of two irreducible curves of types $(1, 0)$ and $(0, 1)$, respectively (see [28, Chap. II, Example 6.6.1], for the notation). Meanwhile, note that M_1 is not defined over k. Indeed, otherwise, since E is defined over k by the configuration, by Lemma 4.2.6 the contraction $\tau' : \widetilde{S} \to \widetilde{S}'$ of E over k provides a Du Val del Pezzo surface of rank one of degree 5 such that its minimal resolution is $\widetilde{S}'$ and of $A_2 + A_1$-type. However, this is a contradiction to Proposition 4.2.1. Hence, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 2 = 3$, so that $\rho_k(V) = \rho_k(\widetilde{S}) - 2 = 1$.

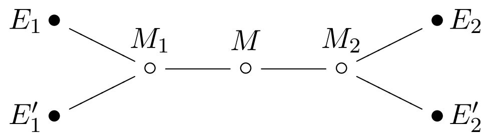
$\widetilde{S}$ *is of* $d = 4$ *and of* $4A_1$*-type:* By [15, Proposition 6.1], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



If there is no singular point on $S_{\overline{k}}$ defined over k, then let $\tau : \widetilde{S} \to V$ be the contraction of $E_{1,2} + E_{1,3} + E_{2,4} + E_{3,4}$. By construction, τ is defined over k and V is a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$. Meanwhile, if all singular points on $S_{\overline{k}}$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, then we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 1 = 2$. Otherwise, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 2 = 3$.

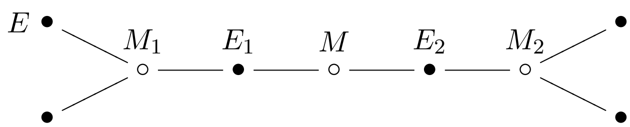
On the other hand, if there is a singular point on $S_{\overline{k}}$ defined over k, assuming without loss of generality that the (-2)-curve M_1 is defined over k, then let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E_{2,4} + E_{3,4}$ and that of the images of $M_2 + M_3$. By construction, τ is defined over k and V is a k-form of $\mathbb{F}_2$. Moreover, the direct images $\tau_*(M_1)$ and $\tau_*(M_4)$ are k-forms of the minimal section and a section with self-intersection number 2 of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_{\overline{k}}$, respectively. Meanwhile, if any singular point on $S_{\overline{k}}$ is defined over k, then we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 4 = 5$. Otherwise, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 3 = 4$.

$\widetilde{S}$ is of $d = 4$ and of $(A_3)_<$-type: By [15, Proposition 6.1], the weighted dual graph of all (-2)-curves and all (-1)-curves is as follows:



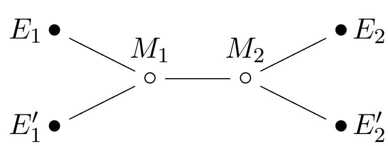
Let $\tau : \widetilde{S} \to V$ be the contraction of $E_1 + E'_1 + E_2 + E'_2$. By construction, τ is defined over k and V is a k-form of $\mathbb{F}_2$. Moreover, the direct images $\tau_*(M)$ and $\tau_*(M_1 + M_2)$ are k-forms of the minimal section and a disjoint union of two closed fibers of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_{\overline{k}}$, respectively. Meanwhile, if any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 3 = 4$. Otherwise, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 2 = 3$.

$\widetilde{S}$ is of $d = 4$ and of $3A_1$-type: By [15, Proposition 6.1], the weighted dual graph of all (-2)-curves and (-1)-curves meeting at least one (-2)-curve is as follows:



Let $\tau : \widetilde{S} \to V$ be the contraction of $E_1 + E_2$ and that of the images of $M_1 + M_2$. By construction, τ is defined over k and V is a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$. Moreover, the direct image $\tau_*(M)$ is a k-form of an irreducible curve of type $(1, 1)$ (see [28, Chap. II, Example 6.6.1], for the notation). Meanwhile, note that M is clearly defined over k but M_1 is not defined over k. Indeed, otherwise, by Lemma 4.2.6 the contraction $\tau' : \widetilde{S} \to \widetilde{S}'$ of a $\mathrm{Gal}(\overline{k}/k)$-orbit of E over k provides a Du Val del Pezzo surface of rank one of degree 5 or 6 such that its minimal resolution is $\widetilde{S}'$ and of $2A_1$-type. However, for both cases, this is a contradiction to Proposition 4.2.1. Hence, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 2 = 3$, so that $\rho_k(V) = 1$.

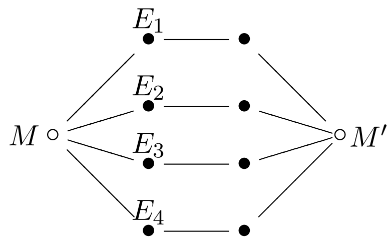
$\widetilde{S}$ is of $d = 4$ and of A_2-type: By [15, Proposition 6.1], the weighted dual graph of all (-2)-curves and (-1)-curves meeting at least one (-2)-curve is as follows:



Let $\tau : \widetilde{S} \to V$ be the contraction of $E_1 + E_2 + E'_1 + E'_2$. By construction, τ is defined over k and V is a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$. Moreover, the direct image $\tau_*(M_1 + M_2)$ is a k-form of a union of two irreducible curves of types $(1, 0)$ and $(0, 1)$, respectively (see [28, Chap. II, Example 6.6.1], for the notation). Meanwhile, note that M_1 is not defined over k. Indeed, otherwise, by Lemma 4.2.6 the contraction $\tau' : \widetilde{S} \to \widetilde{S}'$ of a $\mathrm{Gal}(\overline{k}/k)$-orbit of E_1 over k provides a Du Val del Pezzo surface of rank one of degree 5 or 6 such that its minimal resolution is $\widetilde{S}'$ and of A_1-type or $(A_1)_>$-type, respectively. However, for both cases, this is a contradiction to Proposition 4.2.1. Hence, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 1 = 2$, so that $\rho_k(V) = 1$.

$\widetilde{S}$ is of $d = 4$ and of $(2A_1)_<$-type: By [15, Proposition 6.1], the weighted dual graph of

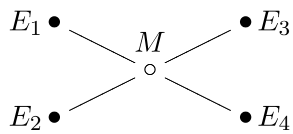
all (-2)-curves and all (-1)-curves is as follows:



If there is no singular point on $S_{\overline{k}}$ defined over k, then $\rho_k(\widetilde{S}) = 2$. Thus, $\widetilde{S}$ is minimal over k by Theorem 1.3.3, in other words, $\widetilde{S}$ is the minimal resolution of an Iskovskih surface (see [15, p. 74]). Hence, we set $V := \widetilde{S}$ and $\tau := id$.

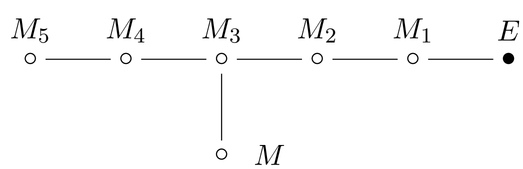
On the other hand, if there is a singular point on $S_{\overline{k}}$ defined over k, then M is defined over k. In particular, so is the disjoint union $E_1 + \cdots + E_4$ of (-1)-curves. Let $\tau : \widetilde{S} \to V$ be the contraction of this disjoint union. By construction, τ is defined over k and V is a k-form of $\mathbb{F}_2$. Moreover, the direct images $\tau_*(M')$ and $\tau_*(M)$ are k-forms of the minimal section and a section with self-intersection number 2 of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_{\overline{k}}$, respectively. Meanwhile, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 2 = 3$.

$\widetilde{S}$ is of $d = 4$ and of A_1-type: By [15, Proposition 6.1], the weighted dual graph of all (-2)-curves and (-1)-curves meeting at least one (-2)-curve is as follows:



Let $\tau : \widetilde{S} \to V$ be the contraction of $E_1 + \cdots + E_4$. By construction, τ is defined over k and V is a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$. Moreover, the direct image $\tau_*(M)$ is a k-form of an irreducible curve of type $(1, 1)$ (see [28, Chap. II, Example 6.6.1], for these notation). Meanwhile, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 1 = 2$, so that $\rho_k(V) = 1$.

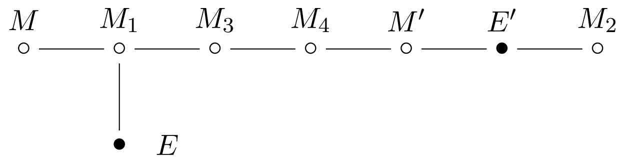
$\widetilde{S}$ is of $d = 3$ and of E_6-type: By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves M_1, M_2, M_3, M_4, M_5 and M on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{5,6}$, $m^0_{4,5}$, $m^0_{3,4}$, $m^0_{2,3}$, $m^0_{1,2}$ and $m^1_{1,2,3}$ in I_3, respectively. Then by using Lemma 2.1.4 there exists exactly one (-1)-curve E on $\widetilde{S}_{\overline{k}}$ corresponding to e_6 in I_3. Hence, we obtain the following weighted dual graph:



Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of E, that of the image of M_1, that of the image of M_2, that of the image of M_3, that of the image of M_4 and finally that of the image of M_5. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Moreover, the direct image $\tau_*(M)$ is a line on $V \simeq \mathbb{P}^2_k$. Meanwhile, since any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 6 = 7$.

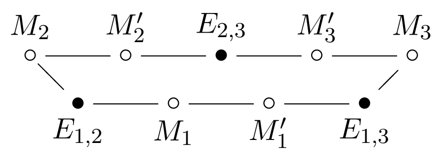
$\widetilde{S}$ is of $d = 3$ and of $A_5 + A_1$-type: By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves M_1, M_2, M_3, M_4, M and M' on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{2,3}$, m^2, $m^0_{3,4}$,

$m^0_{4,5}$, $m^0_{1,2}$ and $m^0_{5,6}$ in I_3, respectively. Then by using Lemma 2.1.4 there exist exactly two (-1)-curve E and E' on $\widetilde{S}_{\overline{k}}$ corresponding to e_6 and $\ell_{1,2}$ in I_3, respectively. Hence, we obtain the following weighted dual graph:



Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E + E'$, that of the images of $M_1 + M_2$, that of the image of M_3 and finally that of the image of M_4. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Moreover, the direct images $\tau_*(M_1)$ and $\tau_*(M_5)$ are two distinct lines on $V \simeq \mathbb{P}^2_k$. Meanwhile, since any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 6 = 7$.

$\widetilde{S}$ is of $d = 3$ and of $3A_2$-type: By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves M_1, M_1', M_2, M_2', M_3 and M_3' on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{2,3}$, $m^0_{4,5}$, $m^0_{5,6}$, $m^1_{1,2,3}$ and $m^1_{4,5,6}$ in I_3, respectively. Then by using Lemma 2.1.4 there exist exactly three (-1)-curves $E_{1,2}$, $E_{1,3}$ and $E_{2,3}$ on $\widetilde{S}_{\overline{k}}$ corresponding to $\ell_{1,4}$, e_3 and e_6 in I_3, respectively. Hence, we obtain the following weighted dual graph:



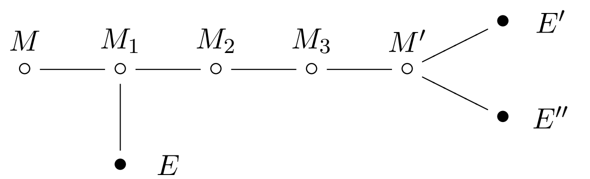
If there is no singular point on $S_{\overline{k}}$ defined over k and $\rho_k(\widetilde{S}) = 2$, then let $\tau : \widetilde{S} \to V$ be the contraction of $E_{1,2} + E_{1,3} + E_{2,3}$. By construction, τ is defined over k and V is a smooth del Pezzo surface of degree 6 with $\rho_k(V) = \rho_k(\widetilde{S}) - 1 = 1$.

If there is no singular point on $S_{\overline{k}}$ defined over k and $\rho_k(\widetilde{S}) = 3$, then $M_1 + M_2 + M_3$ and $M_1' + M_2' + M_3'$ are defined over k. Hence, let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E_{1,2} + E_{1,3} + E_{2,3}$ and that of the images of $M_1' + M_2' + M_3'$. By construction, τ is defined over k and V is a k-form of $\mathbb{P}^2_{\overline{k}}$.

On the other hand, if there is a singular point on $S_{\overline{k}}$ defined over k, assuming without loss of generality that $M_1 + M_1'$ is defined over k, then let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E_{1,2} + E_{1,3}$, that of the images of $M_2 + M_3$ and finally that of the images of $M_2' + M_3'$. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Moreover, the direct images $\tau_*(M_1)$ and $\tau_*(M_1')$ are two distinct lines on $V \simeq \mathbb{P}^2_k$. Meanwhile, if M_1 is defined over k, then we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 6 = 7$. Otherwise, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 3 = 4$.

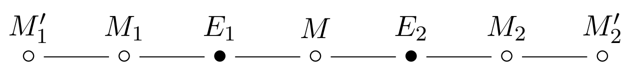
$\widetilde{S}$ is of $d = 3$ and of A_5-type: By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves M_1, M_2, M_3, M and M' on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{2,3}$, $m^0_{3,4}$, $m^0_{4,5}$, $m^0_{1,2}$ and $m^0_{5,6}$ in I_3, respectively. Then by using Lemma 2.1.4 there exist exactly three (-1)-curve E, E' and E'' on $\widetilde{S}_{\overline{k}}$ corresponding to $\ell_{1,2}$, e_6 and c_6 in I_3, respectively. Hence, we obtain the

following weighted dual graph:



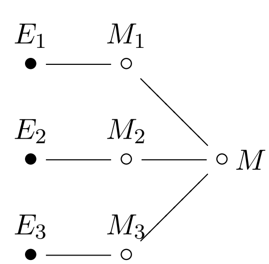
Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E + E' + E''$, that of the image of M_1, that of the image of M_2 and finally that of the image of M_3. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Moreover, the direct images $\tau_*(M_1)$ and $\tau_*(M_5)$ are two distinct lines on $V \simeq \mathbb{P}^2_k$. Meanwhile, since any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 5 = 6$.

$\widetilde{S}$ is of $d = 3$ and of $2A_2 + A_1$-type: By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves M_1, M_1', M_2, M_2' and M on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{2,3}$, $m^0_{1,2}$, $m^0_{5,6}$, $m^0_{4,5}$ and $m^1_{1,2,3}$ in I_3, respectively. Notice that M is defined over k. Then by using Lemma 2.1.4 there exist exactly two (-1)-curves E_1 and E_2 meeting M on $\widetilde{S}_{\overline{k}}$ corresponding to e_3 and $\ell_{4,5}$ in I_3, respectively. Hence, we obtain the following weighted dual graph:



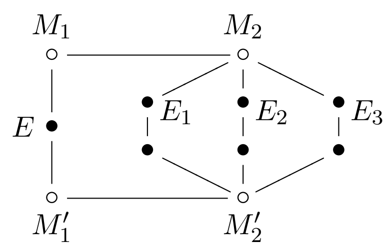
Noticing that the union $E_1 + E_2$ of (-1)-curves is disjoint, let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of this disjoint union, that of the images of $M_1 + M_2$, and finally that of the images of $M_1' + M_2'$. By construction, τ is defined over k and V is a k-form of $\mathbb{P}^2_k$, furthermore, we know $V \simeq \mathbb{P}^2_k$. Indeed, since $\tau_{\overline{k},*}(M)$ is an irreducible conic on $V_{\overline{k}} \simeq \mathbb{P}^2_{\overline{k}}$ defined over k, there exists a general line defined over k on $V_{\overline{k}} \simeq \mathbb{P}^2_{\overline{k}}$. Hence, $V_{\overline{k}}$ has a k-rational point, which is the intersection point for two general lines on $V_{\overline{k}}$, so that $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Moreover, the direct image $\tau_*(M)$ is an irreducible conic on $V \simeq \mathbb{P}^2_k$. Meanwhile, note that M_1 is not defined over k. Indeed, otherwise, we know $\rho_k(\widetilde{S}) - \rho_k(V) = 6$. Thus, $\rho_k(S) = \rho_k(\widetilde{S}) - 5 = \rho_k(V) + 6 - 5 = 2$, which contradicts the assumption $\rho_k(S) = 1$. Hence, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 3 = 4$.

$\widetilde{S}$ is of $d = 3$ and of D_4-type: By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves M_1, M_2, M_3 and M on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{3,4}$, $m^0_{5,6}$ and $m^1_{1,3,5}$ in I_3, respectively. Then by using Lemma 2.1.4 there exist exactly three (-1)-curves E_1, E_2 and E_3 meeting at least one (-2)-curve on $\widetilde{S}_{\overline{k}}$ corresponding to e_2, e_4 and e_6 in I_3, respectively. Hence, we obtain the following weighted dual graph:



Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E_1 + E_2 + E_3$ and that of the images of $M_1 + M_2 + M_3$. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by the similar argument to the case that $\widetilde{S}$ is of $d=6$ and $(3A_1)_<$-type. Moreover, the direct image $\tau_*(M)$ is a line on $V \simeq \mathbb{P}^2_k$. Meanwhile, M_1 is not defined over k. Indeed, otherwise, since E_1 is defined over k, by Lemma 4.2.6 the contraction $\tau' : \widetilde{S} \to \widetilde{S}'$ of E_1 over k provides a Du Val del Pezzo surface of rank one of degree 4 such that its minimal resolution is $\widetilde{S}'$ and of $(A_3)_>$-type. However, this is a contradiction to Proposition 4.2.1. Similarly, M_2 and M_3 are not defined over k. Hence, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 2 = 3$.

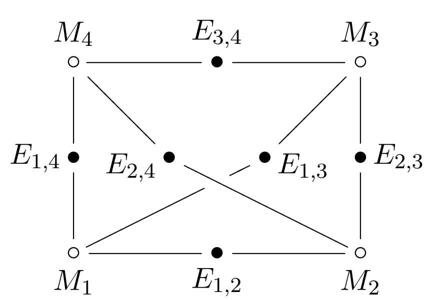
$\widetilde{S}$ is of $d=3$ and of $2A_2$-type: By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves M_1, M_2, M_1' and M_2' on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{2,3}$, $m^0_{4,5}$ and $m^0_{5,6}$ in I_3, respectively. Then by using Lemma 2.1.4 there exist exactly one (-1)-curve E meeting two (-2)-curves M_1 and M_1' on $\widetilde{S}_{\overline{k}}$ corresponding to $\ell_{1,4}$ in I_3. Moreover, there exist exactly three (-1)-curves E_1, E_2 and E_3 meeting only one (-2)-curve M_2 on $\widetilde{S}_{\overline{k}}$ corresponding to e_3, $\ell_{1,2}$ and c_3 in I_3, respectively. Hence, we obtain the following weighted dual graph:



If there is no singular point on $S_{\overline{k}}$ defined over k, then let $\tau : \widetilde{S} \to V$ be the contraction of E. By construction, τ is defined over k and V is a weak del Pezzo surface of rank two and $(2A_1)_<$-type, i.e., the minimal resolution of an Iskovskih surface (see [15, p. 74]), by Theorem 1.3.3 combined with $\rho_k(\widetilde{S}) = 3$.

On the other hand, if there is a singular point on $S_{\overline{k}}$ defined over k, then each (-2)-curve is defined over k. In particular, so is the union $E+E_1+E_2+E_3$ of (-1)-curves. Let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E+E_1+E_2+E_3$ and that of the image of M_1. By construction, τ is defined over k and we see that $V \simeq \mathbb{F}_2$ by using Lemma 2.2.2. Moreover, the direct images $\tau_*(M_2')$, $\tau_*(M_2)$ and $\tau_*(M_1')$ are the minimal section, a section with self-intersection number 2 and a closed fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k, respectively. Meanwhile, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 4 = 5$.

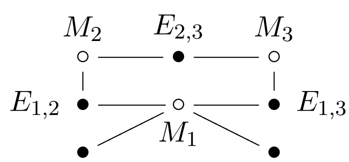
$\widetilde{S}$ is of $d=3$ and of $4A_1$-type: By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves M_1, M_2, M_3 and M_4 on $\widetilde{S}_{\overline{k}}$ correspond to $m^1_{1,2,3}$, $m^1_{1,4,5}$, $m^1_{2,4,6}$ and $m^1_{3,5,6}$ in I_3, respectively. Then by using Lemma 2.1.4 there exist exactly six (-1)-curves $E_{1,2}$, $E_{1,3}$, $E_{1,4}$, $E_{2,3}$, $E_{2,4}$ and $E_{3,4}$ meeting exactly two (-2)-curves on $\widetilde{S}_{\overline{k}}$ corresponding to $e_1, \dots, e_5$ and e_6 in I_3, respectively. Hence, we obtain the following weighted dual graph:



If there is no singular point on $S_{\overline{k}}$ defined over k, then let $\tau : \widetilde{S} \to V$ be the contraction of $E_{1,2} + E_{1,3} + E_{1,4} + E_{2,3} + E_{2,4} + E_{3,4}$. By construction, τ is defined over k and V is a k-form of $\mathbb{P}^2_{\overline{k}}$, furthermore, we know $V \simeq \mathbb{P}^2_k$. Indeed, since $\tau_{\overline{k},*}(M_1 + \cdots + M_4)$ is four distinct lines on $V_{\overline{k}} \simeq \mathbb{P}^2_{\overline{k}}$ defined over k, there exists a general line defined over k on $V_{\overline{k}} \simeq \mathbb{P}^2_{\overline{k}}$. Hence, $V_{\overline{k}}$ has a k-rational point, which is the intersection point for two general lines on $V_{\overline{k}}$, so that $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Meanwhile, note that $M_1, \ldots, M_4$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Indeed, otherwise, assume without loss of generality that $M_1 + M_2$ is defined over k. Since $E_{1,2}$ and $E_{3,4}$ are then defined over k, we know $\rho_k(\widetilde{S}) - \rho_k(V) \geq 3$. Thus, $\rho_k(S) = \rho_k(\widetilde{S}) - 2 \geq \rho_k(V) + 3 - 2 = 2$, which contradicts the assumption $\rho_k(S) = 1$. Hence, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 2 = 3$.

On the other hand, if there is a singular point on $S_{\overline{k}}$ defined over k, assuming without loss of generality that M_1 is defined over k, then let $\tau : \widetilde{S} \to V$ be the compositions of successive contractions of $E_{1,2} + E_{1,3} + E_{1,4}$ and that of the images of $M_2 + M_3 + M_4$. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by the similar argument to the case that $\widetilde{S}$ is of $d = 6$ and $(A_1)_<$-type. Moreover, the direct image $\tau_*(M_1)$ is an irreducible conic on $V \simeq \mathbb{P}^2_k$. Meanwhile, note that M_2 is not defined over k. Indeed, otherwise, since $E_{1,3} + E_{1,4}$ is defined over k, by Lemma 4.2.6 the contraction $\tau' : \widetilde{S} \to \widetilde{S}'$ of $E_{1,3} + E_{1,4}$ over k provides a Du Val del Pezzo surface of rank one of degree 5 such that its minimal resolution is $\widetilde{S}'$ and of A_1-type. However, this is a contradiction to Proposition 4.2.1. Similarly, M_3 and M_4 are not defined over k. Hence, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 2 = 3$.

$\widetilde{S}$ is of $d = 3$ and of $3A_1$-type: By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves M_1, M_2 and M_3 on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$, $m^0_{3,4}$ and $m^0_{5,6}$ in I_3, respectively. Then by using Lemma 2.1.4 there exist exactly three (-1)-curves $E_{1,2}$, $E_{1,3}$ and $E_{2,3}$ meeting exactly two (-2)-curves on $\widetilde{S}_{\overline{k}}$ corresponding to $\ell_{1,3}$, $\ell_{1,5}$ and $\ell_{3,5}$ in I_3, respectively. We note that there is no singular point on $S_{\overline{k}}$ defined over k. Otherwise, assume without loss of generality that M_1 is defined over k. Then there exist exactly two (-1)-curves meeting only one (-2)-curve M_1 on $\widetilde{S}_{\overline{k}}$ corresponding to e_2 and c_2 in I_3. Hence, we obtain the following weighted dual graph:

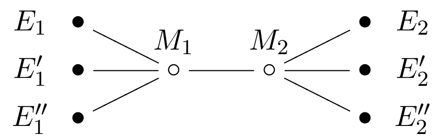


Letting E_1 be one of these (-1)-curves, by Lemma 4.2.6 the contraction $\tau' : \widetilde{S} \to \widetilde{S}'$ of the $\mathrm{Gal}(\overline{k}/k)$-orbit of E_1 over k provides a Du Val del Pezzo surface of rank one of degree 4 or 5 such that its minimal resolution $\widetilde{S}'$ is of $(2A_1)_>$-type or $2A_1$-type, respectively. However, this is a contradiction to Proposition 4.2.1.

Hence, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 1 = 2$. Let $\tau : \widetilde{S} \to V$ be the contraction of $E_{1,2} + E_{1,3} + E_{2,3}$. By construction, τ is defined over k and V is a smooth del Pezzo surface of degree 6 with $\rho_k(V) = 1$.

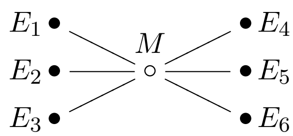
$\widetilde{S}$ is of $d = 3$ and of A_2-type: By [18, p. 446], we can choose a morphism (4.2.1) such that all (-2)-curves M_1 and M_2 on $\widetilde{S}_{\overline{k}}$ correspond to $m^0_{1,2}$ and $m^0_{2,3}$ in I_3, respectively. Then by using Lemma 2.1.4 there exist exactly three (-1)-curves E_1, E_1' and E_1'' meeting M_1 on $\widetilde{S}_{\overline{k}}$ corresponding to $\ell_{1,4}$, $\ell_{1,5}$ and $\ell_{1,6}$ in I_3, respectively. Moreover, there exist exactly three (-1)-curves E_2, E_2' and E_2'' meeting M_1 on $\widetilde{S}_{\overline{k}}$ corresponding to e_3, $\ell_{1,2}$ and c_3 in I_3,

respectively. Hence, we obtain the following weighted dual graph:



Let $\tau : \widetilde{S} \to V$ be the contraction of $E_1 + E_1' + E_1'' + E_2 + E_2' + E_2''$. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by Lemma 2.2.2. Meanwhile, if M_1 is defined over k, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 2 = 3$. Otherwise, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 1 = 2$.

$\widetilde{S}$ *is of* $d = 3$ *and of* A_1*-type:* By [18, p. 446], we can choose a morphism (4.2.1) such that the (-2)-curve M on $\widetilde{S}_{\overline{k}}$ corresponds to m^2 in I_3. Then by using Lemma 2.1.4 there exist exactly six (-1)-curves $E_1, \dots, E_6$ meeting M on $\widetilde{S}_{\overline{k}}$ corresponding to $e_1, \dots, e_6$ in I_3, respectively. Hence, we obtain the following weighted dual graph:



Let $\tau : \widetilde{S} \to V$ be the contraction of $E_1 + \cdots + E_6$. By construction, τ is defined over k and we see $V \simeq \mathbb{P}^2_k$ by the similar argument to the case that $\widetilde{S}$ is of $d = 3$ and $2A_1 + A_1$-type. Moreover, the direct image $\tau_*(M)$ is an irreducible conic on $V \simeq \mathbb{P}^2_k$. Meanwhile, we obtain $\rho_k(\widetilde{S}) = \rho_k(S) + 1 = 2$.

Thus, for all cases, this completes the construction of the birational morphism $\tau : \widetilde{S} \to V$ over k according to the type of $\widetilde{S}$. By using this morphism τ, all Du Val del Pezzo surfaces of rank one over k admitting a singular point defined over k can be summarized in Table 4.1 according to the type of $\widetilde{S}$. Note that this table will play an important role in the proof of Theorem 1.3.9 (1) and (2) in the next subsection. From now on, we shall present the notation in Table 4.1. "$\rho_k(\widetilde{S})$" means possible values of the Picard number of $\widetilde{S}$. "Dual graph" means the weighted dual graph of all (-2)-curves and (-1)-curves, which contracted by τ, on $\widetilde{S}_{\overline{k}}$. By construction of τ, we see that the union of curves corresponding to this weighted dual graph is defined over k, moreover, each curve on $\widetilde{S}$ corresponds to any vertex with no label in this dual graph contracted by τ. Finally, $n°$ is the number between $1°$ and $10°$ assigned by the kind of V and the image via τ of the union of all curves corresponding to this above dual graph. More exactly:

- $1°$: $V \simeq \mathbb{P}^2_k$ and the image via τ of the curve corresponding to the vertex with label L is a line.
- $2°$: $V \simeq \mathbb{P}^2_k$ and the images via τ of the curves corresponding to the vertices with labels L_1 and L_2 are two distinct lines.
- $3°$: $V \simeq \mathbb{P}^2_k$ and the image via τ of the curves corresponding to the vertex with label Q is an irreducible conic.
- $4°$: V is a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ and of rank one, and the images via τ of the union of two curves corresponding to the vertices with labels F_1 and F_2 are a union of k-forms of two irreducible curves of types $(1, 0)$ and $(0, 1)$, respectively.
- $5°$: V is a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ and of rank one, and the image via τ of the curve corresponding to the vertex with label C is a k-form of an irreducible curves of types $(1, 1)$.

- 6°: $V \simeq \mathbb{F}_2$ and the images via τ of the curves corresponding to the vertices with labels M and F are the minimal section and a closed fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k, respectively.
- 7°: $V \simeq \mathbb{F}_2$ and the images via τ of the curves corresponding to the vertices with labels M, F and C are the minimal section, a closed fiber and a section with self-intersection number 2 of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k, respectively.
- 8°: V is a k-form of $\mathbb{F}_2$, and the images via τ of the curves corresponding to the vertices with labels M and C are k-forms of the minimal section and a section with self-intersection number 2 of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_{\overline{k}}$, respectively.
- 9°: V is a k-form of $\mathbb{F}_2$, and $\widetilde{S} = V$.
- 10°: V is a k-form of $\mathbb{F}_2$, and the images via τ of the curves corresponding to the vertices with labels M and F_i are k-forms of the minimal section and a closed fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_{\overline{k}}$, respectively, where the union of two curves corresponding to the vertex with label F_1 and F_2 is defined over k.

4.2.2 Proof of Theorems 1.3.9 (1) and (2)

Let the notation be the same as at beginning of Chapter 4 and assume further that S is of rank one and $d \geq 3$. In this subsection, we shall show Theorems 1.3.9 (1) and (2).

At first, we shall show the "only if" part in Theorem 1.3.9 (2). Assume that d is equal to 3 or 4 and S contains a cylinder $U \simeq \mathbb{A}^1_k \times Z$. The closures in S of fibers of the projection $pr_Z : U \simeq \mathbb{A}^1_k \times Z \to Z$ yields a linear system, say $\mathscr{L}$, on S. By Lemma 2.5.6, $\mathrm{Bs}(\mathscr{L})$ consists of only one singular point on S, which is k-rational, say p. In order to prove the "only if" part in Theorem 1.3.9 (2), we shall show that $p \in S$ is not of type A_1^{++} over k as follows:

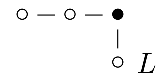
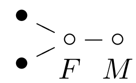
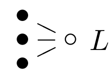
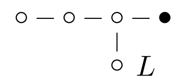
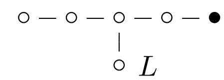
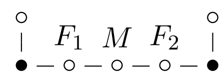
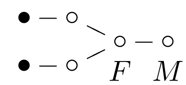
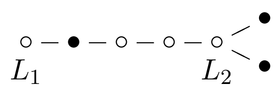
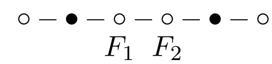
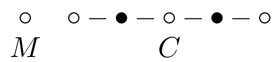
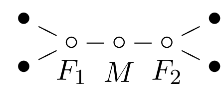
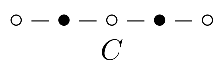
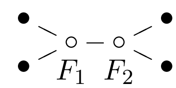
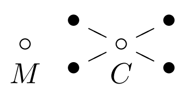
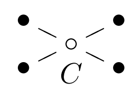
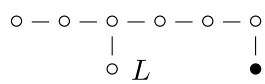
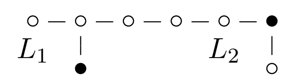
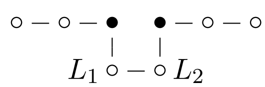
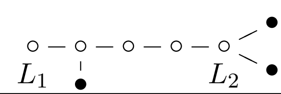
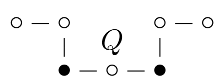
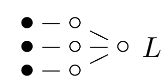
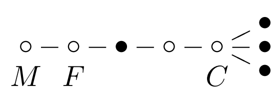
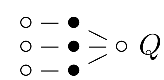
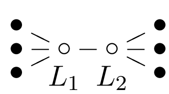
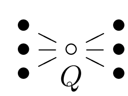
Lemma 4.2.7. Let the notation and the assumptions be the same as above. If the singular point $p \in S_{\overline{k}}$ is of type A_1, then $p \in S$ is of type A_1^+ over k.

Proof. Since $U_{\overline{k}}$ is smooth, $\widetilde{U} := \sigma^{-1}(U) \simeq U$ is a cylinder on $\widetilde{S}$. The closures in $\widetilde{S}$ of fibers of the projection $pr_Z : \widetilde{U} \simeq \mathbb{A}^1_k \times Z \to Z$ yields a linear system, say $\widetilde{\mathscr{L}}$, on $\widetilde{S}$. Since $p \in S_{\overline{k}}$ is of type A_1, the exceptional locus over $\overline{k}$ of the minimal resolution at p consists of only one (-2)-curve, say M. Notice that M is defined over k. By construction of $\widetilde{\mathscr{L}}$, we see that a general member of $\widetilde{\mathscr{L}}_{\overline{k}}$ does not meet any (-2)-curve other than M on $\widetilde{S}_{\overline{k}}$. Hence, we can write $\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}) - bM$ for some $a, b \in \mathbb{Q}$. Noting that the degree d of S is equal to 3 or 4, we have $(\widetilde{\mathscr{L}})^2 = da^2 - 2b^2 \neq 0$ because of $a, b \in \mathbb{Q}$. Thus, $\mathrm{Bs}(\widetilde{\mathscr{L}}) \neq \emptyset$. In particular, $\mathrm{Bs}(\widetilde{\mathscr{L}})$ consists of one point, which is k-rational and lies on M. Thus, we obtain $M(k) \neq \emptyset$, which implies that $p \in S$ is not of type A_1^{++} over k. □

By Lemma 4.2.7, this completes the proof of the "only if" part in Theorem 1.3.9 (2).

Next, we show Theorem 1.3.3 and the "if" part in Theorem 1.3.4. Assume that S has a singular point, which is k-rational, such that it is not of type A_1^{++} over k if d is equal to 3 or 4. Let $\tau : \widetilde{S} \to V$ be the birational morphism over k as in Subsection 4.3.1 over k according to the type of $\widetilde{S}$. Let D be the union of all (-2)-curves on $\widetilde{S}_{\overline{k}}$ and let E be the reduced exceptional divisor of τ, where the support $\mathrm{Supp}(D+E)$ corresponds to the dual graph as in Table 4.1 according to the type of $\widetilde{S}$. We shall construct a cylinder $\widetilde{U}$ on $\widetilde{S}$ according to the number of $n°$ in Table 4.1:

Table 4.1: Types of $\widetilde{S}$ in Theorems 1.3.9 (1) and (2)

d	Type $\rho_k(\widetilde{S})$	n°	Dual graph	d	Type $\rho_k(\widetilde{S})$	n°	Dual graph
8	A_1 2	9°		6	$A_2 + A_1$ 4	1°	
6	A_2 3	6°		6	$(A_1)_<$ 2	1°	
5	A_4 5	1°		4	D_5 6	1°	
4	$A_3 + 2A_1$ 4 or 6	10°		4	D_4 4	6°	
4	$A_3 + A_1$ 5	2°		4	$A_2 + 2A_1$ 3	4°	
4	$4A_1$ 4 or 5	8°		4	$(A_3)_<$ 3 or 4	10°	
4	$3A_1$ 3	5°		4	A_2 2	4°	
4	$(2A_1)_<$ 3	8°		4	A_1 2	5°	
3	E_6 7	1°		3	$A_5 + A_1$ 7	2°	
3	$3A_2$ 4 or 7	2°		3	A_5 6	2°	
3	$2A_2 + A_1$ 4	3°		3	D_4 3	1°	
3	$2A_2$ 5	7°		3	$4A_1$ 3	3°	
3	A_2 2 or 3	2°		3	A_1 2	3°	

$n^\circ = 1^\circ$: In this case, we see that $V \simeq \mathbb{P}^2_k$ and the image of the vertex with a label written L via τ is a line on $V \simeq \mathbb{P}^2_k$, say L. Put $\widetilde{U} := \widetilde{S} \backslash \mathrm{Supp}(D+E)$. Then $\widetilde{U} \simeq V \backslash L \simeq \mathbb{A}^2_k$.

$n^\circ = 2^\circ$: In this case, we see that $V \simeq \mathbb{P}^2_k$ and the images of the vertices with labels written L_1 and L_2 via τ are distinct two lines on $V \simeq \mathbb{P}^2_k$, say L_1 and L_2. Put $\widetilde{U} := \widetilde{S} \backslash \mathrm{Supp}(D+E)$. Then $\widetilde{U} \simeq V \backslash (L_1 \cup L_2) \simeq \mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$ (resp. $\widetilde{U} \simeq V \backslash (L_1 \cup L_2) \simeq \mathbb{A}^1_k \times C_{(1)}$) if L_1 and L_2 are defined over k (resp. L_1 and L_2 are exchanged by the $\mathrm{Gal}(\overline{k}/k)$-action).

$n^\circ = 3^\circ$: In this case, we see that $V \simeq \mathbb{P}^2_k$ and the image of the vertex with a label written Q via τ is an irreducible conic on $V \simeq \mathbb{P}^2_k$, say Q. Notice that Q has a k-rational point. Indeed, the image via σ of $\tau_*^{-1}(Q)$ is a singular point on S of type A_1^+ over k by the assumption. Let L be a line on V such that L and Q tangentially meet at a general k-rational point. Noting that $\tau_*^{-1}(L)$ is defined over k, put $\widetilde{U} := \widetilde{S} \backslash \mathrm{Supp}(D+E+\tau_*^{-1}(L))$. Then $\widetilde{U} \simeq V \backslash (Q \cup L) \simeq \mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$.

$n^\circ = 4^\circ$: In this case, V is a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ and of rank one. Moreover, the images of the vertices with labels written F_1 and F_2 via $\tau_{\overline{k}}$ are k-forms of irreducible curves of types $(1,0)$ and $(0,1)$ on $V_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$, say F_1 and F_2, respectively. Noting that the union $F_1 + F_2$ is defined over k, put $\widetilde{U} := \widetilde{S} \backslash \mathrm{Supp}(D+E)$. Then $\widetilde{U} \simeq V \backslash (F_1 \cup F_2) \simeq \mathbb{A}^2_k$.

$n^\circ = 5^\circ$: In this case, V is a k-form of $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ and of rank one. Moreover, the image of the vertex with a label written C via $\tau_{\overline{k}}$ is a k-form of irreducible curve of type $(1,1)$ on $V_{\overline{k}} \simeq \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$, say C. Notice that C has a k-rational point. Indeed, the image via σ of $\tau_*^{-1}(C)$ is a singular point on S of type A_1^+ over k by the assumption. By Lemma 2.5.1, V contains a cylinder such that this boundary includes C. Hence, we take the pullback $\widetilde{U}$ of this cylinder by τ. Then $\widetilde{U}$ is a cylinder on $\widetilde{S}$ such that this boundary includes $\mathrm{Supp}(D+E)$.

$n^\circ = 6^\circ$: In this case, we see that $V \simeq \mathbb{F}_2$ and the images of the vertices with labels written M and F via τ are the minimal section and a closed fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k, say M and F, respectively. Noting that the union $M+F$ is defined over k, put $\widetilde{U} := \widetilde{S} \backslash \mathrm{Supp}(D+E)$. Then $\widetilde{U} \simeq V \backslash (M \cup F) \simeq \mathbb{A}^2_k$.

$n^\circ = 7^\circ$: In this case, we see that $V \simeq \mathbb{F}_2$ and the images of the vertices with labels written M, F and C via τ are the minimal section, a closed fiber and a section with self-intersection number 2 of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k, say M, F and C, respectively. Noting that the union $M+F+C$ is defined over k, put $\widetilde{U} := \widetilde{S} \backslash \mathrm{Supp}(D+E)$. Then $\widetilde{U} \simeq V \backslash (M \cup F \cup C) \simeq \mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$ by Lemma 2.5.2.

$n^\circ = 8^\circ$: In this case, V is a k-form of $\mathbb{F}_2$. Moreover, the images of the vertices with labels written M and C via τ are the minimal section and a section with self-intersection number 2 of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k, say M and C, respectively. Notice that either M or C has a k-rational point. Indeed, the images via σ of $\tau_*^{-1}(M)$ and $\tau_*^{-1}(C)$ are singular points on $S_{\overline{k}}$ of type A_1. By assumption, one of these is of type A_1^+ over k. Hence, we obtain $V \simeq \mathbb{F}_2$ by using Lemma 2.2.2. Let F be a general fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ defined over k. Noting that the union $M+F+C$ is defined over k, put $\widetilde{U} := \widetilde{S} \backslash \mathrm{Supp}(D+E+\tau_*^{-1}(F))$. Then $\widetilde{U} \simeq V \backslash (M \cup F \cup C) \simeq \mathbb{A}^1_k \times \mathbb{A}^1_{*,k}$ by Lemma 2.5.2.

$n^\circ = 9^\circ$: In this case, $V = \widetilde{S}$ and V is a k-form of $\mathbb{F}_2$. Hence, $\widetilde{S}$ contains a cylinder $\widetilde{U}$, so that $\widetilde{U} \cap \mathrm{Supp}(M) = \emptyset$ (see Lemma 3.3.4).

$n^\circ = 10^\circ$: In this case, V is a k-form of $\mathbb{F}_2$. Moreover, the images of the vertices with labels written M and F_i via τ are k-forms of the minimal section and a closed fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_{\overline{k}}$, say M and F_i, respectively. Then V contains a cylinder such that this boundary includes M, F_1 and F_2 (see Lemma 3.3.4). Hence, we take the pullback $\widetilde{U}$ of this cylinder by τ. Then $\widetilde{U}$ is a cylinder on $\widetilde{S}$ such that this boundary includes $\mathrm{Supp}(D+E)$.

For all cases, we obtain a cylinder $\widetilde{U}$ on $\widetilde{S}$ such that this boundary includes $\mathrm{Supp}(D)$. Therefore, S contains the cylinder $\sigma(\widetilde{U}) \simeq \widetilde{U}$. This completes the proof of Theorem 1.3.9 (1) and the "if" part in Theorem 1.3.9 (2).

Remark 4.2.8. We shall state some remarks on the above argument.

(1) In these cases $n^\circ = 1^\circ$, 4° or 6°, then S always contains the affine plane $\mathbb{A}^2_k$ (compare the fact that the Du Val del Pezzo surface of rank one over $\mathbb{C}$ with $\rho_{\mathbb{C}}(S) = 1$ and of degree $d \geq 3$ contains $\mathbb{C}^2$ if and only if the pair of the degree and the singularities of this surface is $(8, A_1)$, $(6, A_2 + A_1)$, $(5, A_4)$, $(4, D_5)$ or $(3, E_6)$, see [56]).

(2) In these cases $n^\circ = 9^\circ$ or 10°, then $\widetilde{S}$ does not have to admit any k-rational point. However, S always contains a cylinder, say $U \simeq \mathbb{A}^1_k \times Z$ (compare the fact that any smooth del Pezzo surface over k with $\rho_k(S) = 1$ containing a cylinder admits k-rational points, see Theorem 1.2.4). This implies that Z is not necessarily a rational curve over k. Moreover, we also know that V is a trivial k-form of $\mathbb{F}_2$ if and only if $\widetilde{S}_{\overline{k}}$ has a k-rational point.

4.3 Properties of dvisors on weak del Pezzo surfaces

Let the notation be the same as at beginning of Chapter 4. In this section, we shall study some $\mathbb{Q}$-divisors generated by (-2)-curves on $\widetilde{S}_{\overline{k}}$. As an application, we explicitly construct the union of (-1)-curves on $\widetilde{S}_{\overline{k}}$. Furthermore, we determine the condition that each irreducible component of this union is defined over k. This argument will play an important role in determining the existence of Du Val del Pezzo surfaces of rank one with degree ≤ 2 in Section 4.5.

4.3.1 $\mathbb{Q}$-divisors composed of (-2)-curves

In this subsection, let x be a singular point of type A_n, D_5 or E_6 on $S_{\overline{k}}$, which is k-rational, let $M_1, \dots, M_n$ be all irreducible components of the exceptional set on $\widetilde{S}_{\overline{k}}$ by the minimal resolution at x on $S_{\overline{k}}$. Assume that the dual graph of $M_1, \dots, M_n$ is the following graph according to the singularity type of x on $S_{\overline{k}}$:

- Type A_n:

$$\begin{array}{ccccccc} M_1 & & M_2 & & & & M_n \\ \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ \end{array} \tag{4.3.1}$$

- Type D_5:

$$\begin{tikzcd} M_1 \arrow[dr, dash] & & & \\ & M_3 \arrow[r, dash] & M_4 \arrow[r, dash] & M_5 \\ M_2 \arrow[ur, dash] & & & \end{tikzcd} \tag{4.3.2}$$

- Type E_6:

$$\begin{tikzcd} M_1 \arrow[r, dash] & M_3 \arrow[dr, dash] & & \\ & & M_5 \arrow[r, dash] & M_6 \\ M_2 \arrow[r, dash] & M_4 \arrow[ur, dash] & & \end{tikzcd} \tag{4.3.3}$$

Table 4.2: The value of $(M)^2$ in Lemma 4.3.1

$n \backslash j_0$	1	2	3	4	5	6	7	8
1	$-\frac{1}{2}$							
2	$-\frac{2}{3}$	$-\frac{2}{3}$						
3	$-\frac{3}{4}$	-1	$-\frac{3}{4}$					
4	$-\frac{4}{5}$	$-\frac{6}{5}$	$-\frac{6}{5}$	$-\frac{4}{5}$				
5	$-\frac{5}{6}$	$-\frac{4}{3}$	$-\frac{3}{2}$	$-\frac{4}{3}$	$-\frac{5}{6}$			
6	$-\frac{6}{7}$	$-\frac{10}{7}$	$-\frac{12}{7}$	$-\frac{12}{7}$	$-\frac{10}{7}$	$-\frac{6}{7}$		
7	$-\frac{7}{8}$	$-\frac{3}{2}$	$-\frac{15}{8}$	-2	$-\frac{15}{8}$	$-\frac{3}{2}$	$-\frac{7}{8}$	
8	$-\frac{8}{9}$	$-\frac{14}{9}$	-2	$-\frac{20}{9}$	$-\frac{20}{9}$	-2	$-\frac{14}{9}$	$-\frac{8}{9}$

Let M be a $\mathbb{Q}$-divisor on $\widetilde{S_{\overline{k}}}$, which is generated by $M_1, \ldots, M_n$, so that:

$$M = \sum_{j=1}^{n} b_j M_j$$

for some $b_1, \ldots, b_n \in \mathbb{Q}$.

Lemma 4.3.1. With the notation as above, assume further that x is of type A_n on $S_{\overline{k}}$. Let j_0 be an integer with $1 \leq j_0 \leq n$. If $(-M \cdot M_j) = \delta_{j_0,j}$, then we have:

$$M = \frac{n - j_0 + 1}{n+1} \sum_{j=1}^{j_0} j M_j + \frac{j_0}{n+1} \sum_{j=1}^{n-j_0} j M_{n-j+1}$$

and:

$$(M)^2 = -\frac{(n - j_0 + 1) j_0}{n+1}.$$

Proof. For all cases, we can easily show because it is enough to directly compute some intersection numbers. □

In Lemma 4.3.1, if $(-M \cdot M_j) = \delta_{j_0,j}$, then the value of $(M)^2$ is explicitly summarized in Table 4.2 depending on the values of n and j_0.

Lemma 4.3.2. With the notation as above, assume further that x is of type D_5 on $S_{\overline{k}}$. Then we have the following assertions:

(1) If $(-M \cdot M_j) = \delta_{1,j} + \delta_{2,j}$, then we have:

$$M = 2M_1 + 2M_2 + 3M_3 + 2M_4 + M_5$$

and $(M)^2 = -4$.

(2) If $(-M \cdot M_j) = \delta_{1,j}$, then we have:

$$M = \frac{5}{4}M_1 + \frac{3}{4}M_2 + \frac{3}{2}M_3 + M_4 + \frac{1}{2}M_5$$

and $(M)^2 = -\frac{5}{4}$.

(3) If $(-M \cdot M_j) = \delta_{3,j}$, then we have:

$$M = \frac{3}{2}M_1 + \frac{3}{2}M_2 + 3M_3 + 2M_4 + M_5$$

and $(M)^2 = -3$.

(4) If $(-M \cdot M_j) = \delta_{4,j}$, then we have:

$$M = M_1 + M_2 + 2M_3 + 2M_4 + M_5$$

and $(M)^2 = -2$.

(5) If $(-M \cdot M_j) = \delta_{5,j}$, then we have:

$$M = \frac{1}{2}M_1 + \frac{1}{2}M_2 + M_3 + M_4 + M_5$$

and $(M)^2 = -1$.

Proof. For all cases, we can easily show because it is enough to directly compute some intersection numbers. □

Lemma 4.3.3. With the notation as above, assume further that x is of type E_6 on $S_{\overline{k}}$. Then we have the following assertions:

(1) If $(-M \cdot M_j) = \delta_{1,j} + \delta_{2,j}$, then we have:

$$M = 2M_1 + 2M_2 + 3M_3 + 3M_4 + 4M_5 + M_6$$

and $(M)^2 = -4$.

(2) If $(-M \cdot M_j) = \delta_{3,j} + \delta_{4,j}$, then we have:

$$M = 3M_1 + 3M_2 + 6M_3 + 6M_4 + 8M_5 + 4M_6$$

and $(M)^2 = -12$.

(3) If $(-M \cdot M_j) = \delta_{1,j}$, then we have:

$$M = \frac{4}{3}M_1 + \frac{2}{3}M_2 + \frac{5}{3}M_3 + \frac{4}{3}M_4 + 2M_5 + M_6$$

and $(M)^2 = -\frac{4}{3}$.

(4) If $(-M \cdot M_j) = \delta_{3,j}$, then we have:

$$M = \frac{5}{3}M_1 + \frac{4}{3}M_2 + \frac{10}{3}M_3 + \frac{8}{3}M_4 + 4M_5 + 2M_6$$

and $(M)^2 = -\frac{10}{3}$.

(5) If $(-M \cdot M_j) = \delta_{5,j}$, then we have:

$$M = 2M_1 + 2M_2 + 4M_3 + 4M_4 + 6M_5 + 3M_6$$

and $(M)^2 = -6$.

(6) If $(-M \cdot M_j) = \delta_{6,j}$, then we have:

$$M = M_1 + M_2 + 2M_3 + 2M_4 + 3M_5 + 2M_6$$

and $(M)^2 = -2$.

Proof. For all cases, we can easily show because it is enough to directly compute some intersection numbers. □

Lemma 4.3.4. With the notation as above, assume further that $(-K_{\widetilde{S}_{\overline{k}}})^2 = 1$ and one of the following conditions holds:

(1) The dual graph of $M_1, \dots, M_n$ is the same as in (4.3.1) and $M = \sum_{j=1}^{n} M_j$;

(2) $n = 5$, the dual graph of $M_1, \dots, M_5$ is the same as in (4.3.2) and $M = M_1 + M_2 + 2M_3 + 2M_4 + M_5$;

(3) $n = 6$, the dual graph of $M_1, \dots, M_6$ is the same as in (4.3.3) and $M = M_1 + M_2 + 2M_3 + 2M_4 + 3M_5 + 2M_6$.

Then there exists a (-1)-curve E on $\widetilde{S}_{\overline{k}}$ such that $E \sim -K_{\widetilde{S}_{\overline{k}}} - M$ and E is defined over k.

Proof. Noticing the assumption of M, we see $(M)^2 = -2$ by the straightforward calculation. In particular, we obtain $(-K_{\widetilde{S}} - M)^2 = -1$ and $(-K_{\widetilde{S}} - M \cdot -K_{\widetilde{S}}) = 1$. Moreover, for each case, we obtain $(-K_{\widetilde{S}} - M \cdot M_j) \geq 0$ for any i. Indeed, in the case of (1) we have $(-K_{\widetilde{S}} - M \cdot M_j) = \delta_{j,1} + \delta_{j,n}$ (cf. Lemma 4.3.1), in the case of (2) we have $(-K_{\widetilde{S}} - M \cdot M_j) = \delta_{j,4}$ (cf. Lemma 4.3.2 (4)), and in the case of (3) we have $(-K_{\widetilde{S}} - M \cdot M_j) = \delta_{j,6}$ (cf. Lemma 4.3.3 (6)). Meanwhile, $(-K_{\widetilde{S}} - M \cdot M') = 0$ for every (-2)-curve M' on $\widetilde{S}_{\overline{k}}$ other than the irreducible components of M. Hence, by Lemma 2.1.4, there exists a (-1)-curve E on $\widetilde{S}_{\overline{k}}$ such that $E \sim -K_{\widetilde{S}} - M$. Notice that E is included in $\mathrm{Pic}(\widetilde{S}_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}$ because so are $-K_{\widetilde{S}}$ and M. Thus, E is defined over k. This completes the proof. □

Lemma 4.3.5. With the notation as above, assume further that $b_1, \dots, b_n \in \mathbb{Z}$. Then the following assertions hold:

(1) $(M)^2$ is a non-positive even integer.

(2) If x is of type A_n on S and $b_j \geq 1$ for any j, then $(M)^2 \leq -2$, moreover, $(M)^2 = -2$ if and only if $b_j = 1$ for any $j = 1, \dots, n$.

(3) If x is of type A_n on S with $n \geq 3$, $b_1, b_n \geq 1$ and $b_j \geq 2$ for any $j = 2, \dots, n-1$, then $(M)^2 \leq -4$, moreover, $(M)^2 = -4$ if and only if $b_1, b_n = 1$ and $b_j = 2$ for any $j = 2, \dots, n-1$.

(4) If x is of type A_n on S with $n \geq 5$, $b_1, b_n \geq 1$, $b_2, b_{n-1} \geq 2$ and $b_j \geq 3$ for any $j = 3, \dots, n-2$, then $(M)^2 \leq -6$, moreover, $(M)^2 = -6$ if and only if $b_1, b_n = 1$, $b_2, b_{n-1} = 2$ and $b_j = 3$ for any $j = 3, \dots, n-2$.

(5) If x is of type D_5 on $S_{\overline{k}}$ and $b_1, b_2, b_4 \geq 2$, $b_3 \geq 3$ and $b_5 \geq 1$, then $(M)^2 \leq -4$, moreover, $(M)^2 = -4$ if and only if $b_1, b_2, b_4 = 2$, $b_3 = 3$ and $b_5 = 1$.

(6) If x is of type E_6 on $S_{\overline{k}}$ and $b_1, b_2, b_6 \geq 2$, $b_3, b_4 \geq 3$ and $b_5 \geq 4$, then $(M)^2 \leq -4$, moreover, $(M)^2 = -4$ if and only if $b_1, b_2, b_6 = 2$, $b_3, b_4 = 3$ and $b_5 = 4$.

Proof. In (1), since any irreducible component of M is a (-2)-curve and any coefficient of M is an integer, it is clearly seen that $(M)^2$ is an even number. We shall show that $(M)^2 \leq 0$ according to the singularity type of x on $S_{\overline{k}}$:

- *Type A_n:* we have:

$$(M)^2 = -(b_1^2 + b_n^2) - \sum_{j=1}^{n-1}(b_j - b_{j+1})^2. \tag{4.3.4}$$

- *Type D_5:* we have:

$$(M)^2 = -\frac{1}{2}(2b_1 - b_3)^2 - \frac{1}{2}(2b_2 - b_3)^2 - (b_3 - b_4)^2 - (b_4 - b_5)^2 - b_5^2. \tag{4.3.5}$$

- *Type E_6:* we have:

$$\begin{aligned}(M)^2 = -\frac{1}{2}(2b_1 - b_2)^2 - \frac{1}{2}(2b_2 - b_4)^2 \\ - \frac{1}{6}(3b_3 - 2b_5)^2 - \frac{1}{6}(3b_4 - 2b_5)^2 - \frac{1}{6}(2b_5 - 3b_6)^2 - \frac{1}{2}b_6^2.\end{aligned} \tag{4.3.6}$$

Therefore, for all cases, we see that $(M)^2 \leq 0$. This completes the proof of (1).

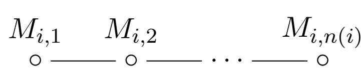
In (2), (3) and (4), it is easy to show by (4.3.4).

In (5), if $b_5 > 1$ then it is easy to see $(M)^2 < -4$ by assumption and (4.3.5). Hence, we assume $b_5 = 1$ in what follows. Now, if $b_4 > 2$, then we also see $(M)^2 < -4$ by an argument similar to the above. Hence, we also assume $b_4 = 2$ in what follows. By sequentially replacing b_4 in the argument by b_3, b_2 and b_1, we obtain the assertion.

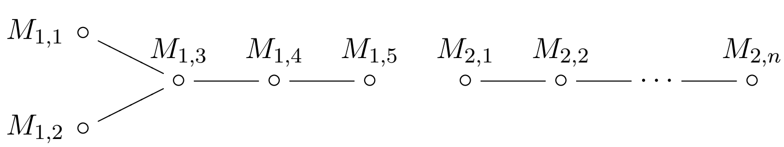
In (6), it can be shown by an argument similar to (5) using (4.3.6) instead of (4.3.5). $\square$

4.3.2 Construction of (-1)-curves on weak del Pezzo surface

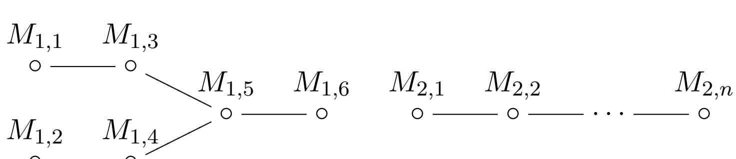
In this subsection, let d be the degree of $\widetilde{S}$, let $x_1, \ldots, x_{r'}$ be all singular points on $S_{\overline{k}}$ let $M_{i,1}, \ldots, M_{i,n(i)}$ be all irreducible components of the exceptional set $\sigma^{-1}(x_i)$ for $i = 1, \ldots, r'$. Here, we assume that $x_1 \in S_{\overline{k}}$ is of type $A_{n(1)}$ with $n(1) \geq 2$ (resp. either $x_1 \in S_{\overline{k}}$ is of type $A_{n(1)}$ with $n(1) \geq 4$ or of type D_5 or E_6) if $d = 2$ (resp. $d = 1$). Moreover, letting r be a positive integer with $r \leq r'$, we also assume that the dual graph of $\sum_{i=1}^{r}\sum_{j=1}^{n(i)} M_{i,j}$ is one of the following graphs (4.3.7), (4.3.8) and (4.3.9):



for $i = 1, \ldots, r$ (4.3.7)



(4.3.8)



(4.3.9)

Table 4.3: Divisor D in Subsection 4.3.2

Name	d	r	Irreducible decomposition of D
(a)	2	2	$(-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=1}^{2}\sum_{j=1}^{n(i)} M_{i,j}$
(b)	2	1	$(-K_{\widetilde{S}_{\overline{k}}}) + (M_{1,1} + M_{1,n(1)}) - 2\sum_{j=1}^{n(1)} M_{1,j}$.
(c)	1	3	$2(-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=1}^{3}\sum_{j=1}^{n(i)} M_{i,j}$
(d)	1	2	$2(-K_{\widetilde{S}_{\overline{k}}}) + (M_{1,1} + M_{1,n(1)}) - 2\sum_{j=1}^{n(1)} M_{1,j} - \sum_{j=1}^{n(2)} M_{2,j}$
(e)	1	1	$2(-K_{\widetilde{S}_{\overline{k}}}) + (M_{1,1} + M_{1,n(1)}) + 2(M_{1,2} + M_{1,n(1)-1}) - 3\sum_{j=1}^{n(1)} M_{1,j}$
(f)	1	2	$2(-K_{\widetilde{S}_{\overline{k}}}) - (2M_{1,1} + 2M_{1,2} + 3M_{1,3} + 2M_{1,4} + M_{1,5}) - \sum_{j=1}^{n} M_{2,j}$
(g)	1	2	$2(-K_{\widetilde{S}_{\overline{k}}}) - (2M_{1,1} + 2M_{1,2} + 3M_{1,3} + 3M_{1,4} + 4M_{1,5} + M_{1,6}) - \sum_{j=1}^{n} M_{2,j}$

Here, in (4.3.7), we shall assume $(d, r) = (2, 2), (2, 1), (1, 3), (1, 2)$ or $(1, 1)$. Furthermore, in (4.3.8) (resp. (4.3.9)), we immediately obtain $n(1) = 5$ (resp. $n(1) = 6$) by the configuration, moreover, we shall assume $(d, r) = (1, 2)$ and put $n(2) := n$.

Let D be the divisor on $\widetilde{S}_{\overline{k}}$ given by one of the lists in Table 4.3 according to the above cases of the dual graph and the pair (d, r). Here, when D is (a), (b), (c), (d) or (e) in Table 4.3, the dual graph of $\sum_{i=1}^{r}\sum_{j=1}^{n(i)} M_{i,j}$ is as in (4.3.7). Moreover, when D is (f) and (g) in Table 4.3, the dual graph of $\sum_{i=1}^{2}\sum_{j=1}^{n(i)} M_{i,j}$ is as in (4.3.8) and (4.3.9), respectively. On the other hand, we assume $n(1) \geq 4$ (resp. $n(1) \geq 6$) if the case of D is either (b) or (d) (resp. (e)).

For all cases, we see $(D)^2 = -2$ and $(D \cdot -K_{\widetilde{S}_{\overline{k}}}) = 2$ by construction, moreover, we have the value of $(D \cdot M_{i,j})$, which is the following according to the cases:

(a): $(D \cdot M_{i,j}) = \delta_{j,1} + \delta_{j,n(i)}$ for $i = 1, 2$.

(b): $(D \cdot M_{i,j}) = \delta_{j,2} + \delta_{j,n(1)-1}$.

(c): $(D \cdot M_{i,j}) = \delta_{j,1} + \delta_{j,n(i)}$ for $i = 1, 2, 3$.

(d): $(D \cdot M_{i,j}) = \delta_{i,1}(\delta_{j,2} + \delta_{j,n(1)-1}) + \delta_{i,2}(\delta_{j,1} + \delta_{j,n(2)})$ for $i = 1, 2$.

(e): $(D \cdot M_{i,j}) = \delta_{j,3} + \delta_{j,n(1)-2}$.

(f): $(D \cdot M_{i,j}) = \delta_{i,1}(\delta_{j,1} + \delta_{j,2}) + \delta_{i,2}(\delta_{j,1} + \delta_{j,n})$ for $i = 1, 2$.

(g): $(D \cdot M_{i,j}) = \delta_{i,1}(\delta_{j,1} + \delta_{j,2}) + \delta_{i,2}(\delta_{j,1} + \delta_{j,n})$ for $i = 1, 2$.

The purpose of this subsection is to prove Proposition 4.3.10. For the following two lemmas, we only treat the case (a) since other cases can be shown by a similar argument.

Lemma 4.3.6. $\dim |D| \geq 0$.

Proof. By the Riemann-Roch theorem and $(D \cdot D - K_{\widetilde{S}_{\overline{k}}}) = 0$, we have $\chi(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}(D)) = \chi(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}})$. Moreover, by the Serre duality theorem combined with $(K_{\widetilde{S}_{\overline{k}}} - D \cdot M_{1,1}) = -1 < 0$, we have $h^2(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}(D)) = h^0(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}(K_{\widetilde{S}_{\overline{k}}} - D)) = 0$. Thus, we have $\dim |D| = h^0(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}(D)) - 1 \geq \chi(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}(D)) - 1$. On the other hand, it is known that $\widetilde{S}_{\overline{k}}$ is a rational surface by Lemma 2.1.3, we see $\chi(\widetilde{S}_{\overline{k}}, \mathscr{O}_{\widetilde{S}_{\overline{k}}}) = 1$. Therefore, we have $\dim |D| \geq 0$. $\square$

By Lemma 4.3.6, we can write $D \sim D_1 + D_2$, whose D_1 and D_2 are effective divisors on $\widetilde{S}_{\overline{k}}$, such that any irreducible component C_1 (resp. C_2) on D_1 (resp. D_2) satisfies $(C_1 \cdot -K_{\widetilde{S}_{\overline{k}}}) > 0$ (resp. $(C_2 \cdot -K_{\widetilde{S}_{\overline{k}}}) = 0$). Hence, $(D_1 \cdot -K_{\widetilde{S}_{\overline{k}}}) = 2$ because of $(D \cdot -K_{\widetilde{S}_{\overline{k}}}) = 2$. Meanwhile, note that D_2 is an effective divisor, which consists of (-2)-curves on $\widetilde{S}_{\overline{k}}$, since $\widetilde{S}_{\overline{k}}$ is a weak del Pezzo surface.

Lemma 4.3.7. $(D_1)^2 \leq -2$.

Proof. By $D_1 \sim D - D_2$, we can write $D_1 \sim (-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=1}^{2}\sum_{j=1}^{n(i)} b_{i,j} M_{i,j} - M'$, where each $b_{i,j}$ is an integer, and M' is an effective divisor consisting of (-2)-curves $\{M_{i,j}\}_{r<i\leq r',\ 1\leq j\leq n(i)}$. By $D_2 \sim D - D_1$, we then have $D_2 = \sum_{i=1}^{2}\sum_{j=1}^{n(i)}(b_{i,j}-1)M_{i,j} + M'$. Since D_2 is effective, we thus see $b_{i,j} \geq 1$ for $i = 1, 2$ and $j = 1, \dots, n(i)$. Thus, we obtain $(D_1)^2 \leq 2 + 2 \cdot (-2) = -2$ by Lemma 4.3.5 (2). □

Remark 4.3.8. The proof of Lemma 4.3.7 uses Lemma 4.3.5 (2) since D is as in (a). On the other hand, if D is as in (b) (resp. (d), (e), (f), (g)), we can show Lemma 4.3.7 by using Lemma 4.3.5 (3) (resp. both (2) and (3), (4), both (2) and (5), both (2) and (6)) instead of Lemma 4.3.5 (2).

Lemma 4.3.9. We shall consider the following formula:

$$-1 = \frac{1}{d} + \sum_{i=1}^{r}(M_i)^2, \tag{4.3.10}$$

where each M_i is an effective $\mathbb{Q}$-divisor generated by $M_{i,1}, \dots, M_{i,n(i)}$ such that $\sum_{j=1}^{n(i)}(-M_i \cdot M_{i,j}) = 1$. Then we have:

- If $(d, r) = (2, 2)$ and the dual graph of $\sum_{i=1}^{r}\sum_{j=1}^{n(i)} M_{i,j}$ is as in (4.3.7), then $\{(M_1)^2, (M_2)^2\} = \{-\frac{5}{6}, -\frac{2}{3}\}$, $\{-\frac{3}{4}, -\frac{3}{4}\}$ or $\{-1, -\frac{1}{2}\}$, so that $\{n(1), n(2)\} = \{5, 2\}$, $\{3, 3\}$ or $\{3, 1\}$, respectively.
- If $(d, r) = (2, 1)$ and the dual graph of $\sum_{i=1}^{r}\sum_{j=1}^{n(i)} M_{i,j}$ is as in (4.3.7), then $(M_1)^2 = -\frac{3}{2}$, so that $n(1) = 5$.
- If $(d, r) = (1, 3)$ and the dual graph of $\sum_{i=1}^{r}\sum_{j=1}^{n(i)} M_{i,j}$ is as in (4.3.7), then $\{(M_1)^2, (M_2)^2, (M_3)^2\} = \{-\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}\}$, so that $\{n(1), n(2), n(3)\} = \{5, 2, 1\}$.
- If $(d, r) = (1, 2)$ and the dual graph of $\sum_{i=1}^{r}\sum_{j=1}^{n(i)} M_{i,j}$ is as in (4.3.7), then $\{(M_1)^2, (M_2)^2\} = \{-\frac{3}{2}, -\frac{1}{2}\}$, $\{-\frac{4}{3}, -\frac{2}{3}\}$, $\{-\frac{3}{2}, -\frac{1}{2}\}$ or $\{-\frac{6}{5}, -\frac{4}{5}\}$, so that $\{n(1), n(2)\} = \{7, 1\}$, $\{5, 2\}$, $\{5, 1\}$ or $\{4, 4\}$, respectively.
- If $(d, r) = (1, 1)$ and the dual graph of $\sum_{i=1}^{r}\sum_{j=1}^{n(i)} M_{i,j}$ is as in (4.3.7), then $(M_1)^2 = -2$, so that $n(1) = 7$.
- If $(d, r) = (1, 2)$ and the dual graph of $\sum_{i=1}^{r}\sum_{j=1}^{n(i)} M_{i,j}$ is as in (4.3.8), then $((M_1)^2, (M_2)^2) = (-\frac{5}{4}, -\frac{3}{4})$ or $(-1, -1)$, so that $n(2) = 3$.
- If $(d, r) = (1, 2)$ and the dual graph of $\sum_{i=1}^{r}\sum_{j=1}^{n(i)} M_{i,j}$ is as in (4.3.9), then $((M_1)^2, (M_2)^2) = (-\frac{4}{3}, -\frac{2}{3})$, so that $n(2) = 2$.

Proof. Notice that the assmption $\sum_{j=1}^{n(i)}(-M_i \cdot M_{i,j}) = 1$ implies that there uniquely exists j_0 such that $(-M_i, \cdot M_{i,j}) = \delta_{j,j_0}$. Hence, for all cases it can be easily shown by using Lemmas 4.3.1, 4.3.2 and 4.3.3 (see also Table 4.2), where we note $n(1) \geq 2$ (resp. $n(1) \geq 4$) if $d = 2$ (resp. $d = 1$). □

Now, we shall consider the following two conditions (†) and (‡) on D:

(‡): There exist two (-1)-curves E_1 and E_2 on $\widetilde{S}_{\overline{k}}$ satisfying $D_1 = E_1 + E_2$ and $(E_1 \cdot E_2) = 0$.

(†): There exists a (-1)-curve E on $\widetilde{S}_{\overline{k}}$ satisfying $D_1 = 2E$.

Then we obtain the following proposition, which will play an important role in Section 4.5:

Proposition 4.3.10. With the notation as above, the following assertions hold:

(1) D satisfies either condition (‡) or (†):

(2) If D satisfies the condition (‡), then we have $D \sim D_1$.

(3) We write $D_1 \sim \frac{2}{d}(-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=1}^{r'} \sum_{j=1}^{n(i)} b_{i,j} M_{i,j}$, where each $b_{i,j}$ is a non-negative integer. Then:

- For any $i \le r$, $b_{i,j} \neq 0$ for some $j \in \{1, \dots, n(i)\}$.
- Each $i = 1, \dots, r'$, if $b_{i,j} \neq 0$ for some $j \in \{1, \dots, n(i)\}$, then $\sum_{j=1}^{n(i)} (E \cdot M_{i,j}) = 1$.

(4) If D is of the case (f) or (g), then D satisfies the condition (‡).

(5) Assume that D satisfies (†), and write $E \sim_{\mathbb{Q}} \frac{1}{d}(-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=1}^{r'} M_i$, where M_i is an effective $\mathbb{Q}$-divisor consisting of $M_{i,1}, \dots, M_{i,n(i)}$. Letting s be the number of $\mathbb{Q}$-divisors M_i as $M_i \neq 0$, then $s \le 2$. Hence, if D is of the case (c), then D satisfies the condition (‡).

(6) Assume that D satisfies the condition (‡). If any irreducible component E of D_1 is contained in $\mathbb{Q}[-K_{\widetilde{S}_{\overline{k}}}] \oplus \left(\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n(i)} \mathbb{Q}[M_{i,j}] \right)$, then each $n(i)$ is one of the following according to the case of D:

- In the case of (a), then $\{n(1), n(2)\} = \{5, 2\}$ or $\{3, 3\}$.
- In the case of (b), then $n(1) = 7$.
- In the case of (c), then $n(1) = 5$ and $\{n(2), n(3)\} = \{2, 1\}$.
- In the case of (d), then $(n(1), n(2)) = (7, 1)$, $(5, 2)$ or $(4, 4)$.
- In the case of (e), then $n(1) = 8$.
- In the case of (f), then $n(2) = 3$ (it is clear that $n(1) = 5$).
- In the case of (g), then $n(2) = 2$ (it is clear that $n(1) = 6$).

(7) Assume that D satisfies the condition (†). If the case of D is (a) or (d), i.e., $r = 2$, then the irreducible component E of D_1 is contained in $\mathbb{Q}[-K_{\widetilde{S}_{\overline{k}}}] \oplus \left(\bigoplus_{i=1}^{2} \bigoplus_{j=1}^{n(i)} \mathbb{Q}[M_{i,j}] \right)$ and each $n(i)$ is as follows according to the case of D:

- In the case of (a), then $(n(1), n(2)) = (3, 1)$.
- In the case of (d), then $(n(1), n(2)) = (5, 1)$.

(8) Assume that D satisfies the condition (†). If the case of D is (b) or (e), i.e, $r = 1$, and the irreducible component E of D_1 is contained in $\mathbb{Q}[-K_{\widetilde{S}_{\overline{k}}}] \oplus \left(\bigoplus_{j=1}^{n(1)} \mathbb{Q}[M_{1,j}] \right)$, then each $n(i)$ is as follows according to the case of D:

- In the case of (b), then $n(1) = 5$.
- In the case of (e), then $n(1) = 7$.

Proof. In (1), note that D_1 consists of at most two irreducible components by the construction of D_1 combined with $(D_1 \cdot -K_{\widetilde{S}_{\overline{k}}}) = 2$. In the case that D_1 consists of exactly one irreducible component, say E. By Lemma 4.3.7, we see $(E)^2 < 0$. Since $(E \cdot -K_{\widetilde{S}_{\overline{k}}}) > 0$ and $\widetilde{S}_{\overline{k}}$ is a weak del Pezzo surface, we know that E is a (-1)-curve. In particular, we obtain $D_1 = 2E$ by virtue of $(D_1 \cdot -K_{\widetilde{S}_{\overline{k}}}) = 2$. Namely, D satisfies the condition (†). In the case that D_1 consists of exactly two irreducible components, say E_1 and E_2. Then $(E_i \cdot -K_{\widetilde{S}_{\overline{k}}}) = 1$ and $D_1 = E_1 + E_2$, so that we have $(D_1)^2 = (E_1)^2 + (E_2)^2 + 2(E_1 \cdot E_2)$. By the above similar argument, E_1 and E_2 are (-1)-curves, in particular, we obtain $(E_1 \cdot E_2) = 0$ by Lemma 4.3.7. Namely, D satisfies the condition (‡).

In (2), assuming that D satisfies the condition (‡), we have $(D_1)^2 = -2$. Hence, we see that this assertion follows from Lemma 4.3.5 according to the case of D (cf. Remark 4.3.8).

In (3), this proof is a bit long and is needed a technical argument. Hence, we will present this proof in the next Subsection 4.3.3.

In what follows, we present the proof under the assumption that (3) is valid.

In (4), we only treat the case where D is of (f), the other cases are similar and left to the reader. Suppose on the contrary that D satisfies the condition (†). In other words, there exists a (-1)-curve E on $\widetilde{S}_{\overline{k}}$ such that $D_1 = 2E$. Then by (3) there uniquely exists $j' \in \{1, \dots, 5\}$ and $j'' \in \{1, \dots, n(2)\}$ such that $(E \cdot M_{1,j}) = \delta_{j,j'}$ and $(E \cdot M_{2,j}) = \delta_{j,j''}$, respectively. Since D_1 is a $\mathbb{Z}$-divisor, $j' \neq 1, 2$ by Lemma 4.3.2. (Note that we shall use Lemma 4.3.3, when we treat the case (g) instead of the case (f).) On the other hand, we write $E \sim_{\mathbb{Q}} (-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=1}^{r'} M_i$, where M_i is an effective $\mathbb{Q}$-divisor consisting of $M_{i,1}, \dots, M_{i,n(i)}$. Then $(M_i)^2 \leq 0$ by using Lemma 4.3.5 (1), moreover, $(M_1)^2 \leq -2$ and $(M_2)^2 < 0$ by Lemmas 4.3.1 and 4.3.2. Hence, we have $-1 = (E)^2 < 1 + (-2) = -1$, which is absurd.

In (5), by the assumption of E, we have:

$$-1 = (E)^2 = \frac{1}{d} + \sum_{i=1}^{r'} (M_i)^2 \tag{4.3.11}$$

Here, if $M_i \neq 0$, we see $(M_i)^2 \leq -\frac{1}{2}$ by (3) and Lemmas 4.3.1 4.3.2 and 4.3.3 (see also Table 4.2). Furthermore, $(M_1)^2 \leq -\frac{2}{3}$ by virtue of $n(1) > 1$. Hence, we have:

$$\frac{1}{d} + \sum_{i=1}^{r'} (M_i)^2 \leq \frac{1}{d} - \frac{2}{3} - (s-1) \cdot \frac{1}{2} \tag{4.3.12}$$

Two formulas (4.3.11) and (4.3.12) imply $s \leq \frac{2}{d} + \frac{5}{3}$. Since s is an integer, we thus obtain $s \leq 2$ and $s \leq 3$ if $d = 2$ and $d = 1$, respectively. In what follows, we consider the case $d = 1$ and suppose $s = 3$. Then we may assume $M_i \neq 0$ for $i = 1, 2, 3$. Notice that each singularity on $S_{\overline{k}}$ corresponding to $\sum_{j=1}^{n(i)} M_{i,j}$ is of type $A_{n(i)}$ for $i = 1, 2, 3$ by virtue of (1) and (4), moreover, note $n(1) \geq 4$. By looking for the triplet $\{(M_1)^2, (M_2)^2, (M_3)^2\}$ with $(M_1)^2 + (M_2)^2 + (M_3)^2 = -2$ in Table 4.2, the triplet is only $\{-\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}\}$, moreover, $n(1) = 5$ and $\{n(2), n(3)\} = \{2, 1\}$. Hence, we may assume:

$$E \sim_{\mathbb{Q}} (-K_{\widetilde{S}_{\overline{k}}}) - \sum_{j=1}^{5} \frac{6-j}{6} M_{1,j} - \sum_{j=1}^{2} \frac{3-j}{3} M_{2,j} - \frac{1}{2} M_{3,1}.$$

However, this contradicts that $D_1 = 2E$ is a $\mathbb{Z}$-divisor.

In (6), assume that D satisfies the condition (‡) and $E \in \mathbb{Q}[-K_{\widetilde{S}_{\overline{k}}}] \oplus \Big(\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n(i)} \mathbb{Q}[M_{i,j}] \Big)$. Hence, we can write $E \sim_{\mathbb{Q}} \frac{1}{2}(-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=1}^{r} M_i$ by noticing $(E \cdot -K_{\widetilde{S}_{\overline{k}}}) = 1$, where M_i is an effective $\mathbb{Q}$-divisor generated by $M_{i,1}, \ldots, M_{i,n(i)}$. By $(E)^2 = -1$, we then have the formula (4.3.10). We shall look for the combination of the values of $(M_1)^2, \ldots, (M_r)^2$ such that the equality (4.3.10) holds according to each case. As an example, we will explain the case of (a). Note that the equality (4.3.10) implies $(M_1)^2 + (M_2)^2 = -\frac{3}{2}$ by $d = 2$ and $r = 2$. Since $(D \cdot M_{i,j}) = \delta_{j,1} + \delta_{j,n(i)}$, we may assume that $(E_1 \cdot M_{i,j}) = \delta_{j,1}$ for $i = 1, 2$ by virtue of (2) and (3). By using Lemma 4.3.9 and looking at the row of $j_0 = 1$ in Table 4.2, we obtain $\{(M_1)^2, (M_2)^2\} = \{-\frac{5}{6}, -\frac{2}{3}\}$ or $\{-\frac{3}{4}, -\frac{3}{4}\}$. This implies that $\{n(1), n(2)\} = \{5, 2\}$ or $\{3, 3\}$. The other cases are left to the reader because these can be shown by a similar argument above.

In (7), assume that D satisfies the condition (†), in other words, there exists a (-1)-curve E on $\widetilde{S}_{\overline{k}}$ such that $D_1 = 2E$. Then $E \in \mathbb{Q}[-K_{\widetilde{S}_{\overline{k}}}] \oplus \Big(\bigoplus_{i=1}^{2} \bigoplus_{j=1}^{n(i)} \mathbb{Q}[M_{i,j}] \Big)$ by virtue of (3) and (5). In particular, we write $E \sim_{\mathbb{Q}} (-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=1}^{2} M_i$, where M_i is an effective $\mathbb{Q}$-divisor generated by $M_{i,1}, \ldots, M_{i,n(i)}$. Hence, we then have the formula (4.3.10) as $r = 2$. We shall look for the combination of the values of $(M_1)^2$ and $(M_2)^2$ such that the equality (4.3.10) holds and $2E \sim D - D_2$ according to each case. As an example, we will explain the case of (a). Note that the equality (4.3.10) implies $(M_1)^2 + (M_2)^2 = -\frac{3}{2}$ by $d = 2$ and $r = 2$. By Lemma 4.3.9, $\{(M_1)^2, (M_2)^2\} = \{-\frac{5}{6}, -\frac{2}{3}\}$, $\{-\frac{3}{4}, -\frac{3}{4}\}$ or $\{-\frac{1}{2}, -1\}$, so that $\{n(1), n(2)\} = \{5, 2\}$, $\{3, 3\}$ or $\{3, 1\}$. However, we note $\{n(1), n(2)\} \neq \{5, 2\}$. Indeed, otherwise, by Lemma 4.3.1 (see also Table 4.2), we see that $2E$ is a $\mathbb{Q}$-divisor but not a $\mathbb{Z}$-divisor (cf. the proof of (5)). This contradicts that $2E = D_1$ is a $\mathbb{Z}$-divisor. By the similar argument, we also see $\{n(1), n(2)\} \neq \{3, 3\}$. Thus, we obtain $\{n(1), n(2)\} = \{3, 1\}$. Namely, $E \sim_{\mathbb{Q}} \frac{1}{2}(-K_{\widetilde{S}_{\overline{k}}}) - (M_{1,1} + 2M_{1,2} + M_{1,3}) - M_{2,1}$. The other cases are left to the reader because these can be shown by a similar argument above.

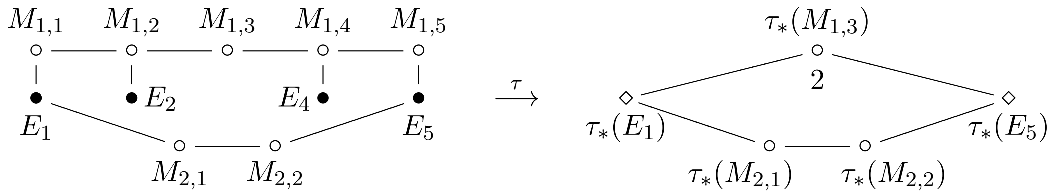
In (8), this proof can be shown by an argument similar to (7) and is left to the reader. □

Now, we shall present the following example about the application of Proposition 4.3.10:

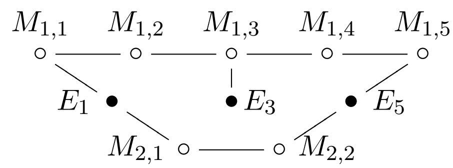
Example 4.3.11. With the notation as above, assume further that $d = 2$ and $\widetilde{S}$ is of $A_5 + A_2$-type. Let $M_{1,1}, \ldots, M_{1,5}$, $M_{2,1}$ and $M_{2,2}$ be all (-2)-curves on $\widetilde{S}_{\overline{k}}$ with the configuration as in (4.3.7). Then we shall consider two divisors $D^{(1,5)}$ and $D^{(3)}$ on $\widetilde{S}_{\overline{k}}$ given by:

$$
\begin{aligned}
D^{(1,5)} &:= -K_{\widetilde{S}_{\overline{k}}} - \sum_{j=1}^{5} M_{1,j} - \sum_{j=1}^{2} M_{2,j}, \\
D^{(3)} &:= -K_{\widetilde{S}_{\overline{k}}} - M_{1,1} - 2(M_{1,2} + M_{1,3} + M_{1,4}) - M_{1,5}.
\end{aligned}
$$

Notice that $D^{(1,5)}$ and $D^{(3)}$ are divisors as in (a) and (b) in Table 4.3, respectively. Hence, since $D^{(1,5)}$ satisfies the condition (‡) by Proposition 4.3.10 (1),(2) and (7), there exist two (-1)-curves E_1 and E_5 on $\widetilde{S}_{\overline{k}}$ such that $D^{(1,5)} \sim E_1 + E_5$. Moreover, $D^{(3)}$ satisfies the condition either (†) or (‡). However, $D^{(3)}$ does not satisfy the condition (‡). Indeed, otherwise, since there exist two (-1)-curves E_2 and E_3 on $\widetilde{S}_{\overline{k}}$ such that $D^{(3)} \sim E_2 + E_4$. Hence, we obtain the compositions $\tau : \widetilde{S}_{\overline{k}} \to V$ of successive contractions of $E_2 + E_4$, that of the images of $M_{1,2} + M_{1,4}$ and finally that of the images of $M_{1,1} + M_{1,5}$ over $\overline{k}$, so that the weighted dual graphs of $\sum_{j=1}^{5} M_{1,j} + \sum_{j'=1}^{2} M_{2,j'} + E_1 + E_2 + E_4 + E_5$ and its image via τ are as follows:



Then $(-K_V)^2 = 8$ and V contains two (-2)-curves $\tau_*(M_{2,1})$ and $\tau_*(M_{2,2})$. This is a contradiction. Thus, $D^{(3)}$ satisfies the condition (†). In other words, there exists a (-1)-curve E_3 on $\widetilde{S}_{\overline{k}}$ such that $D_1^{(3)} = 2E_3$. In particular, we know $(E_3 \cdot M_{i,j}) = \delta_{1,i}\delta_{3,j}$. Since $E_1 + E_5$ and E_3 are defined over k, we see that $\widetilde{S}_{\overline{k}}$ contains a union defined over k corresponding to the following weighted dual graph:



4.3.3 Proof of Proposition 4.3.10 (3)

In this subsection, we shall prove Proposition 4.3.10 (3). With the notation as in Proposition 4.3.10 (3), notice that E is a (-1)-curve on $\widetilde{S}_{\overline{k}}$ by Proposition 4.3.10 (1). Since $D_1 \sim D - D_2$, we can write $E \sim_{\mathbb{Q}} \frac{1}{d}(-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=1}^{r'} M_i$, where each M_i is an effective $\mathbb{Q}$-divisor generated by $M_{i,1}, \ldots, M_{i,n(i)}$. In particular, we note $M_i \neq 0$ for every $i = 1, \ldots, r$.

Lemma 4.3.12. Let $D^{(1)}$ and $D^{(2)}$ be two $\mathbb{Q}$-divisors on $\widetilde{S}_{\overline{k}}$ generated by $M_{i,1}, \ldots, M_{i,n(i)}$. If $(D^{(1)} \cdot M_{i,j}) = (D^{(2)} \cdot M_{i,j})$ for any $j = 1, \ldots, n(i)$, then $D^{(1)} = D^{(2)}$.

Proof. It is enough to show when we assume $D^{(2)} = 0$. We shall write $D^{(1)} = \sum_{j=1}^{n(i)} b_{i,j} M_{i,j}$ for some $b_{i,j} \in \mathbb{Q}$. By assumption, we have the following linear simultaneous equation:

$$\begin{bmatrix} (D^{(1)} \cdot M_{i,1}) \\ \vdots \\ (D^{(1)} \cdot M_{i,n(i)}) \end{bmatrix} = A \begin{bmatrix} b_{i,1} \\ \vdots \\ b_{i,n(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

where A is the intersection matrix with respect to $M_{i,1}, \ldots, M_{i,n(i)}$, i.e., $A = ((M_{i,j} \cdot M_{i,j'}))_{1 \le j,\, j' \le n(i)}$. It is well known that intersection matrix is negative definite ([58]), so that $b_{i,j} = 0$ for any $j = 1, \ldots, n(i)$. Namely, we obtain $D^{(1)} = 0$. □

Lemma 4.3.13. With the notation as above, $\sum_{j=1}^{n(i)} (E \cdot M_{i,j}) \le 1$ for $i = 1, \ldots, r$.

Proof. Let $\Delta_{i,j}$ be the $\mathbb{Q}$-divisor generated by $M_{i,1}, \ldots, M_{i,n(i)}$ such that $(\Delta_{i,j} \cdot M_{i,j'}) = \delta_{j,j'}$ for $j, j' = 1, \ldots, n(i)$ on $\widetilde{S}_{\overline{k}}$. Note that such a $\mathbb{Q}$-divisor $\Delta_{i,j}$ is uniquely exists by using Lemmas 4.3.1, 4.3.2 and 4.3.3. In particular, any coefficient of $\Delta_{i,j}$ is less than or equal to $-\frac{1}{2}$. Hence, we have $(\Delta_{i,j} \cdot \Delta_{i,j'}) \le -\frac{1}{2}$ for any $j, j' = 1, \ldots, n(i)$, where the equal sign holds if and only if $n(i) = 1$. On the other hand, by Lemma 4.3.12, we obtain:

$$M_i = (M_i \cdot M_{i,j})\Delta_{i,j} = (E \cdot M_{i,j})\Delta_{i,j}$$

for any j by virtue of $(E - M_i \cdot M_{i,j}) = 0$. Meanwhile, by using Lemma 4.3.5 (1), we note $(M_i)^2 < 0$ if $M_i \neq 0$. Suppose that $\sum_{j=1}^{n(i)}(E \cdot M_{i,j}) \geq 2$. Notice $(E \cdot M_{i,j}) \geq 0$ for any j. If there exists j_0 such that $(E \cdot M_{i,j_0}) \geq 2$, then we have:

$$-1 = (E)^2 \leq \frac{1}{d} + (E \cdot M_{i,j_0})^2(\Delta_{i,j_0})^2 \leq 1 - 2 = -1,$$

furthermore, we see $(E)^2 < -1$ by virtue of $n(i) \geq 2$ or both $n(i) = 1$ and $i > 1$. This is absurd. Otherwise, by hypothesis there exist two integers j_1 and j_2 such that $(E \cdot M_{i,j_1}) = (E \cdot M_{i,j_2}) = 1$. By virtue of $n(i) \geq 2$, we have:

$$-1 = (E)^2 \leq \frac{1}{d} + (\Delta_{i,j_1})^2 + (\Delta_{i,j_2})^2 + 2(\Delta_{i,j_1} \cdot \Delta_{i,j_2}) < 1 - \frac{1}{2} - \frac{1}{2} - 1 = -1,$$

which is absurd. □

Lemma 4.3.14. With the notation as above, assume further that D satisfies the condition (†). For $i = 1, \dots, r'$, then $\sum_{j=1}^{n(i)}(E \cdot M_{i,j}) \geq 1$ if $M_i \neq 0$.

Proof. Suppose $\sum_{j=1}^{n(i)}(E \cdot M_{i,j}) = 0$ for some $i \in \{1, \dots, r'\}$. Then we note $(E \cdot M_{i,j}) = 0$ for any $j = 1, \dots, n(i)$. Hence, we obtain $M_i = 0$ by Lemma 4.3.12. □

Proposition 4.3.10 (3) can be shown by using Lemmas 4.3.13 and 4.3.14 as follows:

Proof of Proposition 4.3.10 (3). The first assertion of Proposition 4.3.10 (3) follows immediately from the beginning of Subsection 4.3.3. Hence, we shall prove the second assertion of this in what follows. In this proof, we will consider two cases separately:

In the case that D satisfies the condition (†). In other words, we can write $D_1 = 2E$. Hence, we obtain $\sum_{j=1}^{n(i)}(E \cdot M_{i,j}) = 1$ for $i = 1, \dots, r$ by Lemmas 4.3.13 and 4.3.14.

In the case that D satisfies the condition (‡). In other words, there exists a (-1)-curve E' on $\widetilde{S}_{\overline{k}}$ such that $D_1 = E + E'$ and $E \neq E'$. Notice that $\sum_{j=1}^{n(i)}(E' \cdot M_{i,j}) \leq 1$ by Lemma 4.3.13. On the other hand, we see $D \sim E + E'$ by Proposition 4.3.10 (2) and $\sum_{j=1}^{n(i)}(D \cdot M_{i,j}) = 2$. Hence, we see $\sum_{j=1}^{n(i)}(E \cdot M_{i,j}) = \sum_{j=1}^{n(i)}(E' \cdot M_{i,j}) = 1$. □

4.4 Proof of Theorem 1.3.9 (3)

Let the notation be the same as at beginning of Chapter 4 and assume further that S is of rank one and $d \leq 2$. In this section, we shall show Theorem 1.3.9 (3).

4.4.1 Base locus with respect to cylinder

In this subsection, assuming that S contains a cylinder, say $U \simeq \mathbb{A}^1_k \times Z$, where Z is a smooth affine curve defined over k, the closures in S of fibers of the projection $pr_Z : U \simeq \mathbb{A}^1_k \times Z \to Z$ yields a linear system, say $\mathscr{L}$, on S. By Lemma 2.5.6 we see that $\mathrm{Bs}(\mathscr{L})$ consists of exactly one k-rational point, say p, which is a singular point on $S_{\overline{k}}$. Hence, $\mathscr{L}$ is especially a linear pencil on S. On the other hand, $\widetilde{U} := \sigma^{-1}(U) \simeq U$ is a cylinder on $\widetilde{S}$ since $U_{\overline{k}}$ is smooth. The closures in $\widetilde{S}$ of fibers of the projection $pr_Z : \widetilde{U} \simeq \mathbb{A}^1_k \times Z \to Z$ yields a linear system, say $\widetilde{\mathscr{L}}$, on $\widetilde{S}$. The purpose of this subsection is to show the following proposition:

Proposition 4.4.1. With the notation and the assumptions as above, assume further that one of the following conditions holds:

(1) $d = 2$ and p is of type A_n on $S_{\overline{k}}$ but not type A_n^- over k for some $n = 1, \dots, 6$.

(2) $d = 1$ and p is of type A_n on $S_{\overline{k}}$ but not type A_n^- over k for some $n = 1, \dots, 8$.

(3) $d = 1$ and p is of type D_5^+ on $S_{\overline{k}}$.

(4) $d = 1$ and p is of type E_6^+ on $S_{\overline{k}}$.

Then $\mathrm{Bs}(\widetilde{\mathscr{L}})$ consists of only one k-rational point. In particular, the singular point $p \in S$ is not of type A_n^{++} over k except for only one case $(d, n) = (2, 7)$.

In what follows, we shall prove Proposition 4.4.1. Let $M_1, \dots, M_n$ be all irreducible components of the exceptional divisor of $\sigma_{\overline{k}}$ at p such that the dual graph of $M_1, \dots, M_n$ is that as in (4.3.1), (4.3.2) or (4.3.3) according to the singularity type of p on $S_{\overline{k}}$. Now, the following two lemmas hold:

Lemma 4.4.2. With the notation and the assumptions as above, assume further that $p \in S_{\overline{k}}$ is of type A_n. Then we obtain the following assertion:

(1) If $d = 2$, then there exists a curve C on $\widetilde{S}_{\overline{k}}$ such that $C \sim (-K_{\widetilde{S}_{\overline{k}}}) - (M_1 + \cdots + M_n)$. Hence, $M_1 + \cdots + M_n + C$ is a cycle.

(2) If $d = 1$ and $n \geq 3$, then there exists a curve C on $\widetilde{S}$ such that $C \sim 2(-K_{\widetilde{S}_{\overline{k}}}) - 2(M_1 + \cdots + M_n) + (M_1 + M_n)$. Hence, $M_2 + \cdots + M_{n-1} + C$ is a cycle.

Proof. In (1), we take the divisor $D := (-K_{\widetilde{S}_{\overline{k}}}) - (M_1 + \cdots + M_n)$ on $\widetilde{S}_{\overline{k}}$. By construction, we have $(D)^2 = 0$ and $(D \cdot -K_{\widetilde{S}_{\overline{k}}}) = 2$. Hence, we see $\dim |D| \geq 1$ by the Riemann-Roch theorem. Thus, there exists a curve C on $\widetilde{S}_{\overline{k}}$ such that $C \sim D$. Namely, $(C \cdot M_j) = (D \cdot M_j) = \delta_{j,1} + \delta_{j,n}$. This completes the proof of (1).

In (2), it can be shown by the argument similar to (1). □

Lemma 4.4.3. With the notation and the assumptions as above, assume further that $p \in S_{\overline{k}}$ is of type D_5. Then there exists a curve C on $\widetilde{S}_{\overline{k}}$ such that $C \sim 2(-K_{\widetilde{S}_{\overline{k}}}) - (2M_1 + 2M_2 + 3M_3 + 2M_4 + M_5)$. Hence, $M_1 + M_2 + M_3 + C$ is a cycle.

Proof. This lemma can be shown by the argument similar to Lemma 4.4.2. □

Proof of Proposition 4.4.1. Let $\widetilde{L}$ be a general member of $\widetilde{\mathscr{L}}$. Since $\mathrm{Bs}(\mathscr{L}) = \{p\}$, we see that $\widetilde{L}$ meets M_i for some $1 \leq i \leq n$. By construction of $\widetilde{\mathscr{L}}$, if $\widetilde{L}$ meets two distinct irreducible components M_i and M_j, then $\mathrm{Bs}(\widetilde{\mathscr{L}}) = M_i \cap M_j \neq \emptyset$. In what follows, we thus assume that $\widetilde{L}$ meets exactly one irreducible component, say M_{i_0}. Notice that M_{i_0} is defined over k. Let a and b be two positive rational numbers such that $da = (\widetilde{\mathscr{L}} \cdot -K_{\widetilde{S}})$ and $2b = (\widetilde{\mathscr{L}} \cdot M_{i_0})$.

Now, we notice that $\mathrm{Bs}(\widetilde{\mathscr{L}}) \neq \emptyset$ provided that $(\widetilde{\mathscr{L}})^2 \neq 0$. Hence, we shall show $(\widetilde{\mathscr{L}})^2 \neq 0$ according to the cases (1)–(4) in Proposition 4.4.1 in what follows:

In (1) or (2), by the configuration of a dual graph of $M_1 + \cdots + M_n$, we see that n is odd and i_0 is equal to m, where $m := \lceil \frac{n}{2} \rceil$ for simplicity. In particular, M_m corresponds to the central vertex in this graph. Thus, by Lemma 4.3.1, we have $\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}) - bM$, where $M = \sum_{j=1}^{m-1} j(M_j + M_{n-j+1}) + mM_m$. Moreover, we obtain $(\widetilde{\mathscr{L}})^2 = da^2 - 2mb^2$. Suppose that $(\widetilde{\mathscr{L}})^2 = 0$. Note that $m \leq 4$ because of $n \leq 8$. Hence, we obtain $(d, m) = (1, 2)$ or $(2, 1)$

since a, b are rational numbers. In particular, we have $a = (3-d)b$. However, the curve C on $\widetilde{S}_{\overline{k}}$, which is that as in Lemma 4.4.2, then satisfies $(\widetilde{\mathscr{L}} \cdot C) = 0$. This implies that C is included in the boundary of $\widetilde{U}_{\overline{k}}$. Moreover, so is M_i for $i = 1, \ldots, n$. Hence, the boundary of $\widetilde{U}_{\overline{k}}$ includes a cycle $M_1 + C$ (resp. $M_2 + C$) if $d = 2$ (resp. $d = 1$). However, this contradicts Lemma 2.5.3. Therefore, we see $(\widetilde{\mathscr{L}})^2 \neq 0$.

In (3), since $p \in S$ is a singular point of type D_5^+ over k, note that M_1 and M_2 lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, on the other hand, M_3, M_4 and M_5 are defined over k, respectively. Hence, i_0 is equal to 3, 4 or 5. Thus, by Lemma 4.3.2, we have $\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}) - bM$ and $(\widetilde{\mathscr{L}})^2 = a^2 + (M)^2 b^2$, where M is that as in Lemma 4.3.2 (3), (4) or (5) according to the number of i_0. In particular, $(M)^2 = -3$, -2 and -1 if $i_0 = 3$, 4 and 5, respectively. Suppose that $(\widetilde{\mathscr{L}})^2 = 0$. Then $(M)^2 = -1$ since a, b are rational numbers. Hence, we see $i_0 = 5$ and $a = b$. However, the curve C on $\widetilde{S}$, which is that as in Lemma 4.4.3(1), then satisfies $(\widetilde{\mathscr{L}} \cdot C) = 0$. This implies that the boundary of $\widetilde{U}_{\overline{k}}$ includes a cycle $M_1 + M_2 + C$, which contradicts Lemma 2.5.3. Therefore, we see $(\widetilde{\mathscr{L}})^2 \neq 0$.

In (4), since $p \in S$ is a singular point of type E_6^+ over k, note that M_1 and M_2 (resp. M_3 and M_4) lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, on the other hand, M_5 and M_6 are defined over k, respectively. Hence, i_0 is equal to 5 or 6. Thus, by Lemma 4.3.3, we have $\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}) - bM$ and $(\widetilde{\mathscr{L}})^2 = a^2 + (M)^2 b^2$, where M is that as in Lemma 4.3.3 (5) or (6) according to the number of i_0. In particular, $(M)^2 = -6$ and -2 if $i_0 = 5$ and 6, respectively. Thus, we see $(\widetilde{\mathscr{L}})^2 \neq 0$ since a, b are rational numbers. □

4.4.2 Proof of Theorem 1.3.9 (3)(i)–(iii)

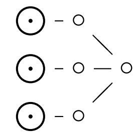
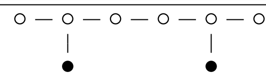
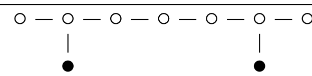
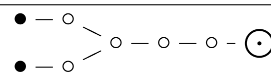
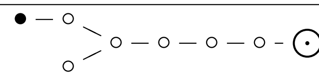
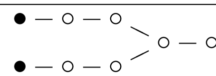
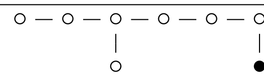
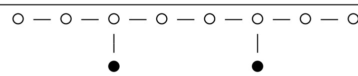
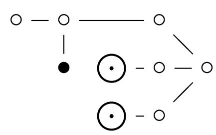
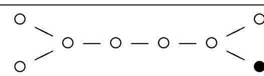
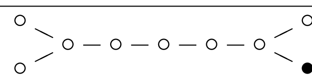
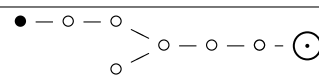
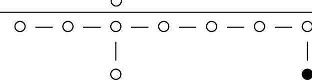
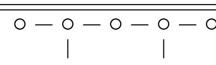
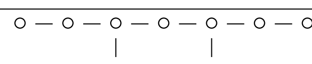
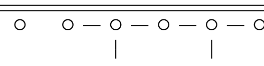
In this subsection, we shall show the assertions (i), (ii) and (iii) in Theorem 1.3.9 (3). In order to prove Theorem 1.3.9 (3)(i) and (ii), we will use Table 4.4. In fact, in this proof, we mainly consider the two morphisms over k. One is the minimal resolution $\sigma : \widetilde{S} \to S$ over k and the other is the contraction $\tau : \widetilde{S} \to W_{d'}$ over k of the union of some (-1)-curves, which can be determined by the weighted dual graph in Table 4.4 according to the type of $\widetilde{S}$ (the detailed configuration of τ will be treated in the following Lemmas 4.4.4, 4.4.5 and 4.4.7). By construction of τ, we will know that $W_{d'}$ contains a cylinder such that the boundary of this pullback via τ includes the union of all (-2)-curves, which is clearly defined over k. Thus, this image via σ is a cylinder in S, namely, we see that S certainly contains a cylinder.

Now, we shall state the notation in Table 4.4. Letting $\tau : \widetilde{S} \to W_{d'}$ be the morphism as above depending on the type of $\widetilde{S}$, we then see that $W_{d'}$ is a weak del Pezzo surface. Then “d'” and “Type of $W_{d'}$” in Table 4.4 mean the degree and the type of $W_{d'}$ according to the type of $\widetilde{S}$, respectively. On the other hand, “$\rho_k(\widetilde{S})$” in Table 4.4 means the Picard number of $\widetilde{S}$ according to the type of $\widetilde{S}$. Notice that this can be obtained by the Picard number of $W_{d'}$, which is explicitly given (see Table 4.1), and the construction of τ. Moreover, “Dual graph” in Table 4.4 means a weighted dual graph on $\widetilde{S}_{\overline{k}}$ according to the type of $\widetilde{S}$, where “$-\odot$” means either “$—\bullet-\circ$” or “$<^{\bullet}_{\bullet}$”, which can be determined according to the type of $\widetilde{S}$. Note that the union of curves on $\widetilde{S}_{\overline{k}}$ corresponding to all vertices on this graph is certainly defined over k by the configuration of $W_{d'}$ (see Table 4.1) and the construction of τ.

Proof of Theorem 1.3.9 (3)(i)

We consider the following two lemmas separately:

Table 4.4: Types of $\widetilde{S}$ in Theorem 1.3.9 (3) (i) and (ii)

d	Type of $\widetilde{S}$	$\rho_k(\widetilde{S})$	Dual graph	d'	Type of $W_{d'}$
2	D_4	3, 4 or 5		8	A_1
2	D_4+A_1	5 or 6			
2	D_4+2A_1	5 or 7			
2	D_4+3A_1	4, 6 or 8			
2	A_6	4		4	A_2+2A_1
2	A_7	5 or 8		4	A_3+2A_1
2	D_5	5		4	$(A_3)_<$
2	D_5+A_1	6		4	A_3+A_1
2	D_6	7		3	A_5
2	D_6+A_1	8		3	A_5+A_1
2	E_6	5		4	D_4
2	E_7	8		3	E_6
1	A_8	5 or 9		3	$3A_2$
1	D_6	6 or 7		2	D_4+A_1
1	D_6+A_1	8		2	D_4+2A_1
1	D_6+2A_1	7 or 9		2	D_4+3A_1
1	D_7	7		2	D_5+A_1
1	D_8	9		2	D_6+A_1
1	E_7	8		2	D_6
1	E_7+A_1	9		2	D_6+A_1
1	E_8	9		2	E_7
2	$(A_5)''$	4		4	$3A_1$
1	$(A_7)''$	5		3	$2A_2+A_1$
2	$(A_5+A_1)''$	5		4	$4A_1$

Lemma 4.4.4. Let the notation and the assumptions be the same as above. If $d = 2$ and $S_{\overline{k}}$ has a singular point of type D_4, then S contains a cylinder.

Proof. Let x be a singular point of type D_4 on $S_{\overline{k}}$. Note that x is k-rational on $S_{\overline{k}}$ by Lemma 2.1.5. Moreover, we see that $\widetilde{S}$ is of $D_4 + nA_1$-type for $n = 0, 1, 2, 3$ and $\widetilde{S}(k) \neq \emptyset$ by the configuration of $\widetilde{S}_{\overline{k}}$ (see Table 4.4). Let $\widetilde{E}$ be the union of reduced curves corresponding to three subgraphs $\odot$ — $\circ$ in the weighted dual graph in Table 4.4. Then we obtain the birational morphism $\tau : \widetilde{S} \to W_8$ over k such that W_8 is a k-form of the Hirzebruch surface $\mathbb{F}_2$ of degree 2 and the direct image $\tau_{\overline{k},*}(\widetilde{E})$ is the disjoint union of three closed fibers, say F_1, F_2 and F_3, of the $\mathbb{P}^1$-bundle $W_{8,\overline{k}} \simeq \mathbb{F}_2 \to \mathbb{P}^1_{\overline{k}}$. In particular, we see $W_8 \simeq \mathbb{F}_2$ because of $\widetilde{S}(k) \neq \emptyset$ by using Proposition 2.2.2. Hence, $\widetilde{U} := \widetilde{S} \backslash \widetilde{E}$ is certainly the cylinder on $\widetilde{S}$ since $\widetilde{U} \simeq \mathbb{F}_2 \backslash (M \cup F_1 \cup F_2 \cup F_3) \simeq \mathbb{A}^1_k \times C_{(2)}$, where M is the (-2)-curve on $\mathbb{F}_2$. Therefore, we see that S contains a cylinder $\sigma(\widetilde{U}) \simeq \widetilde{U}$. □

Lemma 4.4.5. Let the notation and the assumptions be the same as above. If $d = 2$ (resp. $d = 1$) and $S_{\overline{k}}$ has a singular point of type A_6, A_7, D_5, D_6, E_6 or E_7 (resp. type A_8, D_6, D_7, D_8, E_7 or E_8), then S contains a cylinder.

Proof. Let x be a singular point of the type of the one of the above list on $S_{\overline{k}}$. Note that x is k-rational on $S_{\overline{k}}$ by Lemma 2.1.5. Let $\widetilde{E}$ be the union of the (-1)-curves corresponding to all vertices • in the Table 4.4 according to the type of $\widetilde{S}$. Notice that $\widetilde{E}$ is defined over k and $\widetilde{E}_{\overline{k}}$ is either irreducible or disjoint. Hence, we obtain the contraction $\tau : \widetilde{S} \to W_{d'}$ of $\widetilde{E}$ defined over k, so that $W_{d'}$ is a weak del Pezzo surface of degree $d' \in \{2, 3, 4\}$, where d' is determined according to the type of $\widetilde{S}$. If $d' \in \{3, 4\}$, then $W_{d'}$ contains a cylinder such that this boundary includes $\tau(\widetilde{E})$ by the argument in Subsection 4.2.2 (see also Table 4.1). Thus, this pullback, say $\widetilde{U}$, via τ is a cylinder in $\widetilde{S}$ such that this boundary includes the union of all (-2)-curves on $\widetilde{S}_{\overline{k}}$, which is defined over k. Therefore, we see that S contains a cylinder $\sigma(\widetilde{U}) \simeq \widetilde{U}$. If $d' = 2$, then $W_{d'}$ is one of the lists in Table 4.4 and contains a cylinder such that this boundary includes $\tau(\widetilde{E})$ by the above argument. Namely, the above argument can work as well even if $d' = 2$. This completes the proof. □

Remark 4.4.6. We shall state some remarks on Lemma 4.4.5. Let x be the same as in Lemma 4.4.5 and assume $d = 2$. Then:

(1) If x is of type A_6, E_6 or E_7 on $S_{\overline{k}}$, then S always contains the affine plane $\mathbb{A}^2_k$ (compare the fact that the Du Val del Pezzo surface of rank one and of degree 2 over $\mathbb{C}$ contains $\mathbb{C}^2$ if and only if this surface has a singular point of type E_7, see [56]).

(2) If x is of type A_7 on $S_{\overline{k}}$, then $\widetilde{S}$ does not have to admit a k-rational point but always contains a cylinder (compare the fact in Theorem 1.2.4).

Theorem 1.3.9 (3)(i) follows from Lemmas 4.4.4 and 4.4.5.

Proof of Theorem 1.3.9 (3)(ii)

Assume that $\widetilde{S}_{\overline{k}}$ has a singular point x of type $(A_{9-2d})''$ (see Definition 4.1.3, for the definition). Note that x is k-rational on $S_{\overline{k}}$ by Lemma 2.1.5. Notice that $\widetilde{S}$ is only of $(A_5)''$ or $(A_5 + A_1)''$-type (resp. $(A_7)''$-type) if $d = 2$ (resp. $d = 1$). We consider the following two lemmas separately:

Lemma 4.4.7. With the notation and the assumptions as above, assume further that $\widetilde{S}$ is of $(A_{9-2d})''$-type. Then S contains a cylinder if and only if $x \in S$ not of type A_{9-2d}^{++} over k.

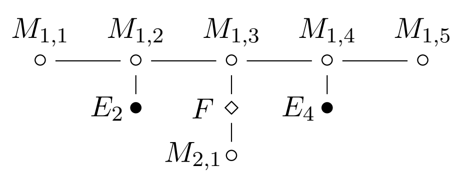
Proof. Assume that S contains a cylinder $U \simeq \mathbb{A}_k^1 \times Z$, where Z is a smooth affine curve defined over k. Then $\widetilde{S}$ contains a cylinder $\widetilde{U} := \sigma^{-1}(U) \simeq U$. The closures in $\widetilde{S}$ of fibers of the projection $pr_Z : \widetilde{U} \simeq \mathbb{A}_k^1 \times Z \to Z$ yields a linear system, say $\widetilde{\mathscr{L}}$, on $\widetilde{S}$. By Proposition 4.4.1, we see $\mathrm{Bs}(\widetilde{\mathscr{L}}) \neq \emptyset$. Thus, $x \in S$ is not of type A_{9-2d}^{++} over k by the assumption and Lemma 2.5.6.

Conversely, assume that $x \in S$ is not of type A_{9-2d}^{++} over k. Let M be the (-2)-curve on $\widetilde{S}_{\overline{k}}$ corresponding to the central vertex on the dual graph with the minimal resolution at x. Notice that M is defined over k, moreover, M has a k-rational point by the assumption. Let $\widetilde{E}$ be the union of the (-1)-curves corresponding to two vertices $\bullet$ in the Table 4.4 according to the type of $\widetilde{S}$. Notice that $\widetilde{E}$ is defined over k and $\widetilde{E}_{\overline{k}}$ is disjoint. Hence, we obtain the contraction $\tau : \widetilde{S} \to W_{d+2}$ of $\widetilde{E}$ defined over k, so that W_{d+2} is a weak del Pezzo surface of degree $d+2$ and $\tau_{\overline{k}}(M)$ is a (-2)-curve. Moreover, since M has a k-rational point, so does the image via τ. Hence, $W_{d+2,\overline{k}}$ contains a (-2)-curve with a k-rational point. This implies that W_{d+2} contains a cylinder by using Theorem 1.3.9 (2) such that this boundary includes $\tau(\widetilde{E})$ (see also Table 4.1). Thus, this pullback, say $\widetilde{U}$, via τ is a cylinder in $\widetilde{S}$ such that this boundary includes the union of all (-2)-curves on $\widetilde{S}_{\overline{k}}$, which is defined over k. Therefore, we see that S contains a cylinder $\sigma(\widetilde{U}) \simeq \widetilde{U}$. □

Lemma 4.4.8. With the notation and the assumptions as above, assume further that $d = 2$ and $\widetilde{S}$ is of $(A_5 + A_1)''$-type. Then S contains a cylinder if and only if $x \in S$ not of type A_5^{++} over k.

Proof. Let $M_{1,1}, \ldots, M_{1,5}$ and $M_{2,1}$ be the (-2)-curves on $\widetilde{S}_{\overline{k}}$ with the configuration as in (4.3.7). By the configuration, $M_{1,3}$ and $M_{2,1}$ are defined over k. By using Proposition 4.3.10, there exist two (-1)-curves E_2 and E_4 on $\widetilde{S}_{\overline{k}}$ such that $(E_i \cdot M_{1,j}) = \delta_{i,j}$ and $(E_i \cdot M_{2,1}) = 0$ for $i = 2, 4$ and $j = 1, \ldots, 5$, moreover, the union $E_2 + E_4$ is defined over k (cf. Example 4.3.11). Let $\tau : \widetilde{S} \to W_8$ be the compositions of successive contractions of a disjoint union $E_2 + E_4$, that of the images of $M_{1,2} + M_{1,4}$ and finally that of the images of $M_{1,1} + M_{1,5}$. By construction, τ is defined over k and W_8 is a k-form of the Hirzebruch surface $\mathbb{F}_2$ of degree 2.

From now on, we prove this lemma. Assume that S contains a cylinder. Let $y \in S_{\overline{k}}$ be the singular point of type A_1 over $\overline{k}$. Then we know that either x is not of type A_5^{++} over k or y is not of type A_1^{++} over k by the similar argument to Lemma 4.4.7. In what follows, we may assume that $y \in S$ is not of type A_1^{++} over k. In other words, $M_{2,1}$ has a k-rational point, hence, so does $\tau_{\overline{k}}(M_{2,1})$. Namely, $W_8 \simeq \mathbb{F}_2$. Hence, there exists uniquely closed fiber of the $\mathbb{P}^1$-bundle $\mathbb{F}_2 \to \mathbb{P}_k^1$ passing through this k-rational point. Let F be the pullback of this fiber by τ. Note that the configuration of the weighted dual graph of $\sum_{j=1}^5 M_{1,j} + M_{2,1} + E_2 + E_4 + F$ is as follows:



In particular, the intersection point of $M_{1,3}$ and F is k-rational, namely, $M_{1,3}(k) \neq \emptyset$. This implies that $x \in S$ is not of type A_5^{++} over k.

Conversely, assume that x is not of type A_5^{++} over k. By putting $M := M_{1,3}$, the proof of this assertion is similar as Lemma 4.4.7 and left to the reader. $\square$

Theorem 1.3.9 (3)(ii) follows from Lemmas 4.4.7 and 4.4.8.

Proof of Theorem 1.3.9 (3)(iii)

The assertion (iii) in Theorem 1.3.9 (3) can be shown by an argument similar to [19, Remark 10] as follows:

Proof of Theorem 1.3.9 (3)(iii). Assume that either $d = 2$ and $S_{\overline{k}}$ allows only singular points of type A_1 or $d = 1$ and $S_{\overline{k}}$ allows only singular points of types A_1, A_2, A_3, D_4. Suppose that S contains a cylinder U. Then we see that $S_{\overline{k}}$ admits an $(-K_{S_{\overline{k}}})$-polar cylinder $U_{\overline{k}}$ because of $\rho_k(S) = 1$ (see Definition 1.1.1, for the definition), which contradicts Theorem 1.1.4. $\square$

As a corollary, we have the following:

Corollary 4.4.9. With the notation and the assumptions as above, assume further that $d = 1$ and $S_{\overline{k}}$ has a singular point of type D_4. Then S does not contain any cylinder.

Proof. Assume that $S_{\overline{k}}$ has a singular point of type D_4. Then $\widetilde{S}$ is only of $2D_4$, $D_4 + A_3$, $D_4 + 3A_1$, $D_4 + A_2$, $D_4 + 2A_1$, $D_4 + A_1$ or D_4-type. Therefore, we see that S does not contain any cylinder by Theorem 1.3.9 (3)(iii). $\square$

4.4.3 Proof for the "only if" part of Theorem 1.3.9 (3)(iv)

In this subsection, we shall show the "only if" part of Theorem 1.3.9 (3)(iv). Assume that S does not satisfy any condition on singularities of (i), (ii) or (iii) in Theorem 1.3.9 (3) and contains a cylinder, say $U \simeq \mathbb{A}^1_k \times Z$. The closures in S of fibers of the projection $pr_Z : U \simeq \mathbb{A}^1_k \times Z \to Z$ yields a linear system, say $\mathscr{L}$, on S. By Lemma 2.5.6 we then see that $\mathrm{Bs}(\mathscr{L}) = \{p\}$ such that p is a singular point on $S_{\overline{k}}$ defined over k. In order to show the "only if" part of Theorem 1.3.9 (3)(iv), we shall prove that p is of type A_n^-, D_n^- or E_n^-. Letting $\widetilde{U}$ be the cylinder in $\widetilde{S}$ defined by $\widetilde{U} := \sigma^{-1}(U) \simeq U$, the closures in $\widetilde{S}$ of fibers of the projection $pr_Z : \widetilde{U} \simeq \mathbb{A}^1_k \times Z \to Z$ yields a linear system, say $\widetilde{\mathscr{L}}$, on $\widetilde{S}$. By Proposition 4.4.1 we then see that $\mathrm{Bs}(\widetilde{\mathscr{L}}) = \{\widetilde{p}\}$ such that $\widetilde{p}$ is k-rational. In other words, the singular point $p \in S$ is not of type A_n^{++} over k for any n. In what follows, suppose that the singularity type of $p \in S$ over k is one of the following according to the degree d:

- $d = 2$: type A_1^+, A_2^+, A_3^+, A_4^+ or $(A_5^+)'$;
- $d = 1$: type A_1^+, A_2^+, A_3^+, A_4^+, A_5^+, A_6^+, $(A_7^+)'$, D_5^+ or E_6^+.

Meanwhile, we will prove Lemmas 4.4.12, 4.4.14 and 4.4.15, which contradict the above hypothesis. Now, we shall treat the following Lemmas 4.4.10 and 4.4.11, which will play a crucial role to show Lemmas 4.4.12 and 4.4.14:

Lemma 4.4.10. With the notation and the assumptions as above, assume further that $\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}) - bM$ for some positive rational numbers a and b, where M is an effective $\mathbb{Q}$-divisor on $\widetilde{S}$ and consists of some irreducible components of exceptional set of σ. Let α, β and γ be three positive rational numbers satisfying $a \geq \alpha b$, $\beta = -(M \cdot M_0)$ and $\gamma = -(M)^2$, where M_0 is an irreducible component of $M_{\overline{k}}$ passing through $\widetilde{p}$. Then the following hold:

(1) If $d=2$, then the following four inequalities do not hold simultaneously:

$$\begin{cases} \alpha - u > 0 \\ \alpha - u - v \geq 0 \\ 2\alpha u + \beta v - \gamma \geq 0 \\ 4u^2 + 4uv + 2v^2 - \gamma \leq 0 \end{cases} \tag{4.4.1}$$

for any rational numbers u, v with $u \geq 0$.

(2) If $d=1$, then the following four inequalities do not hold simultaneously:

$$\begin{cases} \alpha - u > 0 \\ \alpha - u - v \geq 0 \\ \alpha u + \beta v - \gamma \geq 0 \\ 4u^2 + 4uv + 4v^2 - 3\gamma \leq 0 \end{cases} \tag{4.4.2}$$

for any rational numbers u, v with $u \geq 0$.

Proof. We only show (1), because (2) can be shown by the argument similar to (1).

Suppose that there exist $u \in \mathbb{Q}_{\geq 0}$ and $v \in \mathbb{Q}$ such that the all inequalities (4.4.1) hold simultaneously. By virtue of $\alpha - u > 0$, $\alpha - u - v \geq 0$, $b > 0$ and $a \geq \alpha b$, we then see $a - ub > 0$ and $1 - \frac{vb}{a-ub} \geq 0$. Hence, we have:

$$\begin{aligned} \left(\widetilde{\mathscr{L}} \cdot K_{\widetilde{S}} + \frac{vb}{a-ub} M_0 + \frac{1}{a-ub} \widetilde{\mathscr{L}} \right) &= \frac{1}{a-ub} \left\{ -2a(a-ub) + \beta v b^2 + (2a^2 - \gamma b^2) \right\} \\ &= \frac{b}{a-ub} (2ua + \beta v b - \gamma b). \end{aligned} \tag{4.4.3}$$

By virtue of $au \geq \alpha ub$ and $2\alpha u + \beta v - \gamma \geq 0$, we have:

$$\frac{b}{a-ub}(2ua + \beta v b - \gamma b) \geq \frac{b^2}{a-ub}(2\alpha u + \beta v - \gamma) \geq 0. \tag{4.4.4}$$

Notice that the rational map $\Phi_{\widetilde{\mathscr{L}}} \colon \widetilde{S} \dashrightarrow \overline{Z}$ is not a morphism since $\mathrm{Bs}(\widetilde{\mathscr{L}}) = \{\widetilde{p}\}$, where $\overline{Z}$ is the smooth projective model of Z[*2]. Let $\psi : \bar{S} \to \widetilde{S}$ be the shortest succession of blow-ups of $\widetilde{p}$ and its infinitely near points such that the proper transform $\bar{\mathscr{L}} := \psi_*^{-1}(\widetilde{\mathscr{L}})$ of $\widetilde{\mathscr{L}}$ is free of base points. Note that, $\widetilde{p} \in M_0$ and $(\bar{\mathscr{L}} \cdot \bar{M}_0) = 0$ by construction of $\widetilde{\mathscr{L}}$, where $\bar{M}_0$ is the proper transform $\psi_*^{-1}(M_0)$ of M_0. Letting $\{\bar{E}_i\}_{1 \leq i \leq n}$ be the exceptional divisors of ψ with $\bar{E}_n$ the last exceptional one, which is a section of $\bar{\varphi} := \Phi_{\widetilde{\mathscr{L}}} \circ \psi$, we have:

$$K_{\bar{S}} + \frac{vb}{a-ub} \bar{M}_0 + \frac{1}{a-ub} \bar{\mathscr{L}} = \psi^* \left(K_{\widetilde{S}} + \frac{vb}{a-ub} M_0 + \frac{1}{a-ub} \widetilde{\mathscr{L}} \right) + \sum_{i=1}^{n} c_i \bar{E}_i \tag{4.4.5}$$

and

$$(\bar{\mathscr{L}} \cdot \bar{E}_i) = \begin{cases} 0 & (1 \leq i \leq n-1) \\ 1 & (i = n) \end{cases} \tag{4.4.6}$$

[*2]Actually, we further know that $\overline{Z} \simeq \mathbb{P}^1_k$ by using [19, Lemma 7].

for some rational numbers $c_1, \dots, c_n$. Note that the general member of $\bar{\mathscr{L}}$ is a general fiber of the $\mathbb{P}^1$-fibration $\bar{\varphi}$. Hence, we have:

$$\begin{aligned}
-2 &= (\bar{\mathscr{L}} \cdot K_{\bar{S}}) \\
&= \left(\bar{\mathscr{L}} \cdot K_{\bar{S}} + \frac{vb}{a-ub}\bar{M}_0 + \frac{1}{a-ub}\bar{\mathscr{L}}\right) \\
&\underset{(4.4.5)}{=} \left(\bar{\mathscr{L}} \cdot \psi^* \left(K_{\widetilde{S}} + \frac{vb}{a-ub}M_0 + \frac{1}{a-ub}\widetilde{\mathscr{L}}\right)\right) + \sum_{i=1}^{n} c_i(\bar{\mathscr{L}} \cdot \bar{E}_i) \\
&\underset{(4.4.6)}{=} \left(\widetilde{\mathscr{L}} \cdot K_{\widetilde{S}} + \frac{vb}{a-ub}M_0 + \frac{1}{a-ub}\widetilde{\mathscr{L}}\right) + c_n
\end{aligned}$$

Thus, $(\widetilde{S}, \frac{vb}{a-ub}M_0 + \frac{1}{a-ub}\widetilde{\mathscr{L}})$ is not log canonical by (4.4.3) and (4.4.4) (see Definition 2.3.2, for this definition). Furthermore, since $1 - \frac{vb}{a-ub} \geq 0$ and $\frac{1}{a-ub} > 0$, by the variant of Corti's inequality (see Lemma 2.5.3) we have:

$$i(L_1, L_2; p) > 4\left(1 - \frac{vb}{a-ub}\right)(a-ub)^2 = 4\{a-(u+v)b\}(a-ub),$$

where L_1 and L_2 are general members of $\mathscr{L}$. Meanwhile, since L_1 and L_2 meet at only p, we see $i(L_1, L_2; p) = (\widetilde{\mathscr{L}})^2$. Hence, we have:

$$(\widetilde{\mathscr{L}})^2 > 4\{a-(u+v)b\}(a-ub) \iff 0 > 2a^2 - 4(2u+v)ab + \{4u(u+v)+\gamma\}b^2. \quad (4.4.7)$$

On the other hand, we have:

$$2a^2 - 4(2u+v)ab + \{4u(u+v)+\gamma\}b^2 = 2\{a-(2u+v)b\}^2 - (4u^2+4uv+2v^2-\gamma)b^2 \geq 0,$$

by $4u^2 + 4uv + 2v^2 - \gamma \leq 0$. It is a contradiction to (4.4.7). □

Note that the following Lemma 4.4.11 is the special case of Lemma 4.4.10:

Lemma 4.4.11. With the notation and the assumptions as in Lemma 4.4.10, the following two assertions hold:

(1) If $d = 2$, then we obtain $\alpha^2 < \gamma$.

(2) If $d = 1$, then we obtain $3\alpha^2 < 4\gamma$.

Proof. In (1), suppose that $\alpha^2 \geq \gamma$. Then we can easily see that the four inequalities (4.4.1) hold for $(u, v) = (\frac{\gamma}{2\alpha}, 0)$, which contradicts Lemma 4.4.10 (1).

In (2), suppose that $3\alpha^2 \geq 4\gamma$. Then we can easily see that the four inequalities (4.4.2) hold for $(u, v) = (\frac{\gamma}{\alpha}, 0)$, which contradicts Lemma 4.4.10 (2). □

Now, we show Lemmas 4.4.12, 4.4.14 and 4.4.15. For these lemmas, let $M_1, \dots, M_n$ be all irreducible components of the exceptional set over $\overline{k}$ of $\sigma_{\overline{k}}$ at p such that the weighted dual graph of $M_1, \dots, M_n$ is as in (4.3.1), (4.3.2) or (4.3.3) according to the singularities of p on $S_{\overline{k}}$.

Lemma 4.4.12. With the notation and the assumptions as above, the following assertions hold:

(1) If $d = 2$, then the singular point $p \in S$ is not of type A_1^+, A_3^+, A_4^+ nor $(A_5^+)'$ over k.

(2) If $d = 1$, then the singular point $p \in S$ is not of type A_1^+, A_2^+, A_3^+, A_5^+, A_6^+ nor $(A_7^+)'$ over k.

Proof. Suppose that the singularity type of p on S is one of the lists in Lemma 4.4.12. We shall write $m := \lceil \frac{n}{2} \rceil$ for simplicity. By noting $\mathrm{Bs}(\widetilde{\mathscr{L}}) = \{\widetilde{p}\}$, we see $(\widetilde{\mathscr{L}} \cdot M_i) = 0$ for any i other than $i = m$ (resp. $i = m,\ m+1$) if n is odd (resp. even). Indeed, if n is odd (resp. even), then $\widetilde{p}$ lies on M_m (resp. the intersection point of M_m and M_{m+1}). Hence, we can write $\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}) - bM$ for some $a, b \in \mathbb{Q}_{>0}$, where $M = \sum_{j=1}^{m-1} j(M_j + M_{n-j+1}) + m(M_m + \cdots + M_{n-m+1})$ by Lemma 4.3.1.

Let β and γ be two rational numbers defined by $\beta := -(M \cdot M_m)$ and $\gamma := -(M)^2$. Moreover, let α be the positive number defined by $\alpha := (M \cdot E)$, where E is the (-1)-curve on $\widetilde{S}_{\overline{k}}$ according to the degree d and the singularity type of p on $S_{\overline{k}}$ as follows:

- $(d, \mathrm{Singularity}) = (2, A_3),\ (2, A_4),\ (1, A_5),\ (1, A_6)$: By using Proposition 4.3.10, we take a (-1)-curve E on $\widetilde{S}_{\overline{k}}$ such that $(M_j \cdot E) = \delta_{m,j}$ (see also Example 4.3.11).
- $(d, \mathrm{Singularity}) = (2, A_1),\ (1, A_3)$: Notice that $S_{\overline{k}}$ allows a singular point other than p by the assumption of Theorem 1.3.9 (3)(iv). If $S_{\overline{k}}$ admits a cyclic singular point other than p, then we take a (-1)-curve E on $\widetilde{S}_{\overline{k}}$ such that $(M_j \cdot E) = \delta_{m,j}$ by an argument similar to the above. Otherwise, since $d = 1$ and $\widetilde{S}$ is of $D_5 + A_3$-type by the assumption of Theorem 1.3.9 (3)(iv), it is known that there exists a (-1)-curve E on $\widetilde{S}_{\overline{k}}$ such that $(M_j \cdot E) = \delta_{2,j}$ (see, e.g., [56, Figure 1]), so that we take such a (-1)-curve E.
- $(d, \mathrm{Singularity}) = (2, (A_5)'),\ (1, (A_7)')$: By the configuration of singularity of p, we can take the (-1)-curve E such that $(M_j \cdot E) = \delta_{m,j}$.
- $(d, \mathrm{Singularity}) = (1, A_1),\ (1, A_2)$: We take the (-1)-curve E as in Lemma 4.3.4(1). Namely, $(M_1 \cdot E) = 2$ (resp. $(M_1 \cdot E) = (M_2 \cdot E) = 1$) if $p \in S$ is of type A_1^+ (resp. type A_2^+) over k.

By construction of α, we see that $a \geq \alpha b$ because of $0 \leq (\widetilde{\mathscr{L}} \cdot E) = a - \alpha b$. Here, the values of α, β and γ are summarized in Table 4.5 according to the degree d and the singularity type of p on $S_{\overline{k}}$. For all cases except for $(d, \mathrm{Singularity}) = (2, A_1), (1, A_3)$, we thus obtain a contradiction to Lemma 4.4.11 (1) or (2) according to the degree d. In what follows, we consider the remaining cases. In the case of $(d, \mathrm{Singularity}) = (2, A_1)$, setting $(u, v) := (0, 1)$, the inequalities (4.4.1) hold simultaneously, which contradicts Lemma 4.4.10 (1). In the case of $(d, \mathrm{Singularity}) = (1, A_3)$, setting $(u, v) := (1, 1)$, the inequalities (4.4.2) hold simultaneously, which contradicts Lemma 4.4.10 (2). □

Remark 4.4.13. If the pair of the degree d and the singularity type of p on $S_{\overline{k}}$ is $(2, A_2)$ (resp. $(1, A_4)$), there is actually no rational numbers pair (u, v) such that the inequalities (4.4.1) (resp. (4.4.2)) hold simultaneously. We will deal with these cases later (see Lemma 4.4.15).

Lemma 4.4.14. With the notation and the assumptions as above, if $d = 1$ then the singular point $p \in S$ is not of type D_5^+ nor E_6^+ over k.

Proof. Suppose that $p \in S$ is of type D_5^+ or E_6^+ over k. By Proposition 4.4.1, $\mathrm{Bs}(\widetilde{\mathscr{L}})$ consists of only one k-rational point, say $\widetilde{p}$. Note that $\widetilde{p} \in M_3 \cup M_4 \cup M_5$ but $\widetilde{p} \notin M_1 \cup M_2$ (resp. $\widetilde{p} \in M_5 \cup M_6$ but $\widetilde{p} \notin M_1 \cup M_2 \cup M_3 \cup M_4$) provided that $p \in S$ is of type D_5^+ (resp. type E_6^+) over k. Thus, we can write $\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}) - bM$ for some $a, b \in \mathbb{Q}_{>0}$ by Lemmas 4.3.2 and 4.3.3, where M is the effective $\mathbb{Q}$-divisor and is given as in the Table 4.6 depending on one parameter t and according to both the singularity type of p on $S_{\overline{k}}$ and the position of $\widetilde{p}$. Let

Table 4.5: Values of α, β and γ in Lemma 4.4.12

d	Singularity	Irreducible decomposition of M	α	β	γ
2	A_1	M_1	1	2	2
2	A_3	$M_1 + 2M_2 + M_3$	2	2	4
2	A_4	$M_1 + 2M_2 + 2M_3 + M_4$	2	1	4
2	$(A_5)'$	$M_1 + 2M_2 + 3M_3 + 2M_4 + M_5$	3	2	6
1	A_1	M_1	2	2	2
1	A_2	$M_1 + M_2$	2	1	2
1	A_3	$M_1 + 2M_2 + M_3$	2	2	4
1	A_5	$M_1 + 2M_2 + 3M_3 + 2M_4 + M_5$	3	2	6
1	A_6	$M_1 + 2M_2 + 3M_3 + 3M_4 + 2M_5 + M_6$	3	1	6
1	$(A_7)'$	$M_1 + 2M_2 + 3M_3 + 4M_4 + 3M_5 + 2M_6 + M_7$	4	2	8

γ be the positive rational number defined by $\gamma := -(M)^2$. The value of γ and its range are summarized in Table 4.7 depending on one parameter t and according to both the singularity type of p on $S_{\overline{k}}$ and the position of $\widetilde{p}$. Let E be the (-1)-curve on $\widetilde{S}$ that as in Lemma 4.3.4 (2) or (3) according to the singularity type of p on $S_{\overline{k}}$. Noting that $0 \le (\widetilde{\mathscr{L}} \cdot E) = a - 2b$, we shall put $\alpha = 2$.

If $\gamma \le 3$, then we have $3\alpha^2 = 12 \ge 4\gamma$, which contradicts Lemma 4.4.11 (2). Hence, we suppose $\gamma > 3$ in what follows. Then $p \in S_{\overline{k}}$ is of type D_5 and lies on M_5 by Table 4.7. In particular, we see $1 \le t \le 2$. We shall put $\beta := -(M \cdot M_5) = 2t - 2$ and $(u, v) := (-t^2 + 3t - 1, 2t - 3)$. Noting $u = -t^2 + 3t - 1 > 0$, we have:

$$\begin{aligned} \alpha - u &= 2 - (-t^2 + 3t - 1) = \left(t - \frac{3}{2}\right)^2 + \frac{3}{4} > 0, \\ \alpha - u - v &= 2 - (-t^2 + 3t - 1) - (2t - 3) = (t - 2)(t - 3) \ge 0, \\ \alpha u + \beta v - \gamma &= 2(-t^2 + 3t - 1) + (2t - 2)(2t - 3) - (2t^2 - 4t + 4) = 0 \end{aligned}$$

and

$$\begin{aligned} 4u^2 + 4uv + 4v^2 - 3\gamma &= 4(-t^2 + 3t - 1)^2 + 4(-t^2 + 3t - 1)(2t - 3) + 4(2t - 3)^2 - 3(2t^2 - 4t + 4) \\ &= 2(t - 2)^2(2t^2 - 8t + 5) \\ &\le 2(t - 2)^2\{2t^2 - 8t + 5 + (2t - 1)\} \\ &= 4(t - 2)^3(t - 1) \\ &\le 0. \end{aligned}$$

This implies that the inequalities (4.4.2) hold simultaneously, which contradicts Lemma 4.4.10 (2). □

Finally, we treat the case that p is of type A^+_{6-2d} over k. If the singular point $p \in S_{\overline{k}}$ is of A_{6-2d} over $\overline{k}$, then the type of $\widetilde{S}$ is one of the following:

- $d = 2$ and $A_5 + A_2$, $A_4 + A_2$, $A_3 + A_2 + A_1$, $3A_2$, $A_3 + A_2$, $2A_2 + A_1$, $A_2 + 3A_1$, $2A_2$, $A_2 + 2A_1$, $A_2 + A_1$ or A_2-type.

Table 4.6: Effective $\mathbb{Q}$-divisor M in Lemma 4.4.14

Singularity	Position of $\widetilde{p}$	Irreducible decomposition of M	Range of t
D_5	$M_3 \cup M_4$	$tM_1 + tM_2 + 2tM_3 + 2M_4 + M_5$	$1 \leq t \leq \frac{3}{2}$
D_5	M_5	$M_1 + M_2 + 2M_3 + 2M_4 + tM_5$	$1 \leq t \leq 2$
E_6	$M_5 \cup M_6$	$tM_1 + tM_2 + 2tM_3 + 2tM_4 + 3tM_5 + 2M_6$	$1 \leq t \leq \frac{4}{3}$

Table 4.7: Value and range of γ in Lemma 4.4.14

Singularity	Position of $\widetilde{p}$	Range of t	γ	Range of γ
D_5	$M_3 \cup M_4$	$1 \leq t \leq \frac{3}{2}$	$4t^2 - 8t + 6$	$2 \leq \gamma \leq 3$
D_5	M_5	$1 \leq t \leq 2$	$2t^2 - 4t + 4$	$2 \leq \gamma \leq 4$
E_6	$M_5 \cup M_6$	$1 \leq t \leq \frac{4}{3}$	$6t^2 - 12t + 8$	$2 \leq \gamma \leq \frac{8}{3}$

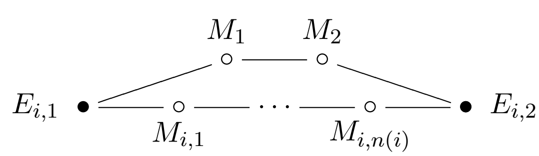
- $d = 1$ and $2A_4$, $A_4 + A_3$, $A_4 + A_2 + A_1$, $A_4 + 3A_1$, $A_4 + A_2$, $A_4 + 2A_1$, $A_4 + A_1$ or A_4-type.

In particular, we see that $S_{\overline{k}}$ has only cyclic singular points. Noting the above argument, we obtain the following lemma:

Lemma 4.4.15. With the notation and the assumptions as above, then the singular point $p \in S$ is not of type A_{6-2d}^{+} over k.

Proof. Suppose that $p \in S$ is of type A_{6-2d}^{+} over k. If $d = 2$ and $\widetilde{S}$ is of A_2-type, then $\widetilde{S}$ is a weak del Pezzo surface of degree 2 with $\rho_k(\widetilde{S}) = 2$. Hence, $\widetilde{S}$ is minimal over k by Theorem 1.3.3. However, by construction $\widetilde{S}$ contains the cylinder $\widetilde{U}$, which is a contradiction to Theorem 1.3.4. In what follows, we shall treat other cases and consider the cases of $d = 2$ and $d = 1$ separately.

In case of $d = 2$: Let $x_1, \ldots, x_r$ be all singular points on $S_{\overline{k}}$ other than p, and let $M_{i,1}, \ldots, M_{i,n(i)}$ be the irreducible components of the exceptional set on $\widetilde{S}$ of the minimal resolution at x_i for $i = 1, \ldots, r$ such that the weighted dual graph of $\sum_{i=1}^{r} \sum_{j=1}^{n(i)} M_{i,j}$ is as in (4.3.7). By using Proposition 4.3.10, for $i = 1, \ldots, r$, there exist two (-1)-curves $E_{i,1}$ and $E_{i,2}$ on $S_{\overline{k}}$ such that the weighted dual graph of $M_1 + M_2 + E_{i,1} + E_{i,2} + \sum_{j=1}^{n(i)} M_{i,j}$ is as follows (cf. Example 4.3.11):



for $i = 1, \ldots, r$. Notice that:

$$
\begin{aligned}
(E_{i,1} + E_{i,2} \cdot E_{i',1} + E_{i',2}) &= \left(-K_{\widetilde{S}} - M_1 - M_2 - \sum_{j=1}^{n(i)} M_{i,j} \cdot -K_{\widetilde{S}} - M_1 - M_2 - \sum_{j'=1}^{n(i')} M_{i',j'} \right) \\
&= (-K_{\widetilde{S}} - M_1 - M_2)^2 + \left(\sum_{j=1}^{n(i)} M_{i,j} \cdot \sum_{j'=1}^{n(i')} M_{i',j'} \right) \\
&= -2\delta_{i,i}.
\end{aligned}
$$

Write $m(i) := \lceil \frac{n(i)}{2} \rceil$ for simplicity. Let $\tau : \widetilde{S} \to V$ be the sequence of contractions of (-1)-curves $\sum_{i=1}^{r}(E_{i,1}+E_{i,2})$ and subsequently (smoothly) contractible curves in Supp $\left(\sum_{i=1}^{r}\sum_{i=1}^{n(i)} M_{i,j}\right)$ such that for each $i = 1, \dots, r$, unions $M_{i,1} + M_{i,n(i)}, \dots, M_{i,m(i)-1} + M_{i,n(i)-m(i)+2}$ are contracted if $m(i) > 1$. By construction, τ is defined over k and V is a smooth del Pezzo surface with $\rho_k(V) = 2$ endowed with a structure of Mori conic bundle $\pi : V \to B$ over k such that each $\tau_{\overline{k}}(M_{i,m(i)})$ is included in a union of some closed fibers of $\pi_{\overline{k}}$. Moreover, $\widetilde{p}$ is a k-rational point on $\widetilde{S}$, so is its image via τ. Thus, $B \simeq \mathbb{P}^1_k$ by Lemma 2.2.2. In particular, we obtain $\mathrm{Pic}(V)_{\mathbb{Q}} = \mathbb{Q}[-K_V] \oplus \mathbb{Q}[F]$, where F is a general fiber of π. Let $\{e_{i,j}\}_{1\le i\le r,\ 1\le j\le m(i)-1}$ be the total transforms of all irreducible components on the exceptional set satisfying $(e_{i,j} \cdot M_{i,j}) < 0$ by τ for $i = 1, \dots, r$ and $j = 1, \dots, m(i)-1$, moreover, we set $e_{i,0} := E_{i,1}+E_{i,2}$. By identifying $\mathrm{Pic}(\widetilde{S})_{\mathbb{Q}}$ with $\mathrm{Pic}(\widetilde{S}_{\overline{k}})_{\mathbb{Q}}^{\mathrm{Gal}(\overline{k}/k)}$, we thus obtain:

$$\mathrm{Pic}(\widetilde{S})_{\mathbb{Q}} \subseteq \mathbb{Q}[-K_{\widetilde{S}}] \oplus \mathbb{Q}[\widetilde{F}] \oplus \left(\bigoplus_{i=1}^{r} \bigoplus_{j=0}^{m(i)-1} \mathbb{Q}[e_{i,j}] \right) \subseteq \mathrm{Pic}(\widetilde{S}_{\overline{k}})_{\mathbb{Q}},$$

where $\widetilde{F}$ is a total transform of F by τ. In particular, we can write:

$$\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}) + b\widetilde{F} + \sum_{i=1}^{r}\sum_{j=0}^{m(i)-1} c_{i,j}e_{i,j}$$

for some $a, b, c_{i,j} \in \mathbb{Q}$. By construction, we obtain that $M_{i,j} + M_{i,m(i)-j+1} \sim e_{i,j} - e_{i,j-1}$ for $j = 1, \dots, m(i)-1$ and $M_{i,m(i)}$ (resp. $M_{i,m(i)}+M_{i,m(i)+1}$) is linearly equivalent to $\widetilde{F}-e_{i,m(i)-1}$ if $n(i)$ is odd (resp. even). Moreover, we notice $(\widetilde{F})^2 = (e_{i,j} \cdot \widetilde{F}) = 0$ for any i, j. Hence, we have $c_{i,j} = 0$ by virtue of $(\widetilde{\mathscr{L}} \cdot M_{i,j}) = 0$ for any i, j. On the other hand, since $E_{i,1}+E_{i,2} \sim e_{i,0}$, we have $a > 0$ by virtue of $0 \le (\widetilde{\mathscr{L}} \cdot e_{i,0}) = 2a$ and $0 < (\widetilde{\mathscr{L}})^2 = 2a(a+2b)$. Moreover, we have $b > 0$ by virtue of $0 < (\widetilde{\mathscr{L}} \cdot M_1 + M_2) = b(\widetilde{F} \cdot M_1 + M_2)$. Thus, we see $\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}) + b\widetilde{F}$ as $a, b > 0$, however, we obtain a contradiction to Lemma 2.5.5.

In case of $d = 1$: By Lemma 4.3.4 (1), there exists a (-1)-curve E_0 on $\widetilde{S}$ such that $(E_0 \cdot M_i) = \delta_{1,i} + \delta_{4,i}$. Hence, we have the contraction $\tau_0 : \widetilde{S} \to W_2$ of E_0 defined over k such that W_2 is a weak del Pezzo surface of degree 2, moreover, this condition is as the above case of $d = 2$. Thus, by an argument similar to the above case with $d = 2$, there exists a 0-curve $\widetilde{F}$ on $\widetilde{S}$ such that we can write:

$$\widetilde{\mathscr{L}} \sim_{\mathbb{Q}} a(-K_{\widetilde{S}}) + b\widetilde{F} + c_0E_0$$

for some $a, b, c_0 \in \mathbb{Q}$. By the configuration of τ_0, we see $(\widetilde{F} \cdot E_0) = 0$ and $M_1 + M_4 \sim \widetilde{F} - 2E_0$. Hence, we have $c_0 = 0$ by virtue of $0 = (\widetilde{\mathscr{L}} \cdot M_1 + M_4) = 2c_0$. Moreover, by an argument similar to the above case with $d = 2$ we see $a, b > 0$, which is a contradiction to Lemma 2.5.5. □

As we already mentioned, the "only if" part in Theorem 1.3.9 (3)(iv) follows from Lemmas 4.4.12, 4.4.14 and 4.4.15.

4.4.4 Assumption for the "if" part of Theorem 1.3.9 (3)(iv)

In this subsection, in order to prove the "if" part in Theorem 1.3.9 (3)(iv), we shall observe the assumption of this precisely. In other words, the purpose of this subsection is to show the following proposition:

Proposition 4.4.16. Let the notation be the same as at beginning Section 4.4, and assume that S does not satisfy any condition of (i), (ii) or (iii) in Theorem 1.3.9 (3). If $S_{\overline{k}}$ has a singular point defined over k of type A_n^-, D_n^- or E_n^- over k, then the type of $\widetilde{S}$ is one of the following:

- $d = 2$: $A_5 + A_2$, $2A_3 + A_1$, $2A_3$, $A_3 + 3A_1$, $3A_2$, $(A_5)'$, $(A_3 + 2A_1)''$, $A_2 + 3A_1$, $(A_3 + A_1)'$, A_3 or A_2-type.
- $d = 1$: $A_7 + A_1$, $E_6 + A_2$, $D_5 + A_3$, $A_5 + A_2 + A_1$, $2A_4$, $(A_7)'$, $D_5 + 2A_1$, $A_5 + A_2$, E_6, $(A_5 + A_1)'$, D_5, A_5 or A_4-type.

In what follows, we will prove Proposition 4.4.16. Let the notation and assumptions be the same as in Proposition 4.4.16. Then we can take a singular point x_0 on $S_{\overline{k}}$, which is k-rational, of type A_n^-, D_n^- or E_n^-. Let r be the number of all singular points other than x_0 on $S_{\overline{k}}$, which are k-rational, and let $x_1, \dots, x_r$ be the singular points other than x_0 on $S_{\overline{k}}$, which are k-rational. We shall consider two cases according to the degree d of S separately.

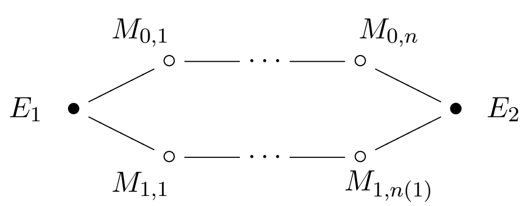
At first, we shall treat the case of $d = 2$. Then $x_0 \in S$ is of type A_n^- over k for some $2 \le n \le 5$, since S does not satisfy any condition of (i) nor (iii) in Theorem 1.3.9 (3). Then all singular points of $S_{\overline{k}}$ other than x_0 are also necessarily of type A, i.e., cyclic quotient singularities. We obtain the following two lemmas:

Lemma 4.4.17. Let the notation and the assumptions be the same as above. If $r > 0$, then $\widetilde{S}$ is of $A_5 + A_2$, $2A_3 + A_1$, $2A_3$, $A_3 + 3A_1$ or $(A_3 + A_1)'$-type.

Proof. Let $n(i)$ be the number such that $x_i \in S_{\overline{k}}$ is of type $A_{n(i)}$ for $i = 1, \dots, r$. Here, we may assume $n(1) \ge n(2) \ge \dots \ge n(r)$ by replacing the subscripts $i = 1, \dots, r$ as needed. Let $\{M_{i,j}\}_{1 \le j \le n(i)}$ be all irreducible components of the exceptional set of the minimal resolution at x_i for $i = 0, 1, \dots, r$ with the configuration as in (4.3.7), where $n(0) := n$, and let D be the divisor on $\widetilde{S}_{\overline{k}}$ defined by $D := (-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=0}^{1} \sum_{j=1}^{n(i)} M_{i,j}$. Since the divisor D is as in (a) in Table 4.3, we see that D satisfies the condition on divisors of either (†) or (‡) by Proposition 4.3.10 (1).

Assume that D satisfies the condition (†). Then the pair $(n, n(1))$ is $(3, 1)$, by Proposition 4.3.10 (7). In particular, we see $r = 1$. Otherwise, supposing $r \ge 2$ and taking the divisor $(-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=1}^{2} \sum_{j=1}^{n(i)} M_{i,j}$ on $\widetilde{S}_{\overline{k}}$, which is the divisor as in (a) in Table 4.3, we have $n(2) = 3$ by the argument similar to the above, however, it is a contradiction to $n(1) \ge n(2)$. Hence, if there exists a singular point on $S_{\overline{k}}$ other than x_0 and x_1, then there exist exactly two singular points of type A_1 on $S_{\overline{k}}$, which lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Indeed, there is no $A_3 + mA_1$-type of $\widetilde{S}$ for $m \ge 4$ by the classification of types of weak del Pezzo surfaces. Namely, $\widetilde{S}$ is then of $A_3 + A_1$ or $A_3 + 3A_1$-type.

Assume that D satisfies the condition (‡). In other words, there exist two (-1)-curves E_1 and E_2 on $\widetilde{S}_{\overline{k}}$ such that $D \sim E_1 + E_2$ (see Proposition 4.3.10 (2)). Hence, the configuration of the weighted dual graph of $\sum_{i=0}^{1} \sum_{j=1}^{n(i)} M_{i,j} + E_1 + E_2$ is as follows:



Since $x_0 \in S$ is of type A_n^- over k with $n \geq 2$, we see that E_1 and E_2 are defined over k, respectively. This implies that the two $\mathbb{Q}$-divisors E_1 and E_2 are included in $\mathbb{Q}[-K_{\widetilde{S}_{\overline{k}}}] \oplus \left(\bigoplus_{i=0}^{1} \bigoplus_{j=1}^{n(i)} \mathbb{Q}[M_{i,j}]\right)$ since $\rho_k(S) = 1$. Hence, the pair $(n, n(1))$ is $(5,2)$, $(2,5)$ or $(3,3)$ by Proposition 4.3.10 (6). If $(n, n(1)) = (5,2)$ or $(2,5)$, then all singular points on $S_{\overline{k}}$ are only x_0 and x_1 since there are at most seven (-2)-curves on $\widetilde{S}_{\overline{k}}$ by Lemma 2.1.5. Namely, $\widetilde{S}$ is of $A_5 + A_2$-type. If $(n, n(1)) = (3,3)$, then there exists at most a singular point of type A_1 on $S_{\overline{k}}$ other than x_0 and x_1 by a similar argument using Lemma 2.1.5. Namely, $\widetilde{S}$ is then of $2A_3$ or $2A_3 + A_1$-type. □

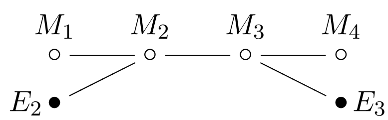
Lemma 4.4.18. Let the notation and the assumptions be the same as above. If $r = 0$, then the following assertions hold:

(1) $x \in S_{\overline{k}}$ is not of type A_4. Namely, $n = 2, 3$ or 5.

(2) $\widetilde{S}$ is not of $A_2 + 2A_1$-type.

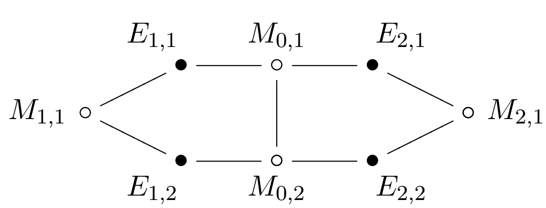
(3) $\widetilde{S}$ is of $(A_5)'$, $(A_3 + 2A_1)''$, A_3, $3A_2$, $A_2 + 3A_1$ or A_2-type.

Proof. In (1), supposing that $n = 4$, let $\{M_j\}_{1 \leq j \leq 4}$ be all irreducible components of the exceptional set of the minimal resolution at x_0 with the configuration as in (4.3.7) and let D be the divisor on $\widetilde{S}_{\overline{k}}$ defined by $D := (-K_{\widetilde{S}_{\overline{k}}}) - (M_1 + 2M_2 + 2M_3 + M_4)$, which is the divisor as in (b) in Table 4.3. By Proposition 4.3.10 (1) and (6), we see that D satisfies the condition (‡). In particular, by Proposition 4.3.10 (2), there exist two (-1)-curves E_2 and E_3 on $\widetilde{S}_{\overline{k}}$ such that $D \sim E_2 + E_3$. Hence, the configuration of the weighted dual graph of $\sum_{j=1}^{4} M_j + E_1 + E_2$ is as follows:



Since $x_0 \in S$ is of type A_4^- over k by assumption, E_2 and E_3 are defined over k, respectively. This implies that the two $\mathbb{Q}$-divisors E_1 and E_2 are included in $\mathbb{Q}[-K_{\widetilde{S}_{\overline{k}}}] \oplus \left(\bigoplus_{j=1}^{4} \mathbb{Q}[M_j]\right)$ since $\rho_k(S) = 1$. However it is a contradiction to Proposition 4.3.10 (6).

In (2), supposing that $\widetilde{S}$ is of $A_2 + 2A_1$-type, let y_1 and y_2 be two singular points of type A_1 on $S_{\overline{k}}$, let $M_{0,1}$ and $M_{0,2}$ (resp. $M_{1,1}$, $M_{2,1}$) be all irreducible components of the exceptional set of the minimal resolution at x_0 (resp. y_1, y_2) with the configuration as in (4.3.7) and let D_i be the divisor on $\widetilde{S}_{\overline{k}}$ defined by $D_i := (-K_{\widetilde{S}_{\overline{k}}}) - (M_{0,1} + M_{0,2}) - M_{i,1}$, which is the divisor as in (a) in Table 4.3 for $i = 1, 2$. By Proposition 4.3.10 (1) and (7), we see that D_i satisfies the condition (‡) for $i = 1, 2$. In particular, by Proposition 4.3.10 (2), there exist two (-1)-curves $E_{i,1}$ and $E_{i,2}$ on $\widetilde{S}_{\overline{k}}$ such that $D_i \sim E_{i,1} + E_{i,2}$. Hence, the configuration of the weighted dual graph of $M_{0,1} + M_{0,2} + M_{1,1} + M_{2,1} + \sum_{i=1}^{2} \sum_{j=1}^{2} E_{i,j}$ is as follows:



Since $x_0 \in S$ is of type A_2^- over k by assumption, $M_{0,1}$ is defined over k. Hence, so is the union $E_{1,1} + E_{2,1}$. This implies that the $\mathbb{Q}$-divisor $E_{1,1} + E_{2,1}$ is contained in $\mathbb{Q}[-K_{\widetilde{S}}] \oplus \left(\bigoplus_{j=1}^{2} \mathbb{Q}[M_{0,j}]\right) \oplus \left(\bigoplus_{i=1}^{2} \mathbb{Q}[M_{i,1}]\right)$ since $\rho_k(S) = 1$. Hence, we have:

$$E_{2,1} + E_{3,1} \sim_{\mathbb{Q}} (-K_{\widetilde{S}}) - \frac{1}{3}(2M_{1,1} + M_{1,2}) - \frac{1}{2}M_{2,1} - \frac{1}{2}M_{3,1}$$

by Lemma 4.3.1 combined with the above graph, however, by the above formula, we then obtain:

$$-2 = (E_{2,1} + E_{3,1})^2 = \left((-K_{\widetilde{S}}) - \frac{1}{3}(2M_{1,1} + M_{1,2}) - \frac{1}{2}M_{2,1} - \frac{1}{2}M_{3,1}\right)^2 = -\frac{5}{3},$$

which is absurd.

In (3), if x_0 is the only singular point of $S_{\overline{k}}$, then we clearly see that $\widetilde{S}$ is of $(A_5)'$, A_3 or A_2-type by the assumptions and (1). In what follows, assume that there exists a singular point y on $S_{\overline{k}}$ other than x_0. Since $r = 0$, there exists a singular point y' other than y on $S_{\overline{k}}$ such that y and y' are included in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Moreover, since there are at most seven (-2)-curves on $\widetilde{S}_{\overline{k}}$ by Lemma 2.1.5, the singular point y is of type A_1 or A_2 on $S_{\overline{k}}$. If singular point y is of type A_2 on $S_{\overline{k}}$, then we see that all singular points on $S_{\overline{k}}$ are only x_0, y and y', namely, $\widetilde{S}$ is then of $3A_2$-type. In what follows, we can thus assume that any singular point on $\widetilde{S}_{\overline{k}}$ other than x_0 is of type A_1. Then $\widetilde{S}$ is of $A_n + sA_1$-type for some integer s. In particular, we precisely see that $\widetilde{S}$ is then of $(A_3 + 2A_1)''$ or $A_2 + 3A_1$-type by the classification of types of weak del Pezzo surfaces (see [69]) combined with (2). □

Next, we shall treat the case of $d = 1$. Notice that $x_0 \in S$ is of type D_5^-, E_6^- or A_n^- over k for some $2 \le n \le 7$, since S does not satisfy any condition of (i) or (iii) in Theorem 1.3.9 (3). If $x_0 \in S$ is of type D_5^- or E_6^- over k, then we obtain the following lemma by the argument similar to Lemma 4.4.17:

Lemma 4.4.19. With the notation and the assumptions as above, assume further that $S_{\overline{k}}$ has a singular point, which is k-rational, of type D_5^- or E_6^- over k, then the type of $\widetilde{S}$ is one of the following according to the number of r:

(1) $r > 0$: $D_5 + A_3$ or $E_6 + A_2$-type.

(2) $r = 0$: $D_5 + 2A_1$, D_5 or E_6-type.

Proof. By assumption of this lemma, we may assume that $x_0 \in S$ is of type D_5^- or E_6^- over k. We only treat the case where the singularity x_0 is of type D_5^- over k, the other cases are similar and left to the reader.

In (1), let $\{M_{i,j}\}_{1 \le j \le n(i)}$ be all irreducible components of the exceptional set of the minimal resolution at x_i for $i = 0, 1$ with the configuration as in (4.3.8), where $n(0) := 5$, and let D be the divisor on $\widetilde{S}_{\overline{k}}$ defined by $D := 2(-K_{\widetilde{S}_{\overline{k}}}) - (M_{0,1} + 2M_{0,2} + 3M_{0,3} + 2M_{0,4} + M_{0,5}) - \sum_{j=1}^{n(1)} M_{1,j}$, which is the divisor as in (f) in Table 4.3. By the argument similar to Lemma 4.4.17, we see that $n(1) = 3$. In particular, all singular points on $S_{\overline{k}}$ are only x_0 and x_1 since there are at most eight (-2)-curves on $\widetilde{S}_{\overline{k}}$ by Lemma 2.1.5. Namely, $\widetilde{S}$ is then of $D_5 + A_3$-type.

In (2), if there exists a singular point other than x_0 on $S_{\overline{k}}$, then there exist exactly two singular points of type A_1 on $S_{\overline{k}}$, which lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, by a similar argument using Lemma 2.1.5. Namely, $\widetilde{S}$ is then of D_5 or $D_5 + 2A_1$-type. Indeed, there is no $D_5 + 3A_1$-type of $\widetilde{S}$ (see [69]). (We also note that there is no $E_6 + 2A_1$-type of $\widetilde{S}$ (see [69]).) □

In what follows, we shall treat the case that S allows only cyclic singular points by noting Lemma 2.1.5. Thus, the singular point $x_0 \in S$ is of type A_n^- over k for some $2 \le n \le 7$. By the argument similar to Lemmas 4.4.17 and 4.4.18, we obtain the following two lemmas:

Lemma 4.4.20. Let the notation and the assumptions be the same as above. If $r > 0$, then the type of $\widetilde{S}$ is one of the following according to the number of r:

(1) $r \ge 2$: $A_5 + A_2 + A_1$-type.

(2) $r = 1$: $(A_5 + A_1)'$, $A_7 + A_1$, $A_5 + A_2$ or $2A_4$-type.

Proof. Let $\{M_{i,j}\}_{1 \le j \le n(i)}$ be all irreducible components of the exceptional set of the minimal resolution at x_i for $i = 0, 1, \dots, r$ with the configuration as in (4.3.7), where $n(0) := n$.

In (1), let D be the divisor on $\widetilde{S}_{\overline{k}}$ defined by $D := 2(-K_{\widetilde{S}_{\overline{k}}}) - \sum_{i=0}^{2} \sum_{j=1}^{n(i)} M_{i,j}$, which is the divisor as in (c) in Table 4.3. By the argument similar to Lemma 4.4.17, we see that $(n, n(1), n(2)) = (5, 2, 1)$ or $(2, 5, 1)$. In particular, all singular points on $S_{\overline{k}}$ are only x_0, x_1 and x_2 since there are at most eight (-2)-curves on $\widetilde{S}_{\overline{k}}$ by Lemma 2.1.5. Namely, $\widetilde{S}$ is then of $A_5 + A_2 + A_1$-type.

In (2), at first we assume that $n \ge 4$. Let D be the divisor on $\widetilde{S}_{\overline{k}}$ defined by $D := 2(-K_{\widetilde{S}_{\overline{k}}}) + (M_{0,1} + M_{0,n}) - 2\sum_{j=1}^{n} M_{0,j} - \sum_{j=1}^{n(1)} M_{1,j}$, which is the divisor as in (d) in Table 4.3. By the argument similar to Lemma 4.4.17, we see that $(n, n(1)) = (5, 1)$ (resp. $(7, 1)$, $(5, 2)$ or $(4, 4)$) if D satisfies the condition (†) (resp. (‡)). In particular, all singular points on $S_{\overline{k}}$ defined over k are only x_0 and x_1 by a similar argument using Lemma 2.1.5. Namely, $\widetilde{S}$ is then of $(A_5 + A_1)'$, $A_7 + A_1$, $A_5 + A_2$ or $2A_4$-type. Here, note that there is no $A_5 + 3A_1$-type of $\widetilde{S}$ (see [69]).

On the other hand, if $n < 4$, then we have $n(1) \ge 4$ since S does not satisfy the condition on singularities of (iii) in Theorem 1.3.9 (3). The same argument as above applies with the role of $i = 0$ and $i = 1$ exchanged. □

Lemma 4.4.21. Let the notation and the assumptions be the same as above. If $r = 0$, then the following assertions hold:

(1) $x \in S_{\overline{k}}$ is not of type A_2, A_3 nor A_6. Namely, $n = 4, 5$ or 7.

(2) $\widetilde{S}$ is not of $A_5 + 2A_1$ nor $A_4 + 2A_1$-type.

(3) $\widetilde{S}$ is of $(A_7)'$, A_5 or A_4-type.

Proof. In (1), since $r = 0$, for any singular point y other than x_0 on $S_{\overline{k}}$, there exists a singular point y' other than y on $S_{\overline{k}}$ such that y and y' are included in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Moreover, since there are at most eight (-2)-curves on $\widetilde{S}_{\overline{k}}$ by Lemma 2.1.5, the singular point $y \in S_{\overline{k}}$ is of type A_1, A_2 or A_3. Hence, we see that $n \ge 4$ since S does not satisfy the condition of (iii) in Theorem 1.3.9 (3).

Supposing that $x_0 \in S_{\overline{k}}$ is of type A_6, let $\{M_j\}_{1 \le j \le 6}$ be all irreducible components of the exceptional set of the minimal resolution at x_0 with the configuration as in (4.3.7). Letting D be the divisor on $\widetilde{S}_{\overline{k}}$ defined by $D := 2(-K_{\widetilde{S}_{\overline{k}}}) - (M_1 + 2M_2 + 3M_3 + 3M_4 + 2M_5 + M_6)$, which is the divisor as in (e) in Table 4.3, we obtain a contradiction by the argument similar to Lemma 4.4.18 (1).

In (2), otherwise, let y_1 and y_2 be two singular points of type A_1 on $S_{\overline{k}}$, let $\{M_{0,j}\}_{1 \le j \le n}$ (resp. $M_{2,1}$, $M_{3,1}$) be all irreducible components of the exceptional set of the minimal resolution at x_0 (resp. y_1, y_2) with the configuration as in (4.3.7). Letting D be the divisor on $\widetilde{S}_{\overline{k}}$

defined by $D := 2(-K_{\widetilde{S}_{\overline{k}}}) - \sum_{j=1}^{n} M_{0,j} - M_{1,1} - M_{2,1}$, which is the divisor as in (c) in Table 4.3, we obtain a contradiction by the argument similar to Lemma 4.4.18 (1).

In (3), by the classification of types of weak del Pezzo surfaces (see [69]) combined with the assumption that $n \geq 4$, $\widetilde{S}$ is of $A_n + sA_1$-type for some integer $s = 0$ or 2. Moreover, we precisely see that $\widetilde{S}$ is then of $(A_7)''$, A_5 or A_4-type by (2) and a similar argument using Lemma 2.1.5. □

Proposition 4.4.16 follows from Lemmas 4.4.17, 4.4.18, 4.4.19, 4.4.20 and 4.4.21.

Conversely, for each type of weak del Pezzo surface in the list of Proposition 4.4.16, there exists certainly a Du Val del Pezzo surface S of rank one over k admitting a singular point of type A_n^-, D_n^- or E_n^- over k such that its minimal resolution $\widetilde{S}$ is of this type. Indeed, for each type of weak del Pezzo surface in the list of Proposition 4.4.16, we can explicitly construct a birational morphism $\tau : \widetilde{S} \to \mathbb{F}_2$ over k and the contraction $\sigma : \widetilde{S} \to S$ of all (-2)-curves over k such that $\widetilde{S}$ is of this type and S is the Du Val del Pezzo surface of rank one (see also Subsection 4.4.5, for detailed constructions of such morphism τ). Here, the Picard number of $\widetilde{S}$ is the number, which is summarized in "$\rho_k(\widetilde{S})$" in Table 4.8 according to the type of $\widetilde{S}$. Furthermore, the singularity types of all singular points on $S_{\overline{k}}$, which are k-rational, are summarized in "k-rat. sing." in Table 4.8 according to the type of $\widetilde{S}$. As an example, in the case $d = 2$ and $3A_2$-type, $S_{\overline{k}}$ has three singular points of type A_2. If $\rho_k(S) = 1$, then one is k-rational and of type A_2^- over k, however, the others lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, namely $\rho_k(\widetilde{S}) = \rho_k(S) + 4 = 5$.

At the end of this subsection, we shall present the notation in Table 4.8. The meanings of "k-rat. sing." and "$\rho_k(\widetilde{S})$" have already been presented. "Dual graph" in Table 4.8 means the weighted dual graph corresponding to the union of all (-2)-curves and some (-1)-curves on $\widetilde{S}$. For all types of $\widetilde{S}$ in the list of Table 4.8, the union of the (-1)-curves on $\widetilde{S}$ corresponding to all vertices $\bullet$ in Table 4.8 certainly exists and is further defined over k. The existence of these curves can be shown by using Proposition 4.3.10 with suitable choices of divisors on S except for the case $d = 2$ and $\widetilde{S}$ is of A_2-type. Moreover, the other case also follows that $\widetilde{S}$ admits a Mori conic bundle with exactly six singular fibers by Theorem 1.3.3 combined with [50, Exercise 3.13]. These dual graphs will be used for the construction of cylinders on the surfaces S in Subsection 4.4.5.

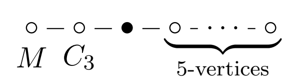
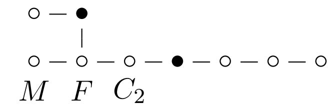
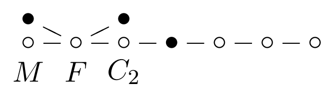
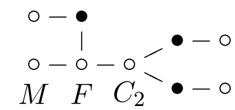
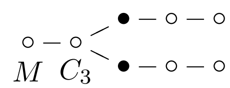
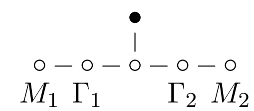
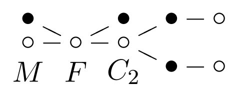
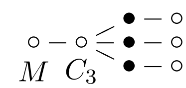
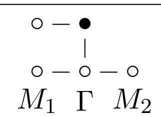
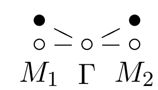
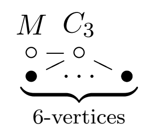
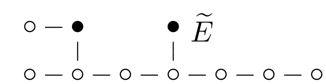
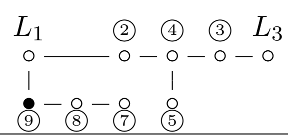
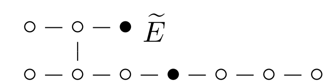
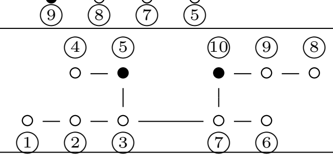
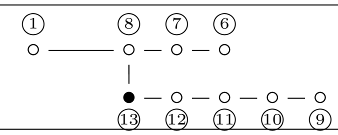
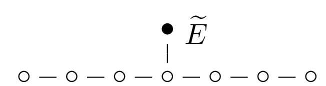
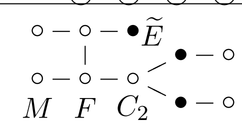
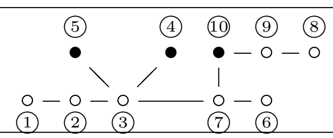
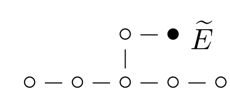
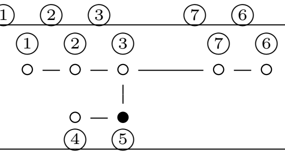
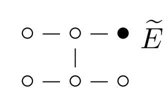
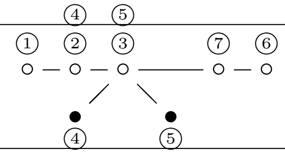
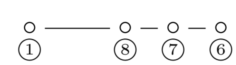
4.4.5 Proof for the "if" part of Theorem 1.3.9 (3)(iv)

Let the notation and assumptions be the same as in Proposition 4.4.16. Then the type of $\widetilde{S}$ is one of those in Table 4.8. In this subsection, we shall show the "if" part of Theorem 1.3.9 (3)(iv). In other words, we will explicitly construct a cylinder on S according to the type in the list of Table 4.8.

Lemma 4.4.22. Let the notation and assumptions be the same as in Proposition 4.4.16. If $d = 2$ or both $d = 1$ and $\widetilde{S}$ is of $A_7 + A_1$, $D_5 + 2A_1$, $(A_7)'$, $D_5 + 2A_1$, E_6 or D_5-type. Then S contains a cylinder.

Proof. In the case of $d = 2$, let D be the union of all (-2)-curves on $\widetilde{S}$. At first, we shall deal with the cases in which $\widetilde{S}$ is of $(A_5)'$, $(A_3 + A_1)'$ and A_3-type. For these cases, we can take a birational morphism $\tau : \widetilde{S} \to W_4$, which is the compositions of the successive contractions of the (-1)-curves corresponding to the vertices $\bullet$ in the weighted dual graph in Table 4.8 and that of the proper transform of the branch components such that all curves corresponding to vertices with no label in the weighted dual graph in Table 4.8 are contracted by τ, according

Table 4.8: Types of $\widetilde{S}$ in the "if" part of Theorem 1.3.9 (3)(iv)

d	Type (k-rat. sing.) $\rho_k(\widetilde{S})$	Dual graph	d	Type (k-rat. sing.) $\rho_k(\widetilde{S})$	Dual graph
2	$A_5 + A_2$ (A_5^-, A_2^-) 8		2	$2A_3 + A_1$ (A_3^-, A_3^-, A_1^+) 8	
2	$2A_3$ (A_3^-, A_3^-) 7		2	$A_3 + 3A_1$ (A_3^-, A_1^+) 6	
2	$3A_2$ (A_2^-) 5		2	$(A_5)'$ (A_5^-) 6	
2	$(A_3 + 2A_1)''$ (A_3^-) 5		2	$A_2 + 3A_1$ (A_2^-) 4	
2	$(A_3 + A_1)'$ (A_3^-, A_1^+) 5		2	A_3 (A_3^-) 4	
2	A_2 (A_2^-) 3		1	$A_7 + A_1$ (A_7^-, A_1^+) 9	
1	$E_6 + A_2$ (E_6^-, A_2^-) 9		1	$D_5 + A_3$ (D_5^-, A_3^-) 9	
1	$A_5 + A_2 + A_1$ (A_5^-, A_2^-, A_1^+) 9		1	$2A_4$ (A_4^-, A_4^-) 9	
1	$(A_7)'$ (A_7^-) 8		1	$D_5 + 2A_1$ (D_5^-) 7	
1	$A_5 + A_2$ (A_5^-, A_2^-) 8		1	E_6 (E_6^-) 7	
1	$(A_5 + A_1)'$ (A_5^-, A_1^+) 7		1	D_5 (D_5^-) 6	
1	A_5 (A_5^-) 6		1	A_4 (A_4^-) 5	

to the type of $\widetilde{S}$, where W_4 is a weak del Pezzo surface of degree 4 and $(2A_1)_<$-type over k. Note that, by construction, τ is defined over k. Moreover, the image of the reduced curves corresponding to all vertices of this weighted dual graph via τ is the union of either $M_1+M_2+\Gamma$ or $M_1+M_2+\Gamma_1+\Gamma_2$, where M_1 and M_2 are (-2)-curves on $W_{4,\overline{k}}$, Γ is a 0-curve on $W_{4,\overline{k}}$ and Γ_1 and Γ_2 are (-1)-curves on $W_{4,\overline{k}}$ meeting transversally at a point. Notice that these curves on $W_{4,\overline{k}}$ are in one-to-one correspondence to these vertices with a label of this weighted dual graph. Since two (-2)-curves on $W_{4,\overline{k}}$ admit a k-rational point respectively, W_4 contains a cylinder, which contains $\tau(D)$ in this boundary (see also Subsection 4.2.2). Thus, $\widetilde{S}$ contains a cylinder $\widetilde{U}$, which contains D in this boundary. Therefore, we see that S contains a cylinder $\sigma(\widetilde{U}) \simeq \widetilde{U}$.

In what follows, we shall deal with the remaining cases. For all remaining cases, we can take a birational morphism $\tau : \widetilde{S} \to \mathbb{F}_2$, which is the compositions of the successive contractions of the (-1)-curves corresponding to the vertices $\bullet$ in the weighted dual graph in Table 4.8 and that of the proper transform of the branch components such that all curves corresponding to vertices with no label in the weighted dual graph in Table 4.8 are contracted by τ, according to the type of $\widetilde{S}$. Note that, by construction, τ is defined over k. Moreover, the image of the reduced curves corresponding to all vertices of this weighted dual graph via τ is the union of either $M+F+C_2$ or $M+C_3$, where M and F are the minimal section and a closed fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k, respectively, and C_n is a rational curve on $\mathbb{F}_2$ with $C_n \sim M+nF$ for $n=2,3$. Notice that these curves on $\mathbb{F}_2$ are in one-to-one correspondence to these vertices with a label of this weighted dual graph. For all cases, $\mathbb{F}_2$ contains a cylinder, whose boundary includes the above union of curves, by Lemma 2.5.2. Thus, we see that S contains a cylinder by an argument similar to the above.

In (2), for all cases, the weighted dual graph in Table 4.8 corresponding to the type of $\widetilde{S}$ contains a vertex with a label written $\widetilde{E}$. This vertex corresponds to a (-1)-curve on $\widetilde{S}_{\overline{k}}$, which is defined over k. Letting $\widetilde{E}$ be this (-1)-curve on $\widetilde{S}$, we can take the contraction $\tau : \widetilde{S} \to W_2$ of $\widetilde{E}$ over k, so that W_2 is a weak del Pezzo surface of degree 2, whose type is one of those in the list of Table 4.8, moreover, the point $\tau(\widetilde{E})$ lies on a curve, which corresponds to a vertex with no label in the weighted dual graph in Table 4.8 according to the type of W_2. Thus, we see that S contains a cylinder by using (1). □

In order to deal with all remaining cases, we shall recall how to construct cylinders in del Pezzo surfaces with Du Val singularities found in [12, §§4.2–4.3]. More precisely, we construct two birational morphisms $g : \check{S} \to \widetilde{S}_{\overline{k}}$ and $h : \check{S} \to \mathbb{P}^2_{\overline{k}}$ over $\overline{k}$ (but not necessarily defined over k) in such a way that there exists a suitable cylinder U in $\mathbb{P}^2_{\overline{k}}$, which would be preserved via $g \circ h^{-1} : \mathbb{P}^2_{\overline{k}} \dashrightarrow \widetilde{S}_{\overline{k}}$ and $(g \circ h^{-1})(U) \cap \mathrm{Supp}(N) = \emptyset$, where N is the union of all (-2)-curves on $\widetilde{S}_{\overline{k}}$. In particular, $S_{\overline{k}}$ contains the cylinder $(\sigma \circ g \circ h^{-1})(U)$. In the following lemmas (Lemmas 4.4.23, 4.4.25 and 4.4.26), in order to show that above argument is still working well over k, we shall prove that g and h are defined over k. In the proofs for Lemmas 4.4.25 and 4.4.26, we look at the corresponding to weighted dual graphs in Table 4.8 and [12, Table 1]. We note that the numbering something like ⓘ in Table 4.8 corresponds to that in [12, Table 1].

Lemma 4.4.23. Let the notation and assumptions be the same as in Proposition 4.4.16. If $d=1$ and $\widetilde{S}$ is of E_6+A_2, $A_5+A_2+A_1$, $2A_4$ or A_5+A_2-type, then S contains a cylinder.

Proof. For all cases, we see that any (-2)-curve on $\widetilde{S}_{\overline{k}}$ is defined over k by the configuration of singular points on $S_{\overline{k}}$ (see also Table 4.8). In particular, any point meeting two (-2)-curves on $\widetilde{S}_{\overline{k}}$ is also defined over k. Then we can construct a birational morphism $g : \check{S} \to \widetilde{S}_{\overline{k}}$, whose

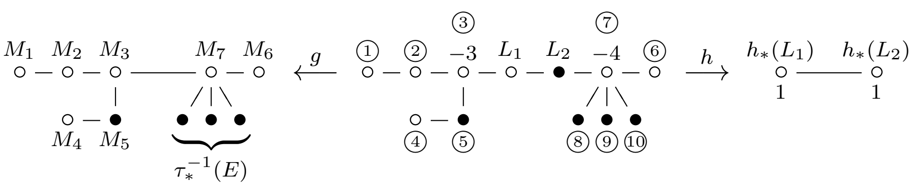
$\check{S}$ is that as in [12, §§4.2] according to the type of $\widetilde{S}$, defined over k. Indeed, we shall consider a sequence of some blow-ups at some k-rational points starting at an intersection point of two (-2)-curves on $\widetilde{S}_{\overline{k}}$ (according to the type of $\widetilde{S}$) and including infinitely near points such that we obtain the configuration of "Construction" in [12, Table 1] according to the types of $\widetilde{S}$. Moreover, we immediately have a birational morphism $h : \check{S}_{\overline{k}} \to \mathbb{P}^2_{\overline{k}}$, which plays same role as h in [12, §§4.2]. This h is clearly defined over k. Therefore, we see that S contains a cylinder. □

Remark 4.4.24. In Lemma 4.4.23, if $\widetilde{S}$ is of $E_6 + A_2$, $A_5 + A_2 + A_1$ or $2A_4$-type, then we could have also inferred the same result from the fact that $g : \check{S} \to \widetilde{S}_{\overline{k}}$ and $h : \check{S} \to \mathbb{P}^2_{\overline{k}}$ are clearly defined over k, where g and h are those as in [12, §§4.2]. Indeed, for these types, all (-1)-curves and (-2)-curves on $\widetilde{S}_{\overline{k}}$ are defined over k since $\rho_k(\widetilde{S}) = \rho_{\overline{k}}(\widetilde{S}_{\overline{k}}) = 9$.

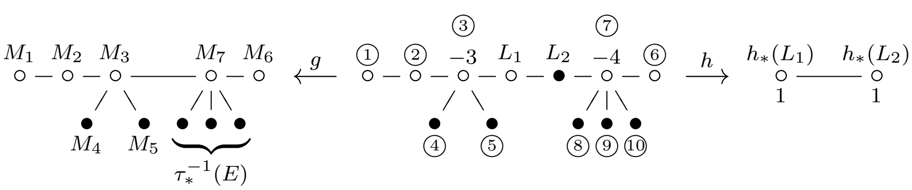
Lemma 4.4.25. Let the notation and assumptions be the same as in Proposition 4.4.16. If $d = 1$ and $\widetilde{S}$ is of $(A_5 + A_1)'$ or A_5-type, then S contains a cylinder.

Proof. Let M_i be the smooth rational curve on $\widetilde{S}_{\overline{k}}$ corresponding to the vertex with a label written (i) in the weighted dual graph of Table 4.8. There exists a (-1)-curve $\widetilde{E}$ on $\widetilde{S}_{\overline{k}}$, which is defined over k, such that $(\widetilde{E} \cdot M_i) = \delta_{1,i} + \delta_{6,i}$ by Lemma 4.3.4 (1). Hence, we obtain the birational morphism $\tau : \widetilde{S} \to W_4$ over k with the reduced exceptional divisor $M_4 + M_5 + \widetilde{E}$, so that W_4 is a weak del Pezzo surface of degree 4 and $(2A_1)_<$-type. Notice that $\tau_*(M_2)$ and $\tau_*(M_7)$ (resp. $\tau_*(M_1)$ and $\tau_*(M_6)$) are (-2)-curves (resp. (-1)-curves) on $W_{4,\overline{k}}$. By the configuration of $W_{4,\overline{k}}$, we know that $\tau_*(M_7)_{\overline{k}}$ meets exactly four (-1)-curves such that one is $\tau_*(M_6)_{\overline{k}}$. Let E be the union of three (-1)-curves meeting $\tau_*(M_7)_{\overline{k}}$ other than $\tau_*(M_6)_{\overline{k}}$ on $W_{4,\overline{k}}$. Noting that E is defined over k, so is $\tau_*^{-1}(E)$. Moreover, all irreducible components of $\tau_*^{-1}(E)$ consist of three (-1)-curve on $\widetilde{S}_{\overline{k}}$ corresponding to curves with a label written (8), (9), (10) in [12, Table 1]. Thus, we can construct two birational morphisms $g : \check{S} \to \widetilde{S}_{\overline{k}}$ and $h : \check{S} \to \mathbb{P}^2_{\overline{k}}$, which play same role as in g and h in [12, §§4.2], defined over k (see the following weighted dual graph):

$(A_5 + A_1)'$-type :



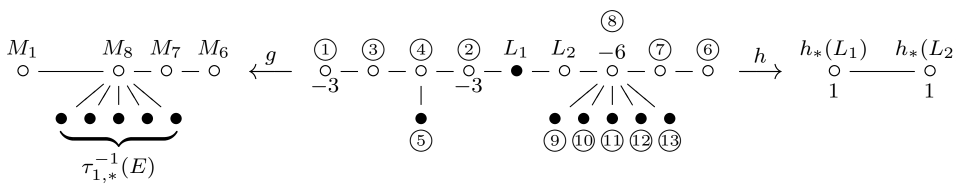
A_5-type :



Here, the numbering something like (i) in the above graph corresponds to that in [12, Table 1]. Therefore, we see that S contains a cylinder. □

Lemma 4.4.26. Let the notation and assumptions be the same as in Proposition 4.4.16. If $d = 1$ and $\widetilde{S}$ is of A_4-type, then S contains a cylinder.

Proof. Let M_i be the (-2)-curve on $\widetilde{S}_{\overline{k}}$ corresponding to the vertex with a label written (i) in the weighted dual graph of Table 4.8. There exists a (-1)-curve $\widetilde{E}$ on $\widetilde{S}_{\overline{k}}$, which is defined over k, such that $(\widetilde{E} \cdot M_i) = \delta_{1,i} + \delta_{6,i}$ by Lemma 4.3.4 (1). Hence, we have the contraction $\tau_1 : \widetilde{S} \to W_2$ of $\widetilde{E}$ over k, so that W_2 is a weak del Pezzo surface of degree 2 and A_2-type. Notice that $\tau_{1,*}(M_7)$ and $\tau_{1,*}(M_8)$ (resp. $\tau_{1,*}(M_1)$ and $\tau_{1,*}(M_6)$) are (-2)-curves (resp. (-1)-curves) on $W_{2,\overline{k}}$. By the configuration of $W_{2,\overline{k}}$, we know that $\tau_{1,*}(M_8)$ meets exactly six (-1)-curves such that one is the $\tau_{1,*}(M_1)$. Let E be the union of five (-1)-curves meeting $\tau_{1,*}(M_8)_{\overline{k}}$ other than $\tau_{1,*}(M_1)_{\overline{k}}$ on $W_{2,\overline{k}}$. Noting that E is defined over k, so is $\tau_{1,*}^{-1}(E)$. Moreover, all irreducible components of $\tau_{1,*}^{-1}(E)$ consist of five (-1)-curves on $\widetilde{S}_{\overline{k}}$ corresponding to curves with a label written 9–13 in [12, Table 1]. On the other hand, we have the contraction $\tau_2 : W_2 \to \mathbb{F}_2$ of $\tau_{1,*}(M_1) + E$ over k. Set $M := \tau_*(M_7)$, $F_0 := \tau_*(M_6)$ and $C_3 := \tau_*(M_8)$, where $\tau := \tau_2 \circ \tau_1 : \widetilde{S} \to W_2 \to \mathbb{F}_2$. Then M and F_0 are the minimal section and a closed fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k, moreover, C_3 is a rational curve on $\mathbb{F}_2$ with $C_3 \sim M + 3F_0$ (cf. Lemma 4.4.22 (1)). Since $(F_0 \cdot C_3) = 1$, F_0 and C_3 meet transversely at a point, say p, which is k-rational. Moreover, we see that there exists a unique rational curve C_2 on $\mathbb{F}_2$ such that $C_2 \sim M + 2F$ and $i(C_2, C_3; p) = 3$, where $i(C_2, C_3; p)$ is the local intersection multiplicity at p of C_2 and C_3. Notice that C_2 is defined over k. Moreover, $\tau_*^{-1}(C_2)$, which is also defined over k, corresponds to the curve with a label written 5 in [12, Table 1]. Thus, we can construct two birational morphisms $g : \check{S} \to \widetilde{S}_{\overline{k}}$ and $h : \check{S} \to \mathbb{P}^2_{\overline{k}}$, which play same role as in g and h in [12, §§4.2], defined over k (see the following weighted dual graph):



Here, the numbering something like i in the above graph corresponds to that in [12, Table 1]. Therefore, we see that S contains a cylinder. $\square$

The "if" part of Theorem 1.3.9 (3)(iv) follows from Proposition 4.4.16 and Lemmas 4.4.22, 4.4.23, 4.4.25 and 4.4.26.

4.5 Examples

In this section, we shall present some examples of Du Val del Pezzo surfaces of Picard rank one and canonical del Pezzo fibrations.

Example 4.5.1. Put $\zeta := \frac{-1+\sqrt{-3}}{2}$ and let S be the cubic surface over $\mathbb{Q}$ defined by:

$$S := \big(12z^2w - 2x^3 - y^3 - 4w^3 + 6xyw = 0\big) \subseteq \mathbb{P}^3_{\mathbb{Q}} = \mathrm{Proj}(\mathbb{Q}[x,y,z,w])$$

Then $S_{\overline{\mathbb{Q}}}$ has exactly three singular points $[\sqrt[3]{2}\zeta^i : \sqrt[3]{4}\zeta^{2i} : 0 : 1] \in \mathbb{P}^3_{\overline{\mathbb{Q}}}$ of type A_1 for $i = 0, 1, 2$ (see also Remark 4.5.2). Let $\sigma : \widetilde{S} \to S$ be the minimal resolution over $\mathbb{Q}$. Then there exists the birational morphism $\tau : \widetilde{S} \to S_6$ over $\mathbb{Q}$ such that S_6 is a smooth del Pezzo surface of degree 6. Hence, $S_{6,\overline{\mathbb{Q}}}$ has six (-1)-curves, say $\{E_i\}_{1 \le i \le 6}$. Moreover, the proper transform of these (-1)-curve by $\tau \circ \sigma^{-1}$ are defined by the following equations:

$$\sqrt[3]{2}\zeta^i x = y,\ x = \pm\frac{\sqrt[3]{2}}{3}\zeta^i(\zeta - 1)z + \sqrt[3]{2}\zeta^i w$$

for $i = 0, 1, 2$. Since all (-1)-curves on $S_{6,\overline{\mathbb{Q}}}$ lie in the same $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit, S_6 is $\mathbb{Q}$-minimal, in particular, we obtain $\rho_{\mathbb{Q}}(S_6) = 1$. By construction of σ and τ, we also obtain $\rho_{\mathbb{Q}}(S) = 1$. Thus, S does not contain any cylinder by Theorem 1.3.9 (2). Indeed, $S_{\overline{\mathbb{Q}}}$ does not allow any singular point which is $\mathbb{Q}$-rational (see also Table 4.1). On the other hand, we know that $S_{\overline{\mathbb{Q}}}$ contains a cylinder by Theorem 1.1.4. This implies that any cylinder on $S_{\overline{\mathbb{Q}}}$ is not defined over $\mathbb{Q}$.

Remark 4.5.2. Let S and ζ be those as in Example 4.5.1 and let A be the square matrix of order 4 defined by:

$$A := \begin{bmatrix} \sqrt[3]{2} & \sqrt[3]{2}\zeta & \sqrt[3]{2}\zeta^2 & 0 \\ \sqrt[3]{4} & \sqrt[3]{4}\zeta^2 & \sqrt[3]{4}\zeta & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix} \in GL(4; \overline{\mathbb{Q}}).$$

Then we obtain the projective transformation $\varphi_A : \mathbb{P}^3_{\overline{\mathbb{Q}}} \xrightarrow{\sim} \mathbb{P}^3_{\overline{\mathbb{Q}}}$ associated to A and we see:

$$\varphi_A^{-1}(S_{\overline{\mathbb{Q}}}) = \left(w^2(x+y+z) + xyz = 0\right) \subseteq \mathbb{P}^3_{\overline{\mathbb{Q}}} = \mathrm{Proj}(\overline{\mathbb{Q}}[x, y, z, w]).$$

It is easily to see that $\varphi_A^{-1}(S_{\overline{\mathbb{Q}}})$ has exactly three singular points $[1{:}0{:}0{:}0], [0{:}1{:}0{:}0], [0{:}0{:}1{:}0] \in \mathbb{P}^3_{\overline{\mathbb{Q}}}$, which are of type A_1.

Example 4.5.3. Let S be the complete intersection of two quadrics over $\mathbb{R}$ in $\mathbb{P}^4_{\mathbb{R}}$ as follows:

$$S := \left(x^2 + y^2 + wv = zw + wv + vz = 0\right) \subseteq \mathbb{P}^4_{\mathbb{R}} = \mathrm{Proj}(\mathbb{R}[x, y, z, w, v]).$$

Then S is a Du Val del Pezzo surface of degree 4 such that $S_{\mathbb{C}}$ has exactly three singular points $p_{\pm} := [1 : \pm\sqrt{-1} : 0 : 0 : 0]$ and $p := [0 : 0 : 1 : 0 : 0]$ in $\mathbb{P}^4_{\mathbb{C}}$, which are of type A_1. Since p_+ and p_- lie in the same $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbit, we see $\rho_{\mathbb{R}}(S) = 1$ (see Subsection 4.3.1). Hence, S contains a cylinder if and only if $p \in S$ is of type A_1^+ over k, by Theorem 1.3.9 (2). However, $p \in S$ is actually of type A_1^{++} over k, that is, S does not contain any cylinder. Indeed, the exceptional set by the minimal resolution at p does not have any $\mathbb{R}$-rational point since it can be written locally as follows:

$$(u^2 + v^2 + 1 = 0) \subseteq \mathbb{A}^2_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[u, v])$$

for some two parameters u and v.

In what follows, we treat three examples of generically canonical del Pezzo fibrations.

Example 4.5.4. Let $f : X \to Y$ be a generically canonical del Pezzo fibration of degree 3 or 4 over a curve Y and let X_η be the generic fiber of f. For simplicity, we put $S := X_\eta$ and $k := \mathbb{C}(Y)$. Assuming that $S_{\overline{k}}$ has a singular point x of type A_1 defined over k, and let $\sigma : \widetilde{S} \to S$ be the minimal resolution at x. Since x is defined over k, so is the exceptional curve $E := \sigma^{-1}(x)$. Note that $E_{\overline{k}}$ is a (-2)-curve. Now, we see that E has a k-rational point since $k = \mathbb{C}(Y)$ is a C_1-field by the Tsen's theorem (see also [29, Theorem 3.12]). In other words, the singular point $x \in S$ is always of type A_1^+ over k (compare the example in Example 4.5.3). Therefore, by Theorem 1.3.9 (2) combined with the above observation, we obtain that f admits a vertical cylinder if and only if $X_{\eta,\overline{\mathbb{C}(Y)}}$ has a singular point defined over $\mathbb{C}(Y)$.

Example 4.5.5. Note that there exists a Du Val del Pezzo surface of Picard rank one with degree $1, \dots, 6$ or 8 such that $\mathrm{Sing}(S) \neq \emptyset$ (see, e.g., [56]). Let S be a Du Val del Pezzo surface of Picard rank one with degree $d \in \{1, \dots, 6, 8\}$ over $\mathbb{C}$ such that $\mathrm{Sing}(S) \neq \emptyset$, let Y be an algebraic variety over $\mathbb{C}$ and let X be the direct product $S \times Y$. Then the second projection $f : X \to Y$ is a generically canonical del Pezzo fibration of degree d. Let X_η be the generic fiber of f. For simplicity, put $k := \mathbb{C}(Y)$. Then all (-1)-curves and (-2)-curves on $X_{\eta,\overline{k}}$ are defined over k. Therefore, f does not admit any vertical cylinder if and only if $d = 1$ and $X_{\eta,\overline{k}}$ allows only singular points of types A_1, A_2, A_3, D_4 by Theorem 1.3.9. This condition is actually equivalent to the condition that S does not contain any cylinder (see [5, Theorem 1.6]).

Example 4.5.6. Let $\mathscr{O}$ be a discrete valuation ring of the rational function field $\mathbb{C}(t)$ such that the maximal ideal of $\mathscr{O}$ is generated by t, and let X be the 3-fold variety over $\mathbb{C}$ defined by:

$$X := (t^n w^2 + x^2 y^2 + x z^3 = 0) \subseteq \mathbb{P}_{\mathscr{O}}(1,1,1,2) = \mathrm{Proj}(\mathscr{O}[x,y,z,w]).$$

Then we obtain the structure morphism $f : X \to \mathrm{Spec}(\mathscr{O})$ as an $\mathscr{O}$-scheme. Letting η be the generic point on $\mathrm{Spec}(\mathscr{O})$, the generic fiber X_η of f can be written as follows:

$$X_\eta = (t^n w^2 + x^2 y^2 + x z^3 = 0) \subseteq \mathbb{P}_{\mathbb{C}(t)}(1,1,1,2) = \mathrm{Proj}(\mathbb{C}(t)[x,y,z,w]).$$

By Example 1.3.10, X_η is a Du Val del Pezzo surface of rank one defined over $\mathbb{C}(Y)$ and of degree 2, moreover, X_η contains a cylinder if and only if n is even. Hence, f is a generically canonical del Pezzo fibration of degree 2, furthermore, f admits a vertical cylinder if and only if n is even.

Chapter 5

Compactifications of the affine plane over non-closed fields

In this chapter, we will prove Theorem 1.3.12. Moreover, we will consider the application of this theorem. Throughout this chapter except for Section 5.2, let k be a field of characteristic zero.

5.1 Compactifications of the affine plane

Let V be a normal projective surface defined over k and let D be a reduced effective divisor on V. Then we say that the pair (V, D) is a *compactification* of the affine plane $\mathbb{A}^2_k$ if $V \backslash \mathrm{Supp}(D) \simeq \mathbb{A}^2_k$. Moreover, for a compactification (V, D) of $\mathbb{A}^2_k$, we say that D is called the *boundary divisor*. In this section, we prepare some facts about compactifications of the affine plane. For any compactification (V, D) of $\mathbb{A}^2_k$, we notice that D has no cycle by Lemma 2.5.3 since $V \backslash \mathrm{Supp}(D)$ is the cylinder in V. Furthermore, we obtain the following lemma:

Lemma 5.1.1. For any compactification (V, D) of $\mathbb{A}^2_k$ over k, we have $\sharp D = \rho_k(V)$.

Proof. Let us put $n := \sharp D$ and $U := V \backslash \mathrm{Supp}(D)$. Let $C_1, \dots, C_n$ be all irreducible components of D, let G be a free abelian group generated by $C_1, \dots, C_n$, and let $f : G \to \mathrm{Cl}(V)$ be the group homomorphism defined by $f(C_i) := [C_i]$. Then f is surjective by virtue of $\mathrm{Cl}(U) \simeq \mathrm{Pic}(U) = 0$ (cf. [40, Lemma 4.6]). In what follows, we shall show that f is injective. Assume that a Weil divisor $a_1C_1 + \dots + a_nC_n$ is a principal divisor $\mathrm{div}(f)$ for some $f \in k(V)^\times$. Since $C_1, \dots, C_n$ are included in $\mathrm{Supp}(D)$, we have $a_1C_1 + \dots + a_nC_n|_U = 0$. Hence, $f|_U \in R^\times$, where R is the coordinate ring of U. Moreover, we see $f \in k^\times$ by virtue of $R^\times = k^\times$. Namely, $\mathrm{div}(f) = 0$. This implies that f is injective. Thus, we have a group isomorphism $G \simeq \mathrm{Cl}(V)$. In particular, $n = \rho_k(V)$. □

In this paper, we will mainly deal with the following special kinds of compactifications of the affine plane:

Definition 5.1.2. Let (S, Δ) be a compactification of the affine plane $\mathbb{A}^2_k$ over k. We say that the pair (S, Δ) is an *lc compactification* of the affine plane $\mathbb{A}^2_k$ over k if S is an lc del Pezzo surface of rank one over k such that $\mathrm{Sing}(S_{\overline{k}}) \neq \emptyset$.

The following lemma will be used in Subsection 5.2.3:

Lemma 5.1.3. Assume that k is algebraically closed. For any lc compactification (S, Δ) of the affine plane $\mathbb{A}^2_k$ over k, then $\sharp\mathrm{Sing}(S) \le 2$.

Proof. See [45, Lemma 4.2]. $\square$

In what follows, we recall minimal normal compactifications of the affine plane.

Definition 5.1.4. Assume that k is algebraically closed. Let (V, D) be a compactification of the affine plane $\mathbb{A}^2_k$ over k such that V is smooth. Then this pair (V, D) is a *minimal normal compactification* of the affine plane $\mathbb{A}^2_k$ if D is an SNC-divisor and any (-1)-curve E in $\mathrm{Supp}(D)$ satisfies $(E \cdot D - E) \ge 3$.

Morrow classified the minimal normal compactifications of the affine plane when the base field is $\mathbb{C}$ ([57]). In this paper, we will mainly use the following facts:

Lemma 5.1.5. Assume that k is algebraically closed. Let (V, D) be a minimal normal compactification of the affine plane $\mathbb{A}^2_k$ over k. Then the following assertions hold:

(1) Any irreducible component of D is a smooth rational curve and the dual graph of D is a linear chain. In particular, $\mathrm{Supp}(D)$ does not contain any (-1)-curve on V.

(2) If $\sharp D = 1$, then $(D)^2 = 1$.

(3) If $\sharp D = 2$, then D contains at least one irreducible component, say Γ, satisfying $(\Gamma)^2 = 0$.

(4) If $\sharp D \ge 3$, then D contains exactly two irreducible components, say Γ_0 and Γ_+, satisfying $(\Gamma_0)^2 = 0$, $(\Gamma_+)^2 > 0$ and $(\Gamma_0 \cdot \Gamma_+) = 1$.

Proof. See [57]. $\square$

Furthermore, [36, Theorem 1.2] proves that the converse of Morrow's result is true. More precisely, the following result holds:

Lemma 5.1.6. Assume that k is algebraically closed. Let V be a smooth protective surface defined over k and let D be a reduced divisor on V such that $V \backslash \mathrm{Supp}(D)$ is affine and each irreducible component of D is a rational curve. If the weighted dual graph of D is the same as that of the boundary divisor of a minimal normal compactification of the affine plane $\mathbb{A}^2_k$, then (V, D) is a minimal normal compactification of $\mathbb{A}^2_k$.

Proof. See [36]. $\square$

5.2 Properties of twigs

Throughout this section, we always assume that all varieties are defined over an algebraically closed field of characteristic zero.

5.2.1 Some definitions of twigs

Let D be an SNC-divisor on a smooth projective surface. Let A be the weighted dual graph of D. If A is given by the following graph, then A is called the *twig* and we write this weighted dual graph $[m_1, \ldots, m_r]$ as A:

$$\underset{-m_1}{\circ} \text{———} \underset{-m_2}{\circ} \text{———} \cdots \text{———} \underset{-m_r}{\circ}$$

In this subsection, we will present some definitions for the twig. The following two definitions are based on [24] (see also [42]):

Definition 5.2.1. Let $A = [m_1, \ldots, m_r]$ be a twig. Then the twig $[m_r, m_{r-1}, \ldots, m_1]$ is called the *transposal* of A and denoted by ${}^tA$. We define also $\overline{A} := [m_2, \ldots, m_r]$ and $\underline{A} := [m_1, \ldots, m_{r-1}]$, where we put $\overline{A} = \underline{A} = \emptyset$ if $r = 1$. We say that A is *admissible* if $m_i \geq 2$ for any $i = 1, \ldots, r$. In what follows, we assume that A is admissible. Then $d(A)$ denotes the absolute value of the determinant of the intersection matrix corresponding to A and is simply called the *determinant* of A, where we put $d(\emptyset) = 1$. We say that $e(A) := d(\overline{A})/d(A)$ is the *inductance* of A. By [24, Corollary (3.8)], e defines a one-to-one correspondence from the set of all admissible twigs to the set of rational numbers in the interval $(0, 1)$. Hence, there exists uniquely an admissible twig A^*, whose inductance is equal to $1 - e({}^tA)$, so that we call the admissible twig A^* the *adjoint* of A.

Example 5.2.2. Consider two admissible twigs $A := [2, 4]$ and $B := [2, 2, 3]$. Then $d({}^tA) = d(A) = 7$ and $d(\overline{{}^tA}) = 2$, so that $e({}^tA) = \frac{2}{7}$. Moreover, $d(B) = 7$ and $d(\overline{B}) = 5$, namely, $e(B) = \frac{5}{7} = 1 - e({}^tA)$. Hence, $B = A^*$.

Furthermore, we will use the following notation in this article:

Definition 5.2.3. (1) Letting $A_1, \ldots, A_s$ be twigs given by $A_i = [m_{i,1}, \ldots, m_{i,r_i}]$ for $i = 1, \ldots, s$, we write $[A_1, \ldots, A_s] := [m_{1,1}, \ldots, m_{1,r_1}, \ldots\ldots, m_{s,1}, \ldots, m_{s,r_s}]$.

(2) For a positive integer t, we write $[t \times 2] := [\underbrace{2, \ldots, 2}_{t\text{-times}}]$.

(3) For a positive integer m and a non-negative integer t, we write two twigs $L(m; t)$ and $R(m; t)$ respectively as follows:

$$L(m;t) := \begin{cases} [[t \times 2], [m]] & \text{if } t > 0 \\ [m] & \text{if } t = 0 \end{cases}, \quad R(m;t) := \begin{cases} [[m], [t \times 2]] & \text{if } t > 0 \\ [m] & \text{if } t = 0 \end{cases}.$$

5.2.2 Twigs contracted to single smooth rational curves

Let D be an SNC-divisor on a smooth projective surface such that any irreducible component of D is a rational curve. We recall the following proposition for later use in Subsection 5.2.3.

Proposition 5.2.4. Let D be the same as above. Assume that the weighted dual graph of D is the twig $[A, [1], B]$ for some admissible twigs A and B. Then D can be contracted to a 0-curve if and only if $B = A^*$.

Proof. See, [24, Proposition (4.7)]. □

In Proposition 5.2.4, since the adjoint of any admissible twig is unique, note that B is uniquely determined according to A. By applying Proposition 5.2.4, we obtain the following proposition:

Proposition 5.2.5. Let D be the same as above. Assume that the weighted dual graph of D is the twig $[[m], A, [1], B]$ for some integer m and admissible twigs A and B. Then D can be contracted to the twig $[m, 1]$ if and only if $B = \underline{A^*}$.

Proof. Since A is admissible, notice that A can be uniquely denoted by $[L(m_r; t_r), \ldots, L(m_1; t_1)]$ for some $m_1 \geq 2$, $m_i \geq 3$ $(i > 1)$ and $t_j \geq 0$ $(1 \leq j \leq r)$. By the induction on r, we see that D can be contracted to the twig $[m, 1]$ if and only if B can be written as follows:

$$B = \begin{cases} [(m_1 - 2) \times 2] & \text{if } r = 1 \\ [[(m_1 - 2) \times 2], R(t_1 + 3; m_2 - 3), \ldots, R(t_{r-1} + 3; m_r - 3)] & \text{if } r > 1 \end{cases}. \tag{5.2.1}$$

Let D' be an SNC-divisor on a smooth projective surface such that the weighted dual graph of D' is the twig $[A,[1],B']$, where B' is the admissible twig defined by:

$$B' := \begin{cases} [[(m_1-2)\times 2],[t_1-2]] & \text{if } r=1 \\ [[(m_1-2)\times 2],R(t_1+3;m_2-3),\dots,R(t_{r-1};m_r-3),[t_r-2]] & \text{if } r>1 \end{cases}.$$

Then we easily see that D' can be contracted to a 0-curve by induction on r. Hence, we have $B'=A^*$ by Proposition 5.2.4. In particular, we see that B is as in (5.2.1) if and only if $B=\underline{B'}=\underline{A^*}$. □

Definition 5.2.6. Let $A=[L(m_r;t_r),\dots,L(m_1;t_1)]$ be an admissible twig, $m_1\geq 2$, $m_i\geq 3$ $(i>1)$ and $t_j\geq 0$ $(1\leq j\leq r)$. In this article, we then put $m_A:=t_r+3$.

Remark 5.2.7. Let A be an admissible twig. By definition of m_A and Lemma 5.2.5, the twig $[[m],A,[1],\underline{A^*},[m_A]]$ can be contracted to the twig $[m,1,2]$ for an arbitrary integer m.

Example 5.2.8. Consider the admissible twig $A:=[2,4]$. By definition, we know $m_A=4$. Meanwhile, since $A^*=[2,2,3]$, we obtain $\underline{A^*}=[2,2]$. For an arbitrary integer m, the twig $[[m],A,[1],\underline{A^*},[m_A]]=[m,2,4,1,2,2,4]$ is then contracted to $[m,1,2]$ as follows:

$$[m,2,4,1,2,2,4]\to[m,2,3,1,2,4]\to[m,2,2,1,4]\to[m,2,1,3]\to[m,1,2].$$

5.2.3 Twigs as boundary divisors of the affine plane

In this subsection, let $(\widetilde{V},\widetilde{D})$ be a compactification of the affine plane $\mathbb{A}^2$ such that $\widetilde{V}$ is a smooth projective surface over $\overline{k}$ and the weighted dual graph of $\widetilde{D}$ is the twig $[m_1,\dots,m_r]$ with $m_i\geq 1$ for any $i=1,\dots,r$. By Lemma 5.1.5, $\widetilde{D}$ consists of irreducible components $\{\widetilde{C}_i\}_{1\leq i\leq r}$ such that $\widetilde{C}_i$ is a $(-m_i)$-curve for $i=1,\dots,r$, moreover, we see that $(\widetilde{V},\widetilde{D})$ is not a minimal normal compactification of $\mathbb{A}^2$. Let $\nu:(\widetilde{V},\widetilde{D})\to(\check{V},\check{D})$ be a sequence of contractions of (-1)-curves and subsequently (smoothly) contractible curves in $\mathrm{Supp}(\widetilde{D})$ such that the pair $(\check{V},\check{D})$ is a minimal normal compactification of $\mathbb{A}^2$, where $\check{D}:=\nu_*(\widetilde{D})$.

Lemma 5.2.9. With the notation as above, then the following three assertions hold:

(1) $r\geq 3$.

(2) There exists at least one integer e with $2\leq e\leq r-1$ such that $m_e=1$.

(3) If $r=3$, then we obtain $m_1=1$ or $m_3=1$.

Proof. In (1), supposing $n\leq 2$, we can easily obtain a contradiction by Lemma 5.1.5.

In (2), noticing $r\geq 3$ by (1), suppose $m_i\geq 2$ for any $i=2,\dots,r-1$. Since $(\widetilde{V},\widetilde{D})$ is not a minimal normal compactification of $\mathbb{A}^2$, we obtain $m_1=1$ or $m_r=1$. Then $\widetilde{D}$ can be contracted to the twig $[m]$ for some non-negative integer m or an admissible twig by straightforward calculation. It contradicts Lemma 5.1.5.

In (3), we note $m_2=1$ by (2). Hence, $\widetilde{D}$ can be contacted to the twig $[m_1-1,m_3-1]$. By Lemma 5.1.5 (2) and (3), we see $m_1-1=0$ or $m_3-1=0$. This completes the proof. □

Lemma 5.2.10. With the notation as above, then the following two assertions hold:

(1) Assume that there exists exactly one integer e with $1\leq e\leq r-1$ such that $m_i=1$ if and only if $i=e$ or $e+1$. Then we obtain $r=3$.

(2) Assume that $r \geq 4$, $m_i = m_{r+1-i}$ for any i and there exists exactly one integer e with $1 \leq e < \frac{r}{2}$ such that $m_i = 1$ if and only if $i = e, e+1, r-e$ or $r+1-e$. Then we obtain $r = 4$. Namely, the weighted dual graph of $\widetilde{D}$ is $[1,1,1,1]$.

Proof. In (1), notice that $r \geq 3$ by Lemma 5.2.9 (1). We may assume $e+1 < r$ by symmetry. Moreover, we can assume that ν starts with the contraction of $\widetilde{C}_e$. Then we see that $\sharp\check{D} \geq 2$, and any irreducible component of $\check{D}$ with self-intersection number ≥ 0 is only $\check{C} := \nu_*(\widetilde{C}_{e+1})$. Thus, we have $\sharp\check{D} = 2$ and $(\check{C})^2 = 0$ by Lemma 5.1.5. This implies $r = 3$ by virtue of $\sharp\widetilde{D} - \sharp\check{D} = (\check{C})^2 - (\widetilde{C}_{e+1})^2 = 1$.

In (2), we can assume that ν starts with the contraction of $\widetilde{C}_e + \widetilde{C}_{r+1-e}$. Then we see that $\sharp\check{D} \geq 2$, and any irreducible component of $\check{D}$ with self-intersection number ≥ 0 is only $\check{C}_1 := \nu_*(\widetilde{C}_{e+1})$ or $\check{C}_2 := \nu_*(\widetilde{C}_{r-e})$. Moreover, by $m_{e+1} = m_{r-e}$ and construction of ν, we obtain $(\check{C}_1)^2 = (\check{C}_2)^2$. Thus, we have $\sharp\check{D} = 2$ and $(\check{C}_1)^2 = (\check{C}_2)^2 = 0$ by Lemma 5.1.5. This implies $r = 4$ by virtue of $\sharp\widetilde{D} - \sharp\check{D} = (\check{C}_1)^2 - (\widetilde{C}_{e+1})^2 + (\check{C}_2)^2 - (\widetilde{C}_{r-e})^2 = 2$. □

Lemma 5.2.10 (1) can be generalized as follows:

Lemma 5.2.11. Let $(\widetilde{V}', \widetilde{D}')$ be a compactification of the affine plane $\mathbb{A}^2$ such that $\widetilde{V}'$ is a smooth projective surface and $\widetilde{D}'$ is an SNC-divisor. Assume that any irreducible component of $\widetilde{D}'$ has self-intersection number ≤ -2 except for exactly two irreducible components E_1 and E_2 such that $(E_1)^2 = (E_2)^2 = -1$ and $(E_1 \cdot E_2) = 1$. Then the weighted dual graph of $\widetilde{D}'$ is the twig $[1,1,m]$ for some integer $m \geq 2$.

Proof. By Lemma 5.1.5 (1), we note $(E_i \cdot \widetilde{D}' - E_i) \leq 2$ for $i = 1, 2$. Let $\nu' : (\widetilde{V}', \widetilde{D}') \to (\check{V}', \check{D}')$ be a sequence of contractions of (-1)-curves and subsequently (smoothly) contractible curves in $\mathrm{Supp}(\widetilde{D}')$ such that the pair $(\check{V}', \check{D}')$ is a minimal normal compactification of $\mathbb{A}^2$, where $\check{D}' := \nu'_*(\widetilde{D}')$. By Lemma 5.1.5 (1), we note $(E_i \cdot \widetilde{D}' - E_i) \leq 2$ for $i = 1, 2$. Noting that $\widetilde{D}'$ is connected and has no cycle by Lemma 2.5.3, the divisor $\widetilde{D}' - E$ can be decomposed into connected components $\widetilde{D}'_1 + \widetilde{D}'_2$ such that $(\widetilde{D}'_i \cdot E_j) = 0$ for $i, j = 1, 2$ with $i \neq j$, where it is not necessarily $\widetilde{D}'_i \neq 0$ for $i = 1, 2$. Since we can assume that ν' starts with the contraction of E_1 (resp. E_2), the weighted dual graph of $\nu'_*(\widetilde{D}'_2 + E_2)$ (resp. $\nu'_*(\widetilde{D}'_1 + E_1)$) is then a twig by Lemma 5.1.5 (1). In particular, the weighted dual graph of $\widetilde{D}'$ is a twig. Hence, this assertion follows from Lemma 5.2.10 (1). □

Lemma 5.2.12. With the notation as above, assume that $r = \sharp D$ is odd, $m_i = m_{r+1-i}$ for any i, and there exists exactly one integer e with $1 \leq e \leq r'$ such that $m_i = 1$ if and only if $i = e$ or $r+1-e$, where $r' := \frac{r+1}{2}$. Then the following assertions hold:

(1) $e \neq r'$.

(2) If $e = r'-1$, then we obtain $m_{r'} = r-2$ and $m_i = 2$ for any $i = 1, \ldots, r$ with $|r'-i| > 1$, namely, the weighted dual graph of $\widetilde{D}$ can be denoted by $[L(1; r'-2), [r-2], R(1; r'-2)]$.

Proof. In (1), suppose $e = r'$. Noting $r \geq 5$ by Lemma 5.2.9, the weighted dual graph of the contraction of $\widetilde{C}_{r'}$ is the twig $[m_1, \ldots, m_{r'-2}, m_{r'-1} - 1, m_{r'+1} - 1, m_{r'+2}, \ldots, m_r]$, which is not admissible by Lemma 5.1.5. Hence, we obtain $m_{r'\pm 1} - 1 = 1$, which contradicts Lemma 5.2.10 (1).

In (2), it is proved by the induction on r, where we note $r \geq 5$ by Lemma 5.2.9. Assume $r = 5$. By assumption, we have $m_2 = m_4 = 1$. Hence, we obtain $m_1 = m_5 = 2$ and $m_3 = 3$ by Lemma 5.2.9 (3) since $\widetilde{D}$ can be contracted to the twig $[m_1 - 1, m_3 - 2, m_5 - 1]$. Assume $r > 5$. Then the weighted dual graph of the contraction of $\widetilde{C}_{r'-1} + \widetilde{C}_{r'+1}$ on $\widetilde{D}$ is the twig

$[m_1, \dots, m_{r'-3}, m_{r'-2}-1, m_{r'}-2, m_{r'+2}-1, m_{r'+3}, \dots, m_r]$. Then we obtain $m_{r'\pm 2}-1 = 1$ by virtue of (1) and Lemma 5.1.5 (4). Moreover, we obtain $m_{r'}-2 > 1$. Indeed, otherwise, by contracting further the direct image of $\widetilde{C}_{r'}$, we have $[m_1, \dots, m_{r'-3}, m_{r'-2}-2, m_{r'+2}-2, m_{r'+3}, \dots, m_r]$, where $m_{r'\pm 2}-2 = 0$. However, this contradicts Lemma 5.1.5 (4). By the inductive hypothesis, we thus obtain $m_{r'}-2 = r-2$ and $m_i = 2$ for any $i = 1, \dots, r$ with $|r'-i| > 2$. This completes the proof. □

Finally, we shall prepare the following proposition, which will play an important role in Section 5.4:

Proposition 5.2.13. With the notation as above, then the following assertions hold:

(1) Assume that there exists exactly one integer e satisfying $m_e = 1$. Then the weighted dual graph of $\widetilde{D}$ can be denoted by $[A, [1], A^*, [m]]$ for some admissible twig A.

(2) Assume that $r = \sharp\widetilde{D}$ is even, $m_i = m_{r+1-i}$ for any i, and there exists exactly one integer e with $1 \le e \le \frac{r}{2}$ such that $m_i = 1$ if and only if $i = e$ or $r+1-e$. Then the weighted dual graph of $\widetilde{D}$ can be denoted by $[{}^t(A^*), [1], {}^tA, A, [1], A^*]$ for some admissible twig A.

(3) Assume that $r = \sharp\widetilde{D}$ is odd, $m_i = m_{r+1-i}$ for any i, and there exist exactly one integer e with $1 \le e \le r'$ such that $m_i = 1$ if and only if $i = e$ or $r+1-e$, where $r' := \frac{r+1}{2}$. Then the weighted dual graph of $\widetilde{D}$ can be denoted by one of the following:
   - $[L(1; r'-2), [r-2], R(1; r'-2)]$.
   - $[L(m_A; t), {}^t(\underline{A^*}), [1], {}^tA, [2t+3], A, [1], \underline{A^*}, R(m_A; t)]$ for some admissible twig A, where m_A is as in Definition 5.2.6 and t is a non-negative integer.

Proof. For each case, we shall prove this proposition by the induction on r.

In (1), notice $r \ge 4$ by Lemma 5.2.9. Assume $r = 4$. We may assume $m_2 = 1$ by symmetry. Since $\widetilde{D}$ can be contracted to the twig $[m_1-1, m_3-1, m_4]$, we obtain $m_1 = m_3 = 2$ by Lemma 5.2.9 (3). Thus, $[m_1, \dots, m_4] = [2, 1, 2, m_4] = [[2], [1], [2]^*, [m_4]]$, where note $[2]^* = [2]$. Assume $r > 4$. Noting $1 < e < r$ by Lemma 5.2.9 (2), the weighted dual graph of the contraction of $\widetilde{C}_e$ on $\widetilde{D}$ is the twig $B' := [m_1, \dots, m_{e-2}, m_{e-1}-1, m_{e+1}-1, m_{e+2}, \dots, m_r]$. Then we obtain $m_{e-1}-1 = 1$ or $m_{e+1}-1 = 1$ since B' is not admissible by Lemma 5.1.5. Moreover, we also obtain $m_{e-1}-1 > 1$ or $m_{e+1}-1 > 1$ by Lemma 5.2.10 (1). By the inductive hypothesis, B' can denote $[A', [1], (A')^*, [m]]$ for some an admissible twig A'. We may assume $m_r = m$ by symmetry. Since the twig $[A', [1], (A')^*]$ can be contracted to the twig $[0]$ by Lemma 5.2.4, so is $[m_1, \dots, m_{r-1}]$. Hence, $[m_1, \dots, m_r] = [A, [1], A^*, [m_r]]$, where $A := [m_1, \dots, m_{e-1}]$.

In (2), notice $r \ge 6$ by Lemmas 5.2.9 (2) and 5.2.10 (1). Assume $r = 6$. Then we have $m_2 = m_5 = 1$ by Lemmas 5.2.9 (2) and 5.2.10 (1). Since $\widetilde{D}$ can be contracted to the twig $[m_1-1, m_3-1, m_4-1, m_6-1]$, which is not admissible by Lemma 5.1.5, we obtain $m_1 = m_3 = m_4 = m_6 = 2$ by Lemmas 5.2.9 (2) and 5.2.10 (1). Thus, $[m_1, \dots, m_6] = [2, 1, 2, 2, 1, 2] = [{}^t([2]^*), [1], {}^t[2], [2], [1], [2]^*]$. Assume $r > 6$. We put $e' := r+1-e$ for simplicity. Note $1 < e$ and $e'-e > 1$ by Lemmas 5.2.9 (2) and 5.2.10 (1). Since the weighted dual graph of the contraction of $\widetilde{C}_e + \widetilde{C}_{e'}$ on $\widetilde{D}$ is a non-admissible twig, say B', by the similar argument to (1), we obtain $m_{e-1}-1 = 1$ or $m_{e+1}-1 = 1$. Furthermore, we also obtain $m_{e-1}-1 > 1$ or $m_{e+1}-1 > 1$ by Lemma 5.2.10 (2). Meanwhile, we note the assumption $m_i = m_{r+1-i}$ for any $i = 1, \dots, r$, so that B' satisfies the hypothesis of (2). Thus, we can show this assertion by the inductive hypothesis combined with a similar argument to (1).

In (3), notice $r \ge 5$ by Lemma 5.2.9 (3). Assume $r = 5$. Then we obtain $m_2 = m_4 = 1$ by Lemmas 5.2.9 (2) and 5.2.12 (1). Hence, it follows from Lemma 5.2.12 (2). Assume $r = 7$.

Then we have either $m_2 = 1$ or $m_3 = 1$ by Lemmas 5.2.9 (2) and 5.2.12 (1). If $m_3 = 1$, then it follows from Lemma 5.2.12 (2). Thus, it is enough to consider the case of $m_2 = 1$. Since $\widetilde{D}$ can be contracted to the twig $[m_1 - 1, m_3 - 1, m_4, m_5 - 1, m_7 - 1]$, we obtain $m_1 = m_7 = 3$, $m_3 = m_5 = 2$ and $m_4 = 3$ by Lemmas 5.2.9 (2) and 5.2.10 (2) combined with the result of $r = 5$. Namely, $[m_1, \ldots, m_7] = [3, 1, 2, 3, 2, 1, 3] = [L(3; 0), {}^t(\underline{[2]}^*), [1], {}^t[2], [3], [2], [1], \underline{[2]}^*, R(3; 0)]$, where note $\underline{[2]}^* = \emptyset$ and $m_{[2]} = 3$ (see Definition 5.2.6). Assume $r > 7$. We put $e' := r + 1 - e$ for simplicity. If $e = r' - 1$, then it follows from Lemma 5.2.12 (2). Thus, it is enough to consider the case of $e < r' - 1$. Namely, $e' - e > 1$. Letting B' be the weighted dual graph of the contraction of $\widetilde{C}_e + \widetilde{C}_{e'}$ on $\widetilde{D}$ is a non-admissible twig, we see that B' satisfies the hypothesis of (3) by a similar argument to (2). If $B' = [L(1; r' - 3), [r - 4], R(1; r' - 3)]$, then $[m_1, \ldots, m_r] = [L(3; r' - 4), [1], [2], [r - 4], [2], [1], R(3; r' - 4)]$ by the assumption $e < r' - 1$, where we note $r - 4 = 2(r' - 4) + 3$. In what follows, we assume that $B' = [L(m_{A'}; t), {}^t(\underline{A'^*}), [1], {}^tA', [2t + 3], A', [1], \underline{A'^*}, R(m_{A'}; t)]$ for some admissible twig A', where t is a non-negative integer. Hence, the twig $[m_{r'}, \ldots, m_{r-t-1}]$ can be contracted to $[[2t+3], A', [1], \underline{A'^*}]$. Since $[[2t+3], A', [1], \underline{A'^*}]$ can be contracted to $[2t+3, 1]$ by Lemma 5.2.5, so is $[m_{r'}, \ldots, m_{r-t-1}]$. Hence, by using Lemma 5.2.5 again, we obtain $[m_{r'}, \ldots, m_{r-t-1}] = [[2t + 3], A, [1], \underline{A^*}]$, where $A := [m_{r'+1}, \ldots, m_{e'-1}]$. Meanwhile, since the twig $[m_{r'}, \ldots, m_r] = [[2t+3], A, [1], \underline{A^*}, [m_{r-t}, \ldots, m_r]]$ can be contracted to $[[2t + 3, 1], R(2; t)]$, we know $[m_{r-t}, \ldots, m_r] = R(m_A; t)$ (cf. Remark 5.2.7). By symmetry, we thus obtain $[m_1, \ldots, m_r] = [L(m_A; t), {}^t(\underline{A^*}), [1], {}^tA, [2t + 3], A, [1], \underline{A^*}, R(m_A; t)]$. □

By Proposition 5.2.13 combined with Propositions 5.2.4 and 5.2.5, if $\widetilde{D}$ satisfies the assumptions of Proposition 5.2.13 (1) (resp. (2), (3)), then we can take the birational morphism $\nu : \widetilde{V} \to \check{V}$ with $\widetilde{V} \backslash \mathrm{Supp}(\widetilde{D}) \simeq \check{V} \backslash \mathrm{Supp}(\check{D}) \simeq \mathbb{A}^2_{\overline{k}}$ such that $\check{V} \simeq \mathbb{F}_m$ for some $m \geq 2$ (resp. $\check{V} \simeq \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$, $\check{V} \simeq \mathbb{P}^2_{\overline{k}}$) and the weighted dual graph of $\check{D}$ is the twig $[0, m]$ (resp. $[0, 0]$, $[-1]$), where $\check{D} := \nu_*(\widetilde{D})$.

5.3 Proof of Theorem 1.3.12 (1) and (2)

Let (S, Δ) be an lc compactification of the affine plane $\mathbb{A}^2_k$ over k (see Definition 5.1.2, for this definition). Let $\{C_i\}_{1 \leq i \leq n}$ be all irreducible components of the divisor $\Delta_{\overline{k}}$ on $S_{\overline{k}}$. By Lemma 5.1.1, we see that $n = \rho_{\overline{k}}(S_{\overline{k}})$, and $C_1, \ldots, C_n$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Let $\sigma : \widetilde{S} \to S$ be the minimal resolution defined over k, let $\widetilde{\Delta}$ be the divisor on $\widetilde{S}$ defined by $\widetilde{\Delta} := \sigma^*(\Delta)_{\mathrm{red.}}$ and let $\widetilde{C}_i$ be the proper transform of C_i by $\sigma_{\overline{k}}$ for $i = 1, \ldots, n$. Let $\mu : \hat{S} \to \widetilde{S}$ be the composite of the shortest sequence of blow-ups such that $\hat{\Delta}_{\overline{k}}$ is an SNC-divisor (see also [45, Lemma 4.1]), where $\hat{\Delta} := \mu^*(\widetilde{\Delta})_{\mathrm{red.}}$. Notice that μ is defined over k. Indeed, μ is a composite of some blow-ups of a $\mathrm{Gal}(\overline{k}/k)$-orbit of one point. Hence, the compactification $(\hat{S}, \hat{\Delta})$ of the affine plane is defined over k. Now, let $\hat{\Delta}_\mu$ be the reduced exceptional divisor of $\mu_{\overline{k}}$, let $\hat{\Delta}_\sigma$ be the proper transform of the reduced exceptional divisor of $\sigma_{\overline{k}}$ by $\mu_{\overline{k}}$, and let $\hat{C}_i$ be the proper transform of $\widetilde{C}_i$ by $\mu_{\overline{k}}$ for $i = 1, \ldots, n$. Namely, $\hat{\Delta}_{\overline{k}} = \hat{\Delta}_\mu + \hat{\Delta}_\sigma + \sum_{i=1}^n \hat{C}_i$.

With the same notation as above, the purpose of this section is to prove Theorem 1.3.12 (1) and (2). For these assertions, the case of $n = 1$ is mostly based on the argument of [42, 45], on the other hand, in order to deal with the case of $n \geq 2$, we need to observe the behavior of the Galois group $\mathrm{Gal}(\overline{k}/k)$ acting naturally on $\hat{S}_{\overline{k}}$. Hence, we shall treat such observations in Subsection 5.3.1, and we shall show Theorem 1.3.12 (1) and (2) in Subsections 5.3.2 and 5.3.3, respectively.

5.3.1 Some observations in case of $n \geq 2$

Let the notation be the same as at beginning Section 5.3, assume further that $n \geq 2$.

By using Lemma 5.1.5 and construction of μ, the following lemma is obvious:

Lemma 5.3.1. With the notation and assumptions as above, the following two assertions hold:

(1) Any irreducible component of $\hat{\Delta}_\sigma$ (resp. $\hat{\Delta}_\mu$) has self-intersection number ≤ -2 (resp. ≤ -1).

(2) Assume that $\mu \neq id$. Then, $\hat{\Delta}_\mu$ contains at least one (-1)-curve. Moreover, any (-1)-curve $\hat{E}$ on $\hat{\Delta}_\mu$ does not meet any (-1)-curve on $\hat{\Delta}_\mu$, and satisfies both $(\hat{E} \cdot \hat{\Delta} - \hat{E}) \geq 3$ and $\sum_{i=1}^n (\hat{E} \cdot \hat{C}_i) > 0$.

By considering the behavior of the $\mathrm{Gal}(\overline{k}/k)$-action, we obtain the following lemma:

Lemma 5.3.2. With the notation and assumptions as above, $(\hat{S}_{\overline{k}}, \hat{\Delta}_{\overline{k}})$ is not a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$. In particular, $\hat{C}_i$ is a (-1)-curve with $(\hat{C}_i \cdot \hat{\Delta} - \hat{C}_i) \leq 2$ for $i = 1, \ldots, n$.

Proof. Suppose that $(\hat{S}_{\overline{k}}, \hat{\Delta}_{\overline{k}})$ is a minimal normal compactification of the affine plane. By virtue of $n \geq 2$ and $\mathrm{Sing}(S_{\overline{k}}) \neq \emptyset$, we have $\sharp \hat{\Delta}_{\overline{k}} \geq 3$. By Lemma 5.1.5 (4), there exist two irreducible components Γ_0 and Γ_+ on $\hat{\Delta}_{\overline{k}}$ such that $(\Gamma_0)^2 = 0$ and $(\Gamma_+)^2 > 0$. By Lemma 5.3.1 (1), Γ_0 and Γ_+ are contained in $\sum_{i=1}^n \hat{C}_i$. However, we see $(\hat{C}_1)^2 = \cdots = (\hat{C}_n)^2$ since $\hat{C}_1, \ldots, \hat{C}_n$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. It is a contradiction. □

Since $\Delta_{\overline{k}}$ has no cycle by Lemma 2.5.3, $C_1, \ldots, C_n$ meet only one point, say p_0. Then we obtain the following lemma:

Lemma 5.3.3. Let the notation and the assumptions be the same as above. If $n = 2$, then p_0 is a singular point on $S_{\overline{k}}$.

Proof. Suppose that p_0 is a non-singular point on $S_{\overline{k}}$. Then we obtain $\sharp \hat{\Delta} \geq 4$ since $S_{\overline{k}}$ has at least two singular points by Lemma 2.5.3. Moreover, we see $(\hat{C}_1 \cdot \hat{C}_2) = 1$. On the other hand, $\hat{C}_1$ and $\hat{C}_2$ are (-1)-curves by Lemma 5.3.2. It is a contradiction to Lemma 5.2.11. □

5.3.2 Properties of boundary divisors

Let the notation be the same as at beginning Section 5.3. In this subsection, we prove Theorem 1.3.12 (1). Since the case of $n = 1$ follows from [42, 45], we shall only treat case of $n \geq 2$.

Now, we prepare the following lemma:

Lemma 5.3.4. Let $(\hat{V}, \hat{D})$ be a compactification of the affine plane $\mathbb{A}^2_{\overline{k}}$ over $\overline{k}$ such that $\hat{V}$ is a smooth projective surface over $\overline{k}$ and $\hat{D}$ is an SNC-divisor on $\hat{V}$, and let $\hat{E}_0, \hat{E}_1, \ldots, \hat{E}_r$ be all (-1)-curves in $\mathrm{Supp}(\hat{D})$. Assume that $r \geq 1$, $(\hat{E}_0 \cdot \hat{E}_i) = 1$ for $i = 1, \ldots, r$ and the union $\sum_{i=1}^r \hat{E}_i$ is disjoint. Then $(\hat{E}_0 \cdot \hat{D} - \hat{E}_0) \leq 2$.

Proof. Suppose that $(\hat{E}_0 \cdot \hat{D} - \hat{E}_0) \geq 3$. Let $\nu : (\hat{V}, \hat{D}) \to (\check{V}, \check{D})$ be a sequence of contractions of (-1)-curves and subsequently (smoothly) contractible curves in $\mathrm{Supp}(\hat{D})$, starting with the contraction of $\sum_{i=1}^r \hat{E}_i$, such that the pair $(\check{V}, \check{D})$ is a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$, where $\check{D} := \nu_*(\hat{D})$. Putting $\check{E} := \nu_*(\hat{E}_0)$, we notice $\check{E} \neq 0$, moreover, $(\check{E} \cdot \check{D} - \check{E}) \leq 2$

and $(\check{E})^2 \geq 0$ by Lemma 5.1.5 (1). Since $\hat{D}$ has no cycle by Lemma 2.5.3, we know that any irreducible component of $\check{D} - \check{E}$ has self-intersection number ≤ -1 (if it exists at all). Thus, we have $\sharp\check{D} \leq 2$ by Lemma 5.1.5 (4). If $\sharp\check{D} = 2$, we obtain $(\check{E})^2 = 0$ by Lemma 5.1.5 (3) and $r+1 \geq (\hat{E}_0 \cdot \hat{D} - \hat{E}_0) \geq 3$ by the assumption. However, we then have $0 = (\check{E})^2 \geq (\hat{E}_0)^2 + r \geq 1$, which is absurd. If $\sharp\check{D} = 1$, we obtain $(\check{E})^2 = 1$ by Lemma 5.1.5 (2) and $r = (\hat{E}_0 \cdot \hat{D} - \hat{E}_0) \geq 3$ by the assumption. However, we then have $1 = (\check{E})^2 \geq (\hat{E}_0)^2 + r \geq 2$, which is absurd. □

In order to prove Theorem 1.3.12 (1), suppose on the contrary that $\mu \neq id$. Then we have:

Claim 5.3.5. The union $\sum_{i=1}^n \hat{C}_i$ is disjoint.

Proof. Suppose that $\hat{C}_i$ and $\hat{C}_j$ meet at a point, say $\hat{q}$, for some i and j with $i \neq j$. Since $\hat{\Delta}_{\overline{k}}$ is an SNC-divisor, any irreducible component of $\hat{\Delta}_{\overline{k}}$ passing through $\hat{q}$ is only $\hat{C}_i$ or $\hat{C}_j$. Hence, $q := (\sigma_{\overline{k}} \circ \mu_{\overline{k}})(\hat{q})$ is a smooth point on $S_{\overline{k}}$, moreover, C_i and C_j pass through q. Let p_0 be the intersection point of $C_1, \ldots, C_n$ on $S_{\overline{k}}$. By noting Lemma 5.3.3, we see $p_0 \neq q$. However, since C_i and C_j also pass through p_0, we see that $C_i + C_j$ is a cycle on $\Delta_{\overline{k}}$. It is a contradiction to Lemma 2.5.3. □

Proof of Theorem 1.3.12 (1). By Lemma 5.3.2 and Claim 5.3.5, we see that $\sum_{i=1}^n \hat{C}_i$ can be contracted. Let $\nu : (\hat{S}_{\overline{k}}, \hat{\Delta}_{\overline{k}}) \to (\check{S}, \check{\Delta})$ be a sequence of contractions of (-1)-curves and subsequently (smoothly) contractible curves in $\mathrm{Supp}(\hat{\Delta})$ over $\overline{k}$, starting with the contraction of $\sum_{i=1}^n \hat{C}_i$, such that the pair $(\check{S}, \check{\Delta})$ is a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$, where $\check{\Delta} := \nu_*(\hat{\Delta})$. For any (-1)-curve $\hat{E}$ on $\hat{\Delta}_\mu$, we have $(\hat{E} \cdot \hat{D} - \hat{E}) \geq 3$ by Lemma 5.3.1 (2), in particular, $\nu_*(\hat{E}) \neq 0$ and $(\nu_*(\hat{E}))^2 \geq 0$. Hence, there exist at most two (-1)-curves on $\hat{\Delta}_\mu$ by Lemma 5.1.5. By Lemma 5.3.4, we further see that there exist exactly two (-1)-curves $\hat{E}_1$ and $\hat{E}_2$ on $\hat{\Delta}_\mu$, where we note that $\hat{E}_1$ and $\hat{E}_2$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. We may assume that $\hat{E}_i$ meets $\hat{C}_i$ for $i = 1, 2$. Let D be the connected component of the reduced exceptional divisor of ν containing $\hat{C}_1$. Letting $\check{E}_i := \nu_*(\hat{E}_i)$ for $i = 1, 2$, we obtain $(\hat{E}_1 \cdot \hat{E}_2) = 0$ and $(\check{E}_1 \cdot \check{E}_2) = 1$ by Lemmas 5.3.1 (2) and 5.1.5, so that $(\hat{E}_2 \cdot D) > 0$. Hence, $\hat{C}_2$ is included in $\mathrm{Supp}(D)$, where we recall the assumption $n \geq 2$. Indeed otherwise, letting $\mathrm{Gal}(\overline{k}/k) \cdot D$ be the $\mathrm{Gal}(\overline{k}/k)$-orbit of D, then $\hat{E}_1 + \hat{E}_2 + \mathrm{Gal}(\overline{k}/k) \cdot D$ has a cycle. However, this is a contradiction to Lemma 2.5.3. Moreover, we know $n = 2$ by the similar argument. This implies $(\check{E}_1 \cdot \check{\Delta} - \check{E}_1) = (\hat{E}_1 \cdot \hat{\Delta} - \hat{E}_1) \geq 3$, which is a contradiction to Lemma 5.1.5 (1). □

Example 5.3.6. Note that the assertion of Theorem 1.3.12 (1) is not always true unless $\rho_k(S) = 1$. We shall construct an example of the lc compactification (S, Δ) of $\mathbb{A}^2_k$ such that $\rho_k(S) > 1$ and $\widetilde{\Delta}_{\overline{k}}$ is not an SNC-divisor. Let C be a cubic curve with a cusp o on $\mathbb{P}^2_k$ and let L be the Zariski tangent line to C at o, i.e., $C_{\overline{k}} \cap L_{\overline{k}} = \{o\}$. By construction, $L \simeq \mathbb{P}^1_k$. Let x_1 be a k-rational point on $C_{\overline{k}} \backslash \{o\}$ and let x_2, x_3, x_4 be three points, whose union $x_1 + x_2 + x_3$ is defined over k, on $L_{\overline{k}} \backslash \{o\}$. Letting $\nu : \widetilde{S} \to \mathbb{P}^2_k$ be a blow-up at four points $x_1, \ldots, x_4$ defined over k, then $\widetilde{S}$ is a weak del Pezzo surface of degree 5 such that $\widetilde{S}_{\overline{k}}$ contains exactly one (-2)-curve $\nu_*^{-1}(L_{\overline{k}})$, which is clearly defined over k. Hence, we obtain a contraction $\sigma : \widetilde{S} \to S$ of the (-2)-curve over k, so that S is a Du Val del Pezzo surface with $\rho_k(S) > 1$ over k. Now, ν can be factorized $\nu' : \widetilde{S} \to \widetilde{S}'$ and $\nu'' : \widetilde{S}' \to \mathbb{P}^2_k$ defined over k such that ν'' is a blow-up at a point x_1. Let $\widetilde{\Delta}'$ be the proper transform of $C + L$ by ν'', and let $\widetilde{\Delta}$ be the reduced effective divisor on $\widetilde{S}$ defined by $\widetilde{\Delta} := \nu'^*(\widetilde{\Delta}')_{\mathrm{red.}}$. Since $(\widetilde{S}', \widetilde{\Delta}')$ is a compactification of $\mathbb{A}^2_k$ ([7]), so are $(\widetilde{S}, \widetilde{\Delta})$ and (S, Δ), where $\Delta := \sigma_*(\widetilde{\Delta})$. However, $\widetilde{\Delta}_{\overline{k}}$ is not an SNC-divisor by construction.

5.3.3 Properties of singularities

In this subsection, we prove Theorem 1.3.12 (2) by using results in Subsections 5.2.1 and 5.2.2. Let the notation be the same as above.

Proof of Theorem 1.3.12 (2)(i). We shall consider two cases whether $n = 1$ or not separately.

In the case of $n = 1$, then Δ is geometrically irreducible on S. Namely, $\Delta_{\overline{k}} = C_1$. Hence, we see $\sharp\mathrm{Sing}(S_{\overline{k}}) \leq 2$ by Lemma 5.1.3. If $\sharp\mathrm{Sing}(S_{\overline{k}}) = 1$, then the assertion is clearly true. If $\sharp\mathrm{Sing}(S_{\overline{k}}) = 2$, then two weighted dual graphs given by the minimal resolution at these singular points on $S_{\overline{k}}$ are different (see [42, 45]). Hence, two singular points on $S_{\overline{k}}$ are k-rational.

In the case of $n \geq 2$, let p_0 be the intersection point of $C_1, \ldots, C_n$ on $S_{\overline{k}}$, so that p_0 is k-rational. If $n = 2$, then p_0 is a singular point on $S_{\overline{k}}$ by Lemma 5.3.3. Moreover, p_0 is also a singular point on $S_{\overline{k}}$ even if $n \geq 3$. Indeed, otherwise, the divisor $\widetilde{\Delta}_{\overline{k}}$ is not normal crossing at the point $\widetilde{p}_0 := \sigma^{-1}(p_0)$, which is a contradiction to Theorem 1.3.12 (1). $\square$

Proof of Theorem 1.3.12 (2)(iii). If $n = 1$, then it follows from Lemma 5.1.3. Indeed, $n = \rho_{\overline{k}}(S_{\overline{k}})$ by Lemma 5.1.1. Hence, we assume $n \geq 2$ in what follows. Let p_0 be the intersection point of $C_1, \ldots, C_n$ on $S_{\overline{k}}$. Then we notice that p_0 is k-rational and singular on $S_{\overline{k}}$ (see Proof of Theorem 1.3.12 (2)(i)). Suppose $\sharp\mathrm{Sing}(S_{\overline{k}}) > 1$. Noting Lemma 2.5.3, there exist n-times of singular points $p_1, \ldots, p_n$ on $S_{\overline{k}}$, which lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, such that $p_i \in C_i \backslash \{p_0\}$ for $i = 1, \ldots, n$. For each $i = 1, \ldots, n$, there exist two irreducible components meeting $\widetilde{C}_i$ on $\widetilde{\Delta}_{\overline{k}} - \left(\sum_{i=1}^{n} \widetilde{C}_i\right)$ such that the images of these via $\sigma_{\overline{k}}$ are two points p_0 and p_i, respectively. On the other hand, $(\widetilde{C}_i \cdot \widetilde{\Delta} - \widetilde{C}_i) \leq 2$ by Lemma 5.3.2, so that $(\widetilde{C}_i \cdot \widetilde{\Delta} - \widetilde{C}_i) = 2$, which implies $\mathrm{Sing}(S_{\overline{k}}) \cap C_i = \{p_0, p_i\}$. Therefore, $\sharp\mathrm{Sing}(S_{\overline{k}}) = n + 1 = \rho_{\overline{k}}(S_{\overline{k}}) + 1$ by Lemma 5.1.1. This completes the proof. $\square$

Proof of Theorem 1.3.12 (2)(ii). By Theorem 1.3.12 (2)(i), we can take a singular point on $S_{\overline{k}}$, which is k-rational, say p_0. If $\sharp\mathrm{Sing}(S_{\overline{k}}) \leq 2$, we see $\sharp\mathrm{Sing}(S_{\overline{k}}) = \sharp\mathrm{Sing}(S)$. In what follows, we shall treat the case of $\sharp\mathrm{Sing}(S_{\overline{k}}) \geq 3$. Then we know that all singular points except for p_0 on $S_{\overline{k}}$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit (see Proof of Theorem 1.3.12 (2)(iii)). This implies that $\mathrm{Sing}(S) = \{p_0\}$. This completes the proof. $\square$

5.4 Proof of Theorem 1.3.12 (3)

In this section, we will prove Theorem 1.3.12 (3). In other words, we shall classify the weighted dual graphs corresponding to lc compactifications of the affine plane whose boundary divisor is not geometrically irreducible. In fact, if this boundary divisor is geometrically irreducible, the weighted dual graph corresponding to this lc compactification of the affine plane was already classified by [42, 45]. The argument for proving this theorem is similar to [42, 45], however, we need to consider a little technical argument. We firstly prepare the following lemma generalizing Lemma 5.2.10 (2):

Lemma 5.4.1. Let $(\widetilde{V}', \widetilde{D}')$ be a compactification of the affine plane $\mathbb{A}^2_k$ over k such that $\widetilde{V}'$ is a smooth projective surface over k, $\widetilde{D}'_{\overline{k}}$ is an SNC-divisor. Assume that any irreducible component of $\widetilde{D}'_{\overline{k}}$ has self-intersection number ≤ -2 except for exactly four irreducible components E'_1, E'_2, E'_3 and E'_4 such that $(E'_i)^2 = -1$ for $i = 1, \ldots, 4$, $(E'_1 \cdot E'_2) = (E'_3 \cdot E'_4) = 1$, and E_1' and E_4' lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Then the weighted dual graph of $\widetilde{D}'$ is the twig $[1,1,1,1]$ (for this notation, see Section 5.2).

Proof. Let $\nu' : (\widetilde{V}'_{\overline{k}}, \widetilde{D}'_{\overline{k}}) \to (\check{V}', \check{D}')$ be a sequence of contractions of (-1)-curves and subsequently (smoothly) contractible curves in $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ such that the pair $(\check{V}', \check{D}')$ is a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$, where $\check{D}' := \nu'_*(\widetilde{D}'_{\overline{k}})$. Since $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ contains a (-1)-curve, we notice $\nu' \neq id$ by Lemma 5.1.5 (1). Hence, we may assume $(E_1' \cdot \widetilde{D}' - E_1') \le 2$.

Now, we suppose $(E_1' \cdot E_4') = 1$. Then $(E_2' \cdot E_3') = 0$ by Lemma 2.5.3, furthermore, $(E_2' \cdot \widetilde{D}' - E_2') \le 2$ and $(E_3' \cdot \widetilde{D}' - E_3') \le 2$. Indeed, we can assume that ν' starts with E_1' (resp. E_4'), so that $\nu'_*(E_2')$ and $\nu'_*(E_4')$ (resp. $\nu'_*(E_3')$ and $\nu'_*(E_1')$) are curves on $\check{V}'$, which transversely meet each other. By Lemma 2.5.3, we then see that E_2' (resp. E_3') meets at most one irreducible component on $\widetilde{D}' - (E_1' + E_2')$ (resp. $\widetilde{D}' - (E_3' + E_4')$) by Lemma 5.1.5 (1), namely, $(E_2' \cdot \widetilde{D}' - E_2') \le 2$ and $(E_3' \cdot \widetilde{D}' - E_3') \le 2$.

Hence, we may assume $(E_1' \cdot E_4') = 0$ in what follows. Indeed, if $(E_1' \cdot E_4') = 1$, we swap the roles of the pairs (E_1', E_2') and (E_4', E_3'). Moreover, since $\widetilde{D}'_{\overline{k}}$ is connected and has no cycle, we may assume that E_2' and E_3' are included in the same connected component of $\widetilde{D}'_{\overline{k}} - (E_1' + E_4')$. Then we can assume that ν' starts with $E_1' + E_4'$, so that $\nu'_*(E_2')$ and $\nu'_*(E_3')$ are curves on $\check{V}'$ with self-intersection number ≥ 0. Hence, $(\nu'_*(E_2') \cdot \nu'_*(E_3')) = 1$ by Lemma 5.1.5 (3) and (4). Meanwhile, by Lemma 2.5.3, we obtain $(E_2' \cdot E_3') = 1$ by the above assumption. Moreover, by symmetry of the weighted dual graph of $\widetilde{D}'_{\overline{k}}$, we can assume further $(\nu'_*(E_2'))^2 = (\nu'_*(E_3'))^2$. By Lemma 5.1.5 (3) and (4), we then obtain $\sharp \check{D}' = 2$ and $(\nu'_*(E_2'))^2 = (\nu'_*(E_3'))^2 = 0$. This implies that the weighted dual graph of $\widetilde{D}'_{\overline{k}}$ is the twig $[1,1,1,1]$. □

The following result will play an important role in the proof of Theorem 1.3.12 (3):

Lemma 5.4.2. Let $(\widetilde{V}, \widetilde{D})$ be a compactification of the affine plane $\mathbb{A}^2_k$ over k such that $\widetilde{V}$ is smooth and $\widetilde{D}_{\overline{k}}$ is an SNC-divisor. Assume that $(\widetilde{V}_{\overline{k}}, \widetilde{D}_{\overline{k}})$ is not a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$. Moreover, letting $\widetilde{E}_1, \ldots, \widetilde{E}_r$ be all (-1)-curves on $\mathrm{Supp}(\widetilde{D}_{\overline{k}})$, assume further that they lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit and the union $\widetilde{E} := \sum_{i=1}^r \widetilde{E}_i$ is disjoint. Hence, we obtain the contraction $\nu' : (\widetilde{V}, \widetilde{D}) \to (\widetilde{V}', \widetilde{D}')$ of $\widetilde{E}$ defined over k by Lemma 5.1.5 (1). Then one of the following three situations holds:

(1) $(\widetilde{V}'_{\overline{k}}, \widetilde{D}'_{\overline{k}})$ is a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$.
(2) The weighted dual graph of $\widetilde{D}'_{\overline{k}}$ is the twig either $[1,1,m]$ for some $m \ge 1$ or $[1,1,1,1]$.
(3) Letting $\widetilde{E}_1', \ldots, \widetilde{E}_{r'}'$ be all (-1)-curves on $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$, they lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit and the union $\sum_{i=1}^{r'} \widetilde{E}_i'$ is disjoint.

Proof. Letting $x_i := \nu'_{\overline{k}}(E_i)$ for $i = 1, \ldots, r$, we see that $x_1, \ldots, x_r$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit by the assumption. Moreover, x_1 lies in at most two (-1)-curves in $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ because of $(\widetilde{E}_1 \cdot \widetilde{D} - \widetilde{E}_1) \le 2$.

Assume that x_1 lies in no (-1)-curve in $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$. Then $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ contains no (-1)-curve. Hence, $(\widetilde{V}'_{\overline{k}}, \widetilde{D}'_{\overline{k}})$ is a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$.

Assume that x_1 lies in exactly two (-1)-curves, say $\widetilde{E}_1'$ and $\widetilde{E}_2'$, in $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$. At first, we consider the case that $\widetilde{E}_1'$ is defined over k. Then we see $x_1, \ldots, x_r \in \widetilde{E}_1'$ since these points lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Thus, we obtain $(\widetilde{E}_1' \cdot \widetilde{D}' - \widetilde{E}_1') \le 2$ by Lemma 5.3.4. In particular, $r \le 2$. If $r = 1$, then $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ contains only two (-1)-curves $\widetilde{E}_1'$ and $\widetilde{E}_2'$. Hence, the weighted dual graph of $\widetilde{D}'_{\overline{k}}$ is the twig $[1,1,m]$ for some $m \geq 2$ by Lemma 5.2.11. If $r = 2$, letting $\nu'' : \widetilde{V}'_{\overline{k}} \to \widetilde{V}''$ be the contraction of $\widetilde{E}'_1$ over $\overline{k}$, then $(\widetilde{V}'', \widetilde{D}'')$ is a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$. Moreover, $\mathrm{Supp}(\widetilde{D}'')$ contains exactly two 0-curves, so that the weighted dual graph of $\widetilde{D}''$ is the twig $[0,0]$ by Lemma 5.1.5 (4). This implies that the weighted dual graph of $\widetilde{D}'_{\overline{k}}$ is the twig $[1,1,1]$. Next, we consider the case that $\widetilde{E}'_1$ and $\widetilde{E}'_2$ are not defined over k. Since $(\widetilde{V}'_{\overline{k}}, \widetilde{D}'_{\overline{k}})$ is not a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$, we have either $(\widetilde{E}'_1 \cdot \widetilde{D}' - \widetilde{E}'_1) \leq 2$ or $(\widetilde{E}'_2 \cdot \widetilde{D}' - \widetilde{E}'_2) \leq 2$. Moreover, each (-1)-curve on $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ is included in the $\mathrm{Gal}(\overline{k}/k)$-orbit of either $\widetilde{E}'_1$ or $\widetilde{E}'_2$ since $x_1, \ldots, x_r$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Suppose that $x_2 \in \widetilde{E}'_1$. Then there exists a (-1)-curve $\widetilde{E}'_3$ other than $\widetilde{E}'_2$ in $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ meeting $\widetilde{E}'_1$ such that $\widetilde{E}'_2$ and $\widetilde{E}'_3$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Since $\widetilde{E}'_1$ is not defined over k, there exists a (-1)-curve $\widetilde{E}'_4$ other than $\widetilde{E}'_1$ in $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ lying in the $\mathrm{Gal}(\overline{k}/k)$-orbit of $\widetilde{E}'_1$. Moreover, there exists a (-1)-curves $\widetilde{E}'_5$ in $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ lying in the $\mathrm{Gal}(\overline{k}/k)$-orbit of $\widetilde{E}'_2$ such that $(\widetilde{E}'_4 \cdot \widetilde{E}'_5) = 1$ and $\widetilde{E}'_5 \neq \widetilde{E}'_2, \widetilde{E}'_3$. If $(\widetilde{E}'_1 \cdot \widetilde{E}'_4) = 0$, then the direct images of $\widetilde{E}'_2$, $\widetilde{E}'_3$ and $\widetilde{E}'_5$ by the contraction of $\widetilde{E}'_1 + \widetilde{E}'_4$ have self-intersection number ≥ 0. It contradicts Lemma 5.1.5 (4). If $(\widetilde{E}'_1 \cdot \widetilde{E}'_4) = 1$, since $(\widetilde{E}'_2 \cdot \widetilde{D}'_{\overline{k}} - \widetilde{E}'_2) \leq 2$ and $\widetilde{D}_{\overline{k}}$ has no cycle, there exists a (-1)-curve $\widetilde{E}'_6$ other than $\widetilde{E}'_5$ meeting $\widetilde{E}'_4$ in $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ and lying in the $\mathrm{Gal}(\overline{k}/k)$-orbit of $\widetilde{E}'_2$, moreover, $\widetilde{E}'_6 \neq \widetilde{E}'_2, \widetilde{E}'_3$ and the union $\widetilde{E}'_2 + \widetilde{E}'_3 + \widetilde{E}'_5 + \widetilde{E}'_6$ is disjoint. Then the direct images of $\widetilde{E}'_1$ and $\widetilde{E}'_4$ by the contraction of $\widetilde{E}'_2 + \widetilde{E}'_3 + \widetilde{E}'_5 + \widetilde{E}'_6$ are two 1-curves. It contradicts Lemma 5.1.5 (3) and (4). Hence, $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ contains exactly $2r$-times of (-1)-curves $\widetilde{E}'_1, \ldots, \widetilde{E}'_{2r}$ such that x_i lies on $\widetilde{E}'_{2i-1}$ and $\widetilde{E}'_{2i}$ for $i = 1, \ldots, r$. Then we note $r \neq 1$. Otherwise, $\widetilde{E}'_1$ and $\widetilde{E}'_2$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, however, this is impossible by Lemma 5.2.11. Meanwhile, suppose $r \geq 3$. Then we may assume $(\widetilde{E}'_1 \cdot \widetilde{D}'_{\overline{k}} - \widetilde{E}'_1) \leq 2$, and $\widetilde{E}'_1$, $\widetilde{E}'_3$ and $\widetilde{E}'_5$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. If the union $\widetilde{E}'_1 + \widetilde{E}'_3 + \widetilde{E}'_5$ is disjoint, then the direct image of $\widetilde{D}'_{\overline{k}}$ by the contraction of $\widetilde{E}'_1 + \widetilde{E}'_3 + \widetilde{E}'_5$ contains three 0-curves. It contradicts Lemma 5.1.5 (4). Otherwise, we may assume $(\widetilde{E}'_1 \cdot \widetilde{E}'_3) = 1$. Then $(\widetilde{E}'_1 \cdot \widetilde{E}'_5) = 0$ since $\widetilde{E}'_1$ meets only $\widetilde{E}'_2$ and $\widetilde{E}'_3$ on $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$, so that the direct image of $\widetilde{D}'_{\overline{k}}$ by the contraction of $\widetilde{E}'_1 + \widetilde{E}'_5$ contains three 0-curves. It contradicts Lemma 5.1.5 (4). Hence, we obtain $r = 2$. Thus, the weighted dual graph of $\widetilde{D}'_{\overline{k}}$ is the twig $[1,1,1,1]$ by Lemma 5.4.1.

Assume that x_1 lies in exactly one (-1)-curve in $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$. Let $\widetilde{E}'_1, \ldots, \widetilde{E}'_{r'}$ be all (-1)-curves on $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$. Then each x_i lies in exactly one (-1)-curve in $\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$ for $i = 1, \ldots, r$, so that $\widetilde{E}'_1, \ldots, \widetilde{E}'_{r'}$ lie the same $\mathrm{Gal}(\overline{k}/k)$-orbit. In what follows, we will show that the union $\sum_{i=1}^{r'} \widetilde{E}'_i$ is disjoint. Suppose on the contrary that $\widetilde{E}'_1$ and $\widetilde{E}'_2$ transversely meet at a point, say x. By the assumption, we obtain $x \neq x_i$ for any $i = 1, \ldots, r$. On the other hand, we obtain $r' > 2$. Indeed, otherwise the weighted dual graph of $\widetilde{D}'_{\overline{k}}$ is the twig $[1,1,m]$ for some $m \geq 2$ by Lemma 5.2.11, however, it contradicts that $\widetilde{E}'_1$ and $\widetilde{E}'_2$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Noticing that the $\mathrm{Gal}(\overline{k}/k)$-orbit of $\widetilde{E}'_1 + \widetilde{E}'_2$ has no cycle by Lemma 2.5.3, we thus have $r' \geq 4$ and the union $\sum_{i=1}^{r'} \widetilde{E}'_i$ is not connected. Hence, we may assume $(\widetilde{E}'_3 \cdot \widetilde{E}'_4) = 1$. Now, note $(\widetilde{E}'_1 + \widetilde{E}'_2 \cdot \widetilde{E}'_3 + \widetilde{E}'_4) = 0$. On the other hand, since $\widetilde{D}'_{\overline{k}}$ is connected, we may assume that there exists a connected divisor $\widetilde{D}'_{1,3}$ on $\widetilde{V}'_{\overline{k}}$ such that $\mathrm{Supp}(\widetilde{E}'_1 + \widetilde{E}'_3) \subseteq \mathrm{Supp}(\widetilde{D}'_{1,3}) \subseteq \mathrm{Supp}(\widetilde{D}'_{\overline{k}})$. Moreover, since $\widetilde{E}'_1 + \widetilde{E}'_2$ and $\widetilde{E}'_3 + \widetilde{E}'_4$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, there exists a connected divisor $\widetilde{D}'_{2,4}$ lying the $\mathrm{Gal}(\overline{k}/k)$-orbit of $\widetilde{D}'_{1,3}$ such that $\mathrm{Supp}(\widetilde{E}'_2 + \widetilde{E}'_4) \subseteq \mathrm{Supp}(\widetilde{D}'_{2,4}) \subseteq$

$\mathrm{Supp}(\widetilde{D}'_{\overline{k}})$. Then $\widetilde{D}'_{1,3} + \widetilde{D}'_{2,4}$ has a cycle by construction. This contradicts Lemma 2.5.3. Thus, $\sum_{i=1}^{r'} \widetilde{E}'_i$ is a disjoint union.

This completes the proof. $\square$

Now, let (S, Δ) be an lc compactification of the affine plane $\mathbb{A}^2_k$ over k (see Definition 5.1.2, for this definition) such that $\rho_{\overline{k}}(S_{\overline{k}}) > 1$, and let $\{C_i\}_{1 \le i \le n}$ be all irreducible components of the divisor $\Delta_{\overline{k}}$ on $S_{\overline{k}}$. By Lemma 5.1.1, note that $n = \sharp \Delta_{\overline{k}}$, and $C_1, \dots, C_n$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Hence, $n \ge 2$ by the assumption. Let $\sigma : \widetilde{S} \to S$ be the minimal resolution over k and let $\widetilde{\Delta}$ be the divisor on $\widetilde{S}$ defined by $\widetilde{\Delta} := \sigma^*(\Delta)_{\mathrm{red.}}$. Notice that $\widetilde{\Delta}_{\overline{k}}$ is an SNC-divisor by Theorem 1.3.12 (1).

Let $\widetilde{C}_i$ be the proper transform of C_i by $\sigma_{\overline{k}}$. By Lemma 5.3.2, any $\widetilde{C}_i$ is a (-1)-curve on $\widetilde{S}_{\overline{k}}$, furthermore, we obtain a contraction $\mathrm{cont}_{E_1} : (\widetilde{S}_1, \widetilde{\Delta}_1) := (\widetilde{S}_{\overline{k}}, \widetilde{\Delta}_{\overline{k}}) \to (\widetilde{S}_2, \widetilde{\Delta}_2 := \mathrm{cont}_{E_1,*}(\widetilde{\Delta}_1))$ of $E_1 := \sum_{i=1}^n \widetilde{C}_i$. If $(\widetilde{S}_{2,\overline{k}}, \widetilde{\Delta}_{2,\overline{k}})$ satisfies the situation (1) in Lemma 5.4.2, then we put $\nu := \mathrm{cont}_{E_1}$. If $(\widetilde{S}_{2,\overline{k}}, \widetilde{\Delta}_{2,\overline{k}})$ satisfies the situation (2) in Lemma 5.4.2, let E_2 be the union of (-1)-curves, which are all terminal component of $\widetilde{\Delta}_{2,\overline{k}}$, and let $\mathrm{cont}_{E_2} : (\widetilde{S}_2, \widetilde{\Delta}_2) \to (\widetilde{S}_3, \widetilde{\Delta}_3 := \mathrm{cont}_{E_2,*}(\widetilde{\Delta}_2))$ be the contraction of E_2, which is defined over k, hence, we put $\nu := \mathrm{cont}_{E_1} \circ \mathrm{cont}_{E_2}$. Otherwise, by using Lemma 5.4.2 repeatably, we can construct a sequence of contractions $\mathrm{cont}_{E_i} \circ \dots \circ \mathrm{cont}_{E_1} : (\widetilde{S}_{\overline{k}}, \widetilde{\Delta}_{\overline{k}}) = (\widetilde{S}_1, \widetilde{\Delta}_1) \to (\widetilde{S}_2, \widetilde{\Delta}_2 = \mathrm{cont}_{E_1,*}(\widetilde{\Delta}_1)) \to \dots \to (\widetilde{S}_{i+1}, \widetilde{\Delta}_{i+1} := \mathrm{cont}_{E_i,*}(\widetilde{\Delta}_i))$ of $\mathrm{Gal}(\overline{k}/k)$-orbits of (-1)-curves over k such that $(\widetilde{S}_{i+1,\overline{k}}, \widetilde{\Delta}_{i+1,\overline{k}})$ satisfies either situation (1) or (2) in Lemma 5.4.2, where E_j is the disjoint union of all (-1)-curves in $\mathrm{Supp}(\widetilde{\Delta}_{j,\overline{k}})$ defined over k for $j = 1, \dots, i$. Hence, we obtain a sequence $\nu = \mathrm{cont}_{E_\ell} \circ \dots \circ \mathrm{cont}_{E_1} : (\widetilde{S}_{\overline{k}}, \widetilde{\Delta}_{\overline{k}}) = (\widetilde{S}_1, \widetilde{\Delta}_1) \to (\widetilde{S}_2, \widetilde{\Delta}_2) \to \dots \to (\widetilde{S}_{\ell+1}, \widetilde{\Delta}_{\ell+1}) =: (\check{S}, \check{\Delta})$ of contractions of $\mathrm{Gal}(\overline{k}/k)$-orbit of (-1)-curves and subsequently (smoothly) contractible curves in $\mathrm{Supp}(\widetilde{\Delta}_{\overline{k}})$ such that the pair $(\check{S}, \check{\Delta})$ is a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$. Notice that ν is defined over k since each cont_{E_i} is defined over k.

By construction of ν, we obtain the following lemma:

Lemma 5.4.3 (cf. [45, Lemma 4.5])**.** With the notation and assumptions as above, then we obtain $\sharp \check{\Delta}_{\overline{k}} \le 2$. Hence, $(\check{S}_{\overline{k}}, \check{\Delta}_{\overline{k}})$ is either $(\mathbb{P}^2_{\overline{k}}, L)$ or $(\mathbb{F}_m, M_m + F)$ for some non-negative integer $m \ne 1$, where L is a line on $\mathbb{P}^2_{\overline{k}}$ and M_m (resp. F) is the minimal section (resp. a fiber) of the structure morphism $\mathbb{F}_m \to \mathbb{P}^1_{\overline{k}}$.

Proof. Suppose that $\sharp \check{\Delta}_{\overline{k}} \ge 3$. Since $(\check{S}_{\overline{k}}, \check{\Delta}_{\overline{k}})$ is a minimal normal compactification of $\mathbb{A}^2_{\overline{k}}$, we see that $\check{\Delta}_{\overline{k}}$ contains two components Γ_0 and Γ_+ such that $(\Gamma_0)^2 = 0$, $(\Gamma_+)^2 > 0$ and $(\Gamma_0 \cdot \Gamma_+) = 1$ by Lemma 5.1.5 (4). Moreover, Γ_0 and Γ_+ are defined over k, respectively. Since ν is defined over k, so is $\nu\left(\sum_{i=1}^n \widetilde{C}_i\right)$. Hence, we know that $(\nu_*^{-1}(\Gamma_0))^2 \ge -1$ or $(\nu_*^{-1}(\Gamma_+))^2 \ge -1$. However, this is a contradiction since any irreducible component of $\widetilde{\Delta}_{\overline{k}} - \sum_{i=1}^n \widetilde{C}_i$ has self-intersection number ≤ -2. $\square$

Let us put $\nu_i := \mathrm{cont}_{E_\ell} \circ \dots \circ \mathrm{cont}_{E_i}$ for $i = 1, \dots, \ell$. By Lemma 5.4.3, we see that $(E_{i,j} \cdot \nu_i^*(\check{\Delta})_{\mathrm{red.}} - E_{i,j})$ is equal to 1 or 2 for any irreducible component $E_{i,j}$ of E_i.

By Theorem 1.3.12 (2)(ii) and Lemma 5.1.1, $S_{\overline{k}}$ contains exactly one singular point p_0, which is k-rational. Moreover, noticing $\sharp \mathrm{Sing}(S_{\overline{k}}) = n + 1$ or 1 by Theorem 1.3.12 (2)(iii), let $p_1, \dots, p_n$ be singular points other than p_0 on $S_{\overline{k}}$ such that $p_i \in C_i$ for $i = 1, \dots, n$ (if it exists) and let $\widetilde{\Delta}^{(i)}$ be the reduced exceptional divisor of the minimal resolution at p_i on $S_{\overline{k}}$ for $i = 0, \dots, n$, where we define $\widetilde{\Delta}^{(i)} := 0$ if p_i does not exist. Namely, $\widetilde{\Delta}_{\overline{k}} = \sum_{i=0}^n \widetilde{\Delta}^{(i)} + \sum_{i=1}^n \widetilde{C}_i$. By the above argument, we obtain the following lemma:

Lemma 5.4.4 (cf. [45, Lemma 4.6])**.** Let the notation and the assumptions be the same as above. For any $i = 1, \ldots, n$, the following three assertions hold:

(1) If $\sharp\mathrm{Sing}(S_{\overline{k}}) = n+1$, then the dual graph of $\widetilde{\Delta}^{(i)}$ is a linear chain.

(2) If $\sharp\mathrm{Sing}(S_{\overline{k}}) = n+1$, then $\widetilde{C}_i$ meets a terminal component of $\widetilde{\Delta}^{(i)}$.

(3) Any irreducible component $\widetilde{\Gamma}_0$ of $\widetilde{\Delta}^{(0)}$ with $(\widetilde{\Gamma}_0 \cdot \widetilde{\Delta}^{(0)} - \widetilde{\Gamma}_0) \geq 3$ does not meet $\widetilde{C}_i$.

Now, the singular point p_0 has the following three possibilities:

(I) p_0 is a cyclic quotient singular point;

(II) p_0 is a non-cyclic quotient singular point;

(III) p_0 is a log canonical but not a quotient singular.

In order to determine the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$, we will consider the above three cases (I)–(III) separately according to the following Subsections 5.4.1–5.4.3.

5.4.1 Case (I): p_0 is a cyclic quotient singularity

Assume that p_0 is a cyclic quotient singular point. Then we will consider the following three Subcases separately:

(I-1) $\widetilde{\Delta}_{\overline{k}}$ is not a linear chain and any irreducible component of $\widetilde{\Delta}^{(0)}$ is defined over k;

(I-2) $\widetilde{\Delta}_{\overline{k}}$ is not a linear chain and there exists an irreducible component of $\widetilde{\Delta}^{(0)}$, which is not defined over k;

(I-3) $\widetilde{\Delta}_{\overline{k}}$ is a linear chain.

Subcase (I-1)

Assume that $\widetilde{\Delta}_{\overline{k}}$ is not a linear chain and any irreducible component of $\widetilde{\Delta}^{(0)}$ is defined over k. Then notice that there exists exactly one irreducible component $\widetilde{\Gamma}$ of $\widetilde{\Delta}^{(0)}$ such that $\sum_{i=1}^{n}(\widetilde{\Gamma}\cdot\widetilde{C}_i) = n$. In particular, we see $(\widetilde{\Gamma}\cdot\widetilde{\Delta}-\widetilde{\Gamma}) \geq 3$. Thus, ν must first repeat the contraction until all irreducible components in $\mathrm{Supp}\left(\sum_{i=1}^{n}\widetilde{\Delta}^{(i)}\right)$ for $i = 1, \ldots, n$ are contracted. In other words, $\widetilde{\Delta}^{(i)}$ is a linear chain consisting entirely of (-2)-curves for $i = 1, \ldots, n$ (see also Lemma 5.4.4). By the above argument combined with Lemma 5.1.5 and Proposition 5.2.13 (1), we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (i) $(i = 1, 2, 3)$ in Appendix A.2, where $n \geq 3$ in the case of (1).

Subcase (I-2)

Assume that $\widetilde{\Delta}_{\overline{k}}$ is not a linear chain and there exists an irreducible component of $\widetilde{\Delta}^{(0)}$ not defined over k. Then there exist exactly two irreducible components $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ of $\widetilde{\Delta}^{(0)}$ such that $(\widetilde{\Gamma}_i \cdot \widetilde{\Delta} - \widetilde{\Gamma}_i) \geq 3$ for $i = 1, 2$ by noting Lemma 5.2.12 (1), in particular, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. By a similar argument to Subcase (I-1), $\widetilde{\Delta}^{(i)}$ is a linear chain consisting entirely of (-2)-curves for $i = 1, \ldots, n$. Hence, by Lemma 5.1.5 and Proposition 5.2.13 (2) and (3), we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (i) $(i = 4, 5, 6, 7)$ in Appendix A.2, where $n' := \frac{n}{2} \geq 2$ in the case of (4).

Subcase (I-3)

Assume that $\widetilde{\Delta}_{\overline{k}}$ is a linear chain. Then we immediately see that $n = 2$ and both $\widetilde{C}_1$ and $\widetilde{C}_2$ meet a distinct terminal component in $\widetilde{\Delta}^{(0)}$, respectively. Thus, by Proposition 5.2.13 (2) and (3), we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (i) $(i = 8, 9, 10)$ in Appendix A.2.

5.4.2 Case (II): p_0 is a non-cyclic quotient singularity

Assume that p_0 is a non-cyclic quotient singular point. In this case, Lemma 5.4.4 and the following lemma play a useful role:

Lemma 5.4.5. Let the notation and the assumptions be the same as above. For any irreducible component $\widetilde{\Gamma}$ of $\widetilde{\Delta}^{(0)}$ satisfying $n' := (\widetilde{\Gamma} \cdot \widetilde{C}_1 + \cdots + \widetilde{C}_n) > 0$, then we have $(\widetilde{\Gamma})^2 < -n'$.

Proof. Suppose that $(\widetilde{\Gamma})^2 \geq -n'$. By the construction of cont_{E_1}, we have $(\mathrm{cont}_{E_1,*}(\widetilde{\Gamma}))^2 \geq -n' + n' = 0$. This means that any irreducible component of $\widetilde{\Delta}^{(0)}$ is not contract by ν. Thus, $\check{\Delta}$ is not a linear chain, which is a contradiction to Lemma 5.1.5 (1). □

Note that the classification of quotient singularities of dimension two is well-known (see Lemma 2.3.9).

By the assumption, $\widetilde{\Delta}^{(0)}$ is not a linear chain, in particular, there is exactly one irreducible component $\widetilde{\Gamma}_0$ of $\widetilde{\Delta}^{(0)}$ such that $(\widetilde{\Gamma}_0 \cdot \widetilde{\Delta}^{(0)} - \widetilde{\Gamma}_0) = 3$. On the other hand, since $\check{\Delta}$ is a linear chain by Lemma 5.1.5 (1), ν is factorized $\nu'' \circ \nu'$ such that $\widetilde{\Gamma}'_0 \neq 0$ and $(\widetilde{\Gamma}'_0 \cdot \widetilde{\Delta}' - \widetilde{\Gamma}'_0) \leq 2$, where $\widetilde{\Gamma}'_0 := \nu'_*(\widetilde{\Gamma}_0)$ and $\widetilde{\Delta}' := \nu'_*(\widetilde{\Delta}_{\overline{k}})$. We can assume that ν' is defined over k since ν is a sequence of contractions of the $\mathrm{Gal}(\overline{k}/k)$-orbit consisting of (-1)-curves and subsequently $\mathrm{Gal}(\overline{k}/k)$-orbits consisting of (smoothly) contractible curves in $\mathrm{Supp}(\widetilde{\Delta}_{\overline{k}})$. In other words, any irreducible component $\widetilde{\Delta}'$ except for $\widetilde{\Gamma}'_0$ has self-intersection number ≤ -2. Hence, ν' is uniquely determined. Now, we will consider the following three Subcases (II-1)–(II-3) separately:

(II-1) $(\widetilde{\Gamma}'_0 \cdot \widetilde{\Delta}' - \widetilde{\Gamma}'_0) = 2$ holds;

(II-2) $(\widetilde{\Gamma}'_0 \cdot \widetilde{\Delta}' - \widetilde{\Gamma}'_0) = 1$ holds;

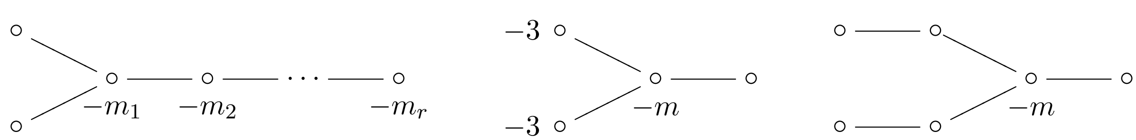
(II-3) $(\widetilde{\Gamma}'_0 \cdot \widetilde{\Delta}' - \widetilde{\Gamma}'_0) = 0$ holds.

Subcase (II-1)

At first, we shall treat the case of $(\widetilde{\Gamma}'_0 \cdot \widetilde{\Delta}' - \widetilde{\Gamma}'_0) = 2$. By virtue of $\sharp\widetilde{\Delta}' \geq 3$ combined with Lemma 5.1.5 (4), we see that $\widetilde{\Gamma}'_0$ is a (-1)-curve. Thus, by Proposition 5.2.13 (1) and Lemmas 5.4.4 and 5.4.5 combined with the classification of quotient singularities of dimension two, we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (i) $(i = 11, \ldots, 17)$ in Appendix A.2.

Subcase (II-2)

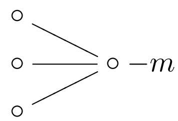
Next, we shall treat the case of $(\widetilde{\Gamma}'_0 \cdot \widetilde{\Delta}' - \widetilde{\Gamma}'_0) = 1$. By using the classification of quotient singularities of dimension two, the weighted dual graph of $\widetilde{\Delta}^{(0)}$ is then one of the following three weighted dual graphs, where m_i and m are integers such that $m_i \geq 2$ and $m \geq 2$:



Note that $\widetilde{\Gamma}'_0$ is not a (-1)-curve by Lemma 5.2.9 (2). Hence, we obtain $\nu' = \nu$. Moreover, we see that $\sharp\widetilde{\Delta}' = 2$ and $\widetilde{\Gamma}'_0$ is a 0-curve by Lemma 5.1.5 (3) and (4). Thus, by Lemmas 5.4.4 and 5.4.5, we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (i) $(i = 18, \ldots, 25)$ in Appendix A.2.

Subcase (II-3)

Finally, we shall treat the case of $(\widetilde{\Gamma}'_0 \cdot \widetilde{\Delta}' - \widetilde{\Gamma}'_0) = 0$. By using the classification of quotient singularities of dimension two, the weighted dual graph of $\widetilde{\Delta}^{(0)}$ is then as follows, where m is an integer with $m \geq 2$:



By the assumption, we see $\nu = \nu'$ and $\sharp\widetilde{\Delta}' = 1$. In particular, $\widetilde{\Gamma}'_0$ is a 1-curve by Lemma 5.1.5 (2). Thus, by Lemmas 5.4.4 and 5.4.5, we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (26) or (27) in Appendix A.2.

5.4.3 Case (III): p_0 is a log canonical but not a quotient singularity

Assume that p_0 is a log canonical but not a quotient singular point. Note that the classification of log canonical singularities of dimension two is known (see Theorem 2.3.11), where it is enough to treat only the rational singularities (cf. [45, Theorem 1.1(2)]). By the classification of rational log canonical but not quotient singularities of dimension two, there exists at least one irreducible component $\widetilde{\Gamma}_0$ of $\widetilde{\Delta}^{(0)}$ satisfying $(\widetilde{\Gamma}_0 \cdot \widetilde{\Delta}^{(0)} - \widetilde{\Gamma}_0) \geq 3$. More precisely, one of the following three Subcases holds:

(III-1) There exists exactly one irreducible component $\widetilde{\Gamma}_0$ of $\widetilde{\Delta}^{(0)}$ satisfying $(\widetilde{\Gamma}_0 \cdot \widetilde{\Delta}^{(0)} - \widetilde{\Gamma}_0) = 4$.

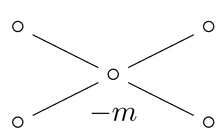
(III-2) There exist exactly two irreducible components $\widetilde{\Gamma}_{0,1}$ and $\widetilde{\Gamma}_{0,2}$ satisfying $(\widetilde{\Gamma}_{0,i} \cdot \widetilde{\Delta}^{(0)} - \widetilde{\Gamma}_{0,i}) = 3$ for $i = 1, 2$.

(III-3) There exists exactly one irreducible component $\widetilde{\Gamma}_0$ of $\widetilde{\Delta}^{(0)}$ satisfying $(\widetilde{\Gamma}_0 \cdot \widetilde{\Delta}^{(0)} - \widetilde{\Gamma}_0) = 3$.

Notice that Lemma 5.4.5 works verbatim for Subcases (III-1)–(III-3). In what follows, we will consider Subcases (III-1)–(III-3) separately.

Subcase (III-1)

Assume that there exists exactly one irreducible component $\widetilde{\Gamma}_0$ of $\widetilde{\Delta}^{(0)}$ satisfying $(\widetilde{\Gamma}_0 \cdot \widetilde{\Delta}^{(0)} - \widetilde{\Gamma}_0) = 4$. Then the weighted dual graph of $\widetilde{\Delta}^{(0)}$ is as follows, where m is an integer with $m > 2$:



By the similar argument to Subsection 5.4.2, ν is factorized $\nu'' \circ \nu'$ such that ν' is as in Subsection 5.4.2. In particular, we can assume that any irreducible component $\widetilde{\Delta}'$ except for $\widetilde{\Gamma}'_0$ has self-intersection number ≤ -2, where $\widetilde{\Delta}' := \nu'_*(\widetilde{\Delta}_{\overline{k}})$ and $\widetilde{\Gamma}'_0 := \nu'_*(\widetilde{\Gamma}_0)$. By Lemma

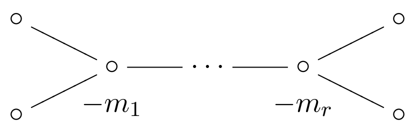
5.4.4 (3), $\widetilde{C}_i$ meets a terminal component of $\widetilde{\Delta}^{(0)}$ for any $i = 1, \ldots, n$. Moreover, $n \leq 4$ by Lemma 5.4.5, in particular, $n \neq 2$ by Lemmas 5.1.5 and 5.2.12 (1). In what follows, we shall consider the two cases of $n = 3$ and $n = 4$ separately.

If $n = 3$, then we see $\nu' = \nu$ and $\sharp\widetilde{\Delta}' = 2$. Moreover, $\widetilde{\Gamma}'_0$ is a 0-curve by Lemma 5.1.5 (3). Therefore, we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (28) or (29) in Appendix A.2.

If $n = 4$, then we see $\nu' = \nu$ and $\sharp\widetilde{\Delta}' = 1$. Moreover, $\widetilde{\Gamma}'_0$ is a 1-curve by Lemma 5.1.5 (2). Therefore, we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (30) or (31) in Appendix A.2.

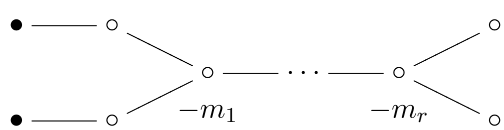
Subcase (III-2)

Assume that there exist exactly two irreducible components $\widetilde{\Gamma}_{0,1}$ and $\widetilde{\Gamma}_{0,2}$ of $\widetilde{\Delta}^{(0)}$ satisfying $(\widetilde{\Gamma}_{0,i} \cdot \widetilde{\Delta}^{(0)} - \widetilde{\Gamma}_{0,i}) = 3$ for $i = 1, 2$. Then the weighted dual graph of $\widetilde{\Delta}^{(0)}$ is as follows, where each m_i is an integer with $m_i \geq 2$ (furthermore, at least one m_i is strictly more than 2 and $r > 1$):

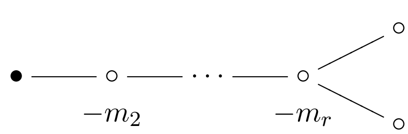


Since $\check{\Delta}$ is a linear chain by Lemma 5.1.5 (1), ν is factorized $\nu'' \circ \nu'$ such that $\widetilde{\Gamma}'_{0,i} \neq 0$ for $i = 1, 2$ and $(\widetilde{\Gamma}'_{0,1} \cdot \widetilde{\Delta}' - \widetilde{\Gamma}'_{0,1}) \leq 2$ by replacing $\widetilde{\Gamma}_{0,1}$ and $\widetilde{\Gamma}_{0,2}$ as needed, where $\widetilde{\Gamma}'_{0,i} := \nu'_*(\widetilde{\Gamma}_{0,i})$ for $i = 1, 2$ and $\widetilde{\Delta}' := \nu'_*(\widetilde{\Delta}_{\overline{k}})$. For the same reason as in Subsection 5.4.2, we can assume that ν' is defined over k. Then ν' is uniquely determined, and any irreducible component $\widetilde{\Delta}'$ except for $\widetilde{\Gamma}'_{0,i}$ for $i = 1, 2$ has self-intersection number ≤ -2. By noticing $\widetilde{\Gamma}_{0,i} \neq 0$ for $i = 1, 2$ combined with Lemma 5.4.4(3), $\widetilde{C}_i$ meets a terminal component of $\widetilde{\Delta}^{(0)}$ for any $i = 1, \ldots, n$. Moreover, $n \leq 4$ by Lemma 5.4.5, in particular, $n \neq 3$ by considering the symmetry of the weighted dual graph of $\widetilde{\Delta}^{(0)}$. In what follows, we shall consider the two cases of $n = 2$ and $n = 4$ separately.

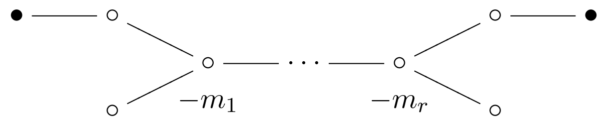
In the case of $n = 2$, suppose that the weighted dual graph of $\widetilde{\Delta}^{(0)} + \sum_{i=1}^{2} \widetilde{C}_i$ is as follows:



Then we see that $(\widetilde{\Gamma}'_{0,1} \cdot \widetilde{\Delta}' - \widetilde{\Gamma}'_{0,1}) = 1$ and $\widetilde{\Delta}'$ is not a linear chain, so that $\widetilde{\Gamma}'_{0,1}$ is a (-1)-curve. Moreover, $r > 2$ by Lemma 5.2.9 (3). Namely, the weighted dual graph of $\widetilde{\Delta}'$ is as follows:



By contracting of $\widetilde{\Delta}' - \widetilde{\Gamma}'_{0,1}$ over $\overline{k}$, we have a log del Pezzo surface of rank one with exactly one quotient singular point of type D, which is a contradiction to [41, Theorem 3.1(1)]. Hence, the weighted dual graph of $\widetilde{\Delta}^{(0)} + \sum_{i=1}^{2} \widetilde{C}_i$ is as follows:



Then we see that $\sharp\widetilde{\Delta}' \geq 4$ and $\widetilde{\Delta}'$ is a linear chain. Moreover, $\widetilde{\Gamma}'_{0,i}$ is a (-1)-curve for $i = 1, 2$ by noting Lemma 5.1.5 (4). Thus, the weighted dual graph of $\widetilde{\Delta}'$ satisfies the condition of Proposition 5.2.13 (2) or (3). In particular, by Proposition 5.2.13 (2) and (3) and Lemma 5.4.4 we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (i) $(i = 32, 33, 34)$ in Appendix A.2.

In the case of $n = 4$, note that $\widetilde{\Gamma}'_{0,i}$ is not a (-1)-curve for $i = 1, 2$ by Lemma 5.2.9 (2). Hence, we obtain $\nu' = \nu$. Moreover, we see that $\sharp\widetilde{\Delta}' = 2$ and $\widetilde{\Gamma}'_{0,i}$ is a 0-curve for $i = 1, 2$ by Lemma 5.1.5 (3) and (4). Therefore, by Lemma 5.4.4 we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (35) in Appendix A.2.

Subcase (III-3)

Assume that there exists exactly one irreducible component $\widetilde{\Gamma}_0$ of $\widetilde{\Delta}^{(0)}$ satisfying $(\widetilde{\Gamma}_0 \cdot \widetilde{\Delta}^{(0)} - \widetilde{\Gamma}_0) = 3$. By the similar argument as in Subsection 5.4.2 combined with the classification of rational log canonical singularities of dimension two, we see that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is given as (i) $(i = 36, \ldots, 52)$ in Appendix A.2.

The argument in Subsections 5.4.1–5.4.3 completes the proof of Theorem 1.3.12 (3).

5.5 Applications of Theorem 1.3.12

5.5.1 Existing conditions for the affine plane in lc del Pezzo surfaces of rank one

In this subsection, we shall prove Theorems 1.3.14 and 1.3.15 by applying Theorem 1.3.12. Let S be an lc del Pezzo surface of rank one defined over k such that $\mathrm{Sing}(S_{\overline{k}}) \neq \emptyset$, and let $\sigma : \widetilde{S} \to S$ be the minimal resolution over k.

At first, Theorem 1.3.14 can be shown as follows:

Proof of Theorem 1.3.14. Notice that (A) implies (B) in Theorem 1.3.14 is obvious by Theorem 1.3.12 (3). Hence, we shall prove the converse of this. Assume that there exists a reduced effective divisor $\widetilde{\Delta}$ on $\widetilde{S}$ as in Theorem 1.3.14 (B). By the configuration of the weighted dual graph of $\widetilde{\Delta}$, we can construct the birational morphism $\nu : \widetilde{S} \to \check{S}$ over k such that $\widetilde{S}\backslash\mathrm{Supp}(\widetilde{\Delta}) \simeq \check{S}\backslash\mathrm{Supp}(\check{\Delta})$ and the weighted dual graph of $\check{\Delta}$ is either:

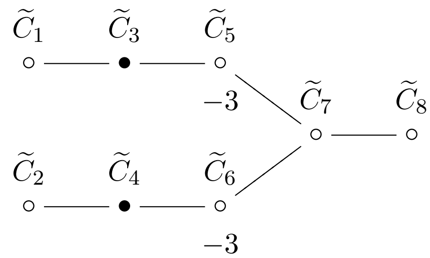
$$\overset{1}{\circ} \quad \text{or} \quad \overset{m}{\circ}\!-\!\!-\!\!-\!\overset{0}{\circ} \quad (m \neq -1)$$

(see also Example 5.5.2, for an example on the construction of ν), where $\check{\Delta} := \nu_*(\widetilde{\Delta})$. Meanwhile, letting $\Delta := \sigma_*(\widetilde{\Delta})$, we see $\widetilde{S}\backslash\mathrm{Supp}(\widetilde{\Delta}) \simeq S\backslash\mathrm{Supp}(\Delta)$ since the exceptional locus of σ is included in $\mathrm{Supp}(\widetilde{\Delta})$. Moreover, since Δ is $\mathbb{Q}$-ample because of $\rho_k(S) = 1$, we know that $S\backslash\mathrm{Supp}(\Delta)$ is affine by [27, Theorem 1]. By Lemma 5.1.6, we thus obtain $\check{S}_{\overline{k}}\backslash\mathrm{Supp}(\check{\Delta}_{\overline{k}}) \simeq \mathbb{A}^2_{\overline{k}}$. Since there is no non-trivial k-form of $\mathbb{A}^2_{\overline{k}}$ ([33]), we then have $\check{S}\backslash\mathrm{Supp}(\check{\Delta}) \simeq \mathbb{A}^2_k$. Therefore,

S contains the affine plane $S \backslash \mathrm{Supp}(\Delta) \simeq \mathbb{A}^2_k$. This implies that the assertion (A) in Theorem 1.3.14 holds. $\square$

Remark 5.5.1. Notice that each weighted dual graph in [42, Appendix C] (except for (1)), [45, Fig. 1] or Appendix A.2 includes at least one vertex corresponding to a (-1)-curve. Hence, it is quite subtle to determine whether S contains the affine plane or not by using only singularities on $S_{\overline{k}}$. In fact, we can construct some examples that two lc del Pezzo surfaces of rank one with the same singularities such that one contains the affine plane but the other does not (see Subsection 5.6.3).

Example 5.5.2. With the notation as above, we shall consider a case that there exists a reduced effective divisor $\widetilde{\Delta}$ on $\widetilde{S}$ such that the exceptional locus of σ is included in $\mathrm{Supp}(\widetilde{\Delta})$ and the weighted dual graph of $\widetilde{\Delta}$ is as (21) in Appendix A.2. Let $\widetilde{C}_1, \ldots, \widetilde{C}_8$ be all irreducible component of $\widetilde{\Delta}_{\overline{k}}$ named as follows:



By the symmetry of the above graph, $\widetilde{C}_1 + \widetilde{C}_2$, $\widetilde{C}_3 + \widetilde{C}_4$, $\widetilde{C}_5 + \widetilde{C}_6$, $\widetilde{C}_7$ and $\widetilde{C}_8$ are defined over k, respectively. Then we construct the compositions of successive contractions $\nu : \widetilde{S} \to \check{S}$ of a disjoint union $\widetilde{C}_3 + \widetilde{C}_4$, that of the images of $\widetilde{C}_1 + \widetilde{C}_2$ and finally that of the images of $\widetilde{C}_5 + \widetilde{C}_6$. By construction, ν is defined over k. Moreover, putting $\check{\Delta} := \nu_*(\widetilde{\Delta})$, then $\check{\Delta}_{\overline{k}}$ consists of two irreducible components $\check{C}_7 := \nu_*(\widetilde{C}_7)$ and $\check{C}_8 := \nu_*(\widetilde{C}_8)$ such that $\check{C}_7$ and $\check{C}_8$ are a 0-curve and a (-2)-curve, respectively. Since $\check{S} \backslash (\check{C}_7 \cup \check{C}_8)$ is affine by [27, Theorem 1] combined with $\check{S} \backslash (\check{C}_7 \cup \check{C}_8) \simeq S \backslash \sigma_*(\widetilde{\Delta})$, we obtain $\check{S}_{\overline{k}} \simeq \mathbb{F}_2$ and $\check{S}_{\overline{k}} \backslash (\check{C}_{7,\overline{k}} \cup \check{C}_{8,\overline{k}}) \simeq \mathbb{A}^2_{\overline{k}}$ by Lemma 5.1.6. Furthermore, we see $\check{S} \simeq \mathbb{F}_2$ and $\check{S} \backslash (\check{C}_7 \cup \check{C}_8) \simeq \mathbb{A}^2_k$ by [33]. Namely, $S \backslash \mathrm{Supp}(\sigma_*(\widetilde{\Delta})) \simeq \mathbb{A}^2_k$.

From now on, we shall prove Theorem 1.3.15 by using Theorem 1.3.14. Thus, assume that S has at most Du Val singularities, and let d be the degree of S, i.e., $d := (-K_S)^2$. Then Theorem 1.3.15 can be shown as follows:

Proof of Theorem 1.3.15. We shall consider the two cases of $\rho_{\overline{k}}(S_{\overline{k}}) = 1$ or $\rho_{\overline{k}}(S_{\overline{k}}) > 1$ separately.

In the case of $\rho_{\overline{k}}(S_{\overline{k}}) = 1$, then looking for all weighted dual graphs in [42, Appendix C] such that each vertex corresponds to either a (-1)-curve or a (-2)-curve, we know that such the graphs are summarized in (1), (14), (2), (3), (5), (7) and (12) in [42, Appendix C], where we assume $n = 2$ for graphs (1), (14), (2) and (3), and that the subgraph A consists of only one vertex corresponding to a (-2)-curve for graphs of (14) and (2). Notice that the these graphs correspond to the pair of the degree and singularity type of $S_{\overline{k}}$ $(8, A_1)$, $(6, A_2 + A_1)$, $(5, A_4)$, $(4, D_5)$, $(3, E_6)$, $(2, E_7)$ and $(1, E_8)$, respectively. Moreover, for each graph except for (1), the union of (-1)-curves corresponding to all vertices $\bullet$ is always defined over k. Meanwhile, for the graph (1), a curve corresponding to the vertex with the weight zero is defined over k if and only if the singularity type of S is type A_1^+ over k, which is equivalent to $S \simeq \mathbb{P}(1, 1, 2)$.

In the case of $\rho_{\overline{k}}(S_{\overline{k}}) > 1$, then looking for all weighted dual graphs in Appendix A.2 such that each vertex corresponds to either a (-1)-curve or a (-2)-curve, we know that such the graphs are summarized in (1), (2), (4), (5), (8), (18), (24) and (26) in Appendix A.2, where we assume $(t,n) = (0,3)$ (resp. $(t,n,m) = (0,2,2)$, $(t,n') = (0,2)$, $(t,n') = (0,1)$, $m = 2$) for graphs of (1) (resp. (2), (4), (5), (18)) and that the subgraph A consists of only one vertex corresponding to a (-2)-curve for graphs of (5) and (8). Notice that the these graphs correspond to the pair of the degree and singularity type of $S_{\overline{k}}$ $(6, A_1)$ (with 3 lines), $(6, A_2)$, $(4, A_2)$, $(2, A_6)$, $(4, A_2 + 2A_1)$, $(4, D_4)$, $(2, E_6)$ and $(3, D_4)$, respectively. Moreover, for each graph, the union of (-1)-curves corresponding to all vertices $\bullet$ is always defined over k.

By the argument in Subsection 4.2.2 (see also Table 4.1), we then note that the pair of the degree and singularity type of $S_{\overline{k}}$ is $(8, A_1)$, $(6, A_2 + A_1)$, $(6, A_2)$, $(6, (A_1)_<)$ or $(5, A_4)$ provided that $d \geq 5$. Thus, by using Theorem 1.3.14 we obtain this assertion of Theorem 1.3.15. □

Remark 5.5.3. Let S be a Du Val del Pezzo surface of rank one over k. If k is algebraically closed, we can determine whether S contains the affine plane or not by using only the singular type on $S_{\overline{k}}$ ([56, Theorem 1]). However, in general, it seems to need the singularity type on $S_{\overline{k}}$ and further the degree. See also Example 5.6.4.

5.5.2 Application to singular del Pezzo fibrations

Let $f : X \to Y$ be a generically canonical del Pezzo fibration defined over $\mathbb{C}$ (see Definition 1.3.1) and let X_η be the generic fiber of f. By Lemma 1.2.2, recall that f admits a vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinder (see Definition 1.2.1) if and only if X_η contains the affine plane $\mathbb{A}^2_{\mathbb{C}(Y)}$. Hence, by Theorem 1.3.15 we then obtain the following corollary:

Corollary 5.5.4. Let $f : X \to Y$ be a generically canonical del Pezzo fibration of degree $d \in \{1, \ldots, 6, 8\}$ and let X_η be the generic fiber of f such that $\mathrm{Sing}(X_{\eta,\overline{\mathbb{C}(Y)}}) \neq \emptyset$. Then we have the following:

(1) If $d = 8$, then f admits a vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinder if and only if the singularity type of $X_{\eta,\overline{\mathbb{C}(Y)}}$ is A_1^+. (see Section 4.1, for this definition)

(2) If $d = 5, 6$, then f always admits a vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinder.

(3) If $d \leq 4$, f admits a vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinder if and only if the pair of the degree d and the singularity type of $X_{\eta,\overline{\mathbb{C}(Y)}}$ is one of the following:

$$(4, D_5),\ (4, D_4),\ (4, A_2 + 2A_1),\ (4, A_2),\ (3, E_6),\ (3, D_4),\ (2, E_7),\ (2, E_6),\ (2, A_6),\ (1, E_8).$$

Remark 5.5.5. If a generically canonical del Pezzo fibration $f : X \to Y$ of degree d, whose generic fiber X_η of f satisfies $\mathrm{Sing}(X_{\eta,\overline{\mathbb{C}(Y)}}) = \emptyset$, then by Theorem 1.2.4 we know that f admits a vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinder if and only if $d \geq 8$ and X_η has a $\mathbb{C}(Y)$-rational point.

Example 5.5.6. Let $\mathscr{O}$ be a DVR of $\mathbb{C}(t)$ such that the maximal ideal of $\mathscr{O}$ is generated by t and let X be the three-dimensional algebraic variety over $\mathbb{C}$ defined by:

$$X := (tw^2 + xy^3 + z^4 + yzw = 0) \subseteq \mathbb{P}_{\mathscr{O}}(1,1,1,2) = \mathrm{Proj}(\mathscr{O}[x,y,z,w]).$$

Let $f : X \to \mathrm{Spec}(\mathscr{O})$ be the structure morphism as an $\mathscr{O}$-scheme and let η be the generic point of $\mathrm{Spec}(\mathscr{O})$. Then the generic fiber X_η of f is an irreducible quartic hypersurface of the weighted projective space given by:

$$X_\eta = (tw^2 + xy^3 + z^4 + yzw = 0) \subseteq \mathbb{P}_{\mathbb{C}(t)}(1,1,1,2) = \mathrm{Proj}(\mathbb{C}(t)[x,y,z,w]).$$

Then $X_{\eta,\overline{\mathbb{C}(t)}}$ is a Du Val del Pezzo surface of degree 2 with exactly one singular point $p := [1:0:0:0]$ of type E_6. Note that the weighted dual graph of all (-1)-curves and (-2)-curves on the minimal resolution of $X_{\eta,\overline{\mathbb{C}(Y)}}$ is as (11^o) in [14, p. 349]. Hence, we see that X_η is of rank one by straightforward calculation. Thus, f is a generically canonical del Pezzo fibration of degree 2. Moreover, it admits a vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinder by Corollary 5.5.4 (see also Lemma 4.4.5).

Similarly, notice that Theorem 1.3.14 also provides a way to determine whether generically lt del Pezzo fibrations and generically lc del Pezzo fibrations admit vertical $\mathbb{A}^2_{\mathbb{C}}$-cylinders or not.

5.6 Remarks on Theorem 1.3.12

5.6.1 Existence of lc compactifications of the affine plane

In this subsection, we shall discuss whether there exists indeed an lc compactification of the affine plane corresponding to the weighted dual graph (i) in Appendix A.2 for each $i = 1, \ldots, 52$.

At first, we consider the situation that an affine line ℓ defined over k meeting transversely at exactly n-times of curves lying in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Letting $a_1, \ldots, a_n$ be these intersection points, we know that they lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. This implies that the minimal polynomial of a_1 over k is of degree n.

Now, we prepare the following condition $(*)$ with respect to the base field k:

$$\begin{array}{l}\text{For any } n \in \mathbb{Z}_{>0}, \text{ there exist } a_n, b_n \in \overline{k}, \text{ which are not Galois conjugate over } k, \\ \text{such that their minimal polynomials over } k \text{ are of degree 2 and } n, \text{ respectively.}\end{array} \qquad (*)$$

Letting A be one of the graphs in Appendix A.2, assume that there exists an lc compactification (S, Δ) of the affine plane $\mathbb{A}^2_k$ corresponding to this graph A. In other words, letting $\sigma : \widetilde{S} \to S$ be the minimal resolution and letting $\widetilde{\Delta} := \sigma^*(\Delta)_{\mathrm{red.}}$, then the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is the same as A. By Lemmas 5.1.1 and 5.3.2, we notice that all (-1)-curves, which are included in $\mathrm{Supp}(\widetilde{\Delta}_{\overline{k}})$, lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Hence, we can completely see the configuration of $\mathrm{Gal}(\overline{k}/k)$-orbits of each irreducible component of $\widetilde{\Delta}_{\overline{k}}$. More precisely, one of the following four situations holds:

Situation 1: There exist connecting two vertices, which lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. Moreover, there exist exactly two vertices v_1 and v_2 such that these two vertices are connected to n_2-times of vertices respectively, in which $2n_2$-times of curves corresponding to these vertices lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, where $n_2 \geq 2$. Note that the following graphs in Appendix A.2 show this situation:

- (4), (5), where $n_2 := n'$ with $n' \geq 2$;
- (35), where $n_2 := 2$.

Situation 2: There exist connecting two vertices, which lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit. However, there is no vertices v_1 and v_2 as in Situation 1. Note that graphs (5) with $n' = 1$, (8) and (32) in Appendix A.2 show this situation.

Situation 3: There exists a unique vertex, which corresponds to a curve defined over k, connecting n_1-times of vertices corresponding to curves, which lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, where $n_1 \geq 2$. Moreover, there exist exactly n_1-times of vertices $v_1, \ldots, v_{n_1}$ such that these

n_1-times of vertices are connected to n_2-times of vertices respectively, in which $n_1 n_2$-times of curves corresponding to these vertices lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, where $n_2 \geq 2$. Note that the following graphs in Appendix A.2 show this situation:

- (6), (7), where $(n_1, n_2) := (2, n')$ with $n' \geq 2$;
- (20), (35), (45), where $(n_1, n_2) := (2, 2)$;
- (37), where $(n_1, n_2) := (2, 3)$;
- (48), where $(n_1, n_2) := (3, 2)$.

Situation 4: There exists a unique vertex, which corresponds to a curve defined over k, connecting n_1-times of vertices corresponding to curves, which lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, where $n_1 \geq 2$. However, there is no vertices $v_1, \ldots, v_{n_1}$ as in Situation 3. Note that the following graphs in Appendix A.2 show this situation:

- (1) (resp. (2), (3)), where $n_1 := n$ with $n \geq 3$ (resp. $n \geq 2$);
- (36), where $n_1 := 5$;
- (14), (30), (31), where $n_1 := 4$;
- (13), (26), (27), (28), (29), (49), (50), (51), (52), where $n_1 := 3$;
- Otherwise, where $n_1 := 2$.

Thus, if for any graph A one of those as in Appendix A.2 there exists an lc compactification (S, Δ) of the affine plane $\mathbb{A}^2_k$ corresponding to this graph A, then the base field k satisfies the condition $(*)$.

Example 5.6.1. Let (S, Δ) be an lc comapctification of the affine plane $\mathbb{A}^2_{\mathbb{R}}$ over the real number field $\mathbb{R}$, let $\sigma : \widetilde{S} \to S$ be the minimal resolution over $\mathbb{R}$ and let us put $\widetilde{\Delta} := \sigma^*(\Delta)_{\mathrm{red.}}$. Notice that $\mathbb{R}$ does not satisfy the condition $(*)$ because of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$. In particular, the weighted dual graph of $\widetilde{\Delta}_{\mathbb{C}}$ does not appear in the list of Situations 1 or 3. If the weighted dual graph of $\widetilde{\Delta}_{\mathbb{C}}$ occurs Situation 2, then this graph is one of (5) with $n' = 1$, (8) and (32) in Appendix A.2. If the weighted dual graph of $\widetilde{\Delta}_{\mathbb{C}}$ occurs Situation 4, then this graph is one of (5) with $n' = 1$, (2)–(3) with $n = 2$, (6)–(7) with $n' = 1$, (9)–(12), (15)–(19), (21)–(25), (33)–(34), (38)–(44), (46) or (47) in Appendix A.2.

Conversely, assuming that k satisfies the condition $(*)$, let A be one of the graphs in Appendix A.2. Then we can explicitly construct an lc compactification (S, Δ) of the affine plane $\mathbb{A}^2_k$. We shall explain the method of this construction. Let $(\check{S}, \check{\Delta})$ be a minimal normal compactification of the affine plane $\mathbb{A}^2_k$ over k according to the configuration of A as follows:

- If A is as the graph (1), (6), (7), (9), (10), (26), (27), (30), (31), (33), (34), (48), (49), (50), (51) or (52) in Appendix A.2, then $(\check{S}, \check{\Delta}) := (\mathbb{P}^2_k, L)$, where L is a general line on $\check{S} \simeq \mathbb{P}^2_k$;
- If A is as the graph (4), (5), (8), (32) or (35) in Appendix A.2, then $\check{S}$ is a k-form of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ of rank one and $\check{\Delta} := F_1 + F_2$, where F_1 and F_2 is a k-form of an irreducible curve of type $(1, 0)$ and $(0, 1)$, respectively. Notice that $\check{\Delta}$ is defined over k;
- If A is as the graph (11), (13), (14), (15), (16), (17), (20), (21), (22), (23), (24), (25), (28), (29), (36), (37), (38), (39), (43) or (44) in Appendix A.2, then $(\check{S}, \check{\Delta}) := (\mathbb{F}_2, M + F)$, where M and F is the minimal section and a general fiber of the structure morphism $\mathbb{F}_2 \to \mathbb{P}^1_k$ over k;

- If A is as the graph (12), (40), (41), (42), (45), (46) or (47) in Appendix A.2, then $(\check{S}, \check{\Delta}) := (\mathbb{F}_3, M+F)$, where M and F is the minimal section and a general fiber of the structure morphism $\mathbb{F}_3 \to \mathbb{P}^1_k$ over k;
- If A is as the graph (2), (3), (18) or (19) in Appendix A.2, then $(\check{S}, \check{\Delta}) := (\mathbb{F}_m, M+F)$, where M and F is the minimal section and a general fiber of the structure morphism $\mathbb{F}_m \to \mathbb{P}^1_k$ over k.

Then we can construct two birational morphisms $\nu : \widetilde{S} \to \check{S}$ and $\sigma : \widetilde{S} \to S$ over k such that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is the same as A and (S, Δ) is an lc compactification of the affine plane $\mathbb{A}^2_k$, where $\widetilde{\Delta} := \nu^*(\check{\Delta})_{\mathrm{red.}}$ and $\Delta := \sigma_*(\widetilde{\Delta})$. As for how to construct the above two birational morphisms for concrete examples, see the following example (notice that we can construct by the similar way for other cases):

Example 5.6.2. Assume that the base field k satisfies the condition $(*)$, and let A be the weighted dual graph of $\widetilde{\Delta}$ is as (37) in Appendix A.2. Since $(*)$ holds, there exist two elements $a_3, b_3 \in \overline{k}$, which are not Galois conjugate over k, such that there are exactly two (resp. three) elements $a_3^{(1)}, a_3^{(2)} \in \overline{k}$ (resp. $b_3^{(1)}, b_3^{(2)}, b_3^{(3)} \in \overline{k}$), which are Galois conjugates of a_3 (resp. b_3) over k, where $a_3^{(1)} := a_3$ and $b_3^{(1)} := b_3$. Let $P(t) \in k[t]$ be the minimal polynomial for a_3 over k. Now, put $\check{S} := \mathbb{F}_2$, and let F and M be a fiber and the minimal section of the structure morphism $\check{S} \simeq \mathbb{F}_2 \to \mathbb{P}^1_k$ over k, respectively. Then we shall take an affine open neighborhood $U \simeq \mathrm{Spec}(k[x,y])$ such that $\ell := F \cap U \simeq (x=0) \subseteq \mathbb{A}^2_k$. Let $\nu' : \check{S}' \to \check{S}_{\overline{k}}$ be the blow-up at two points $(0, a_3^{(i)}) \in \mathbb{A}^2_{\overline{k}}$ for $i = 1, 2$. Note that ν' is defined over k. Then the pullback $\nu'^{-1}(\ell)$ and the exceptional set E of ν' can be written by $(u=0)$ and $(P(y)=0)$ in $\mathbb{A}^1_k \times \mathbb{P}^1_k = \mathrm{Spec}(k[y]) \times \mathrm{Proj}(k[u,v])$. Hence, we can construct the blow-up $\nu'' : \widetilde{S} \to \check{S}'_{\overline{k}}$ at six points $a_3^{(i)} \times [1 : b_3^{(j)}] \in \mathbb{A}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ for $i = 1, 2$ and $j = 1, 2, 3$. Noticing ν'' is defined over k, so is $\nu := \nu' \circ \nu''$. Now, let $\widetilde{E}$ be the reduced exceptional divisor of ν, and put $\widetilde{F} := \nu_*^{-1}(F)$ and $\widetilde{M} := \nu_*^{-1}(M)$. Then the weighted dual graph of the reduced divisor $\widetilde{\Delta} := \widetilde{E} + \widetilde{F} + \widetilde{M}$ on $\widetilde{S}$ is as in A. Moreover, we know that ${\nu''}_*^{-1}(E) + \widetilde{F} + \widetilde{M}$ can be contracted, hence, we obtain this contraction $\sigma : \widetilde{S} \to S$ over k. By construction, letting $\Delta := \sigma_*(\widetilde{\Delta})$, we see that (S, Δ) is certainly an lc compactification of $\mathbb{A}^2_k$.

5.6.2 Maximal number of singular points on lc compactifications of the affine plane

Let (S, Δ) be an lc compactification of the affine plane over k. If $k = \mathbb{C}$, then $\sharp\mathrm{Sing}(S_{\overline{k}}) \leq 2$ by Lemma 5.1.3. Meanwhile, by Theorem 1.3.12 (2)(iii) we see $\sharp\mathrm{Sing}(S_{\overline{k}}) \leq \rho_{\overline{k}}(S_{\overline{k}}) + 1$, which can be regarded as a generalization of the case of $k = \overline{k}$. In particular, assuming that k satisfies $(*)$, for any positive integer n, there exists a log del Pezzo surface S_n of rank one defined over k containing $\mathbb{A}^2_k$ such that $\sharp\mathrm{Sing}(S_{n,\overline{k}}) = n+1$. Indeed, it follows from [42] (resp. the weighted dual graph (2) in Appendix A.2) if $n = 1$ (resp. $n \geq 2$). On the other hand, we see $\sharp\mathrm{Sing}(S_{\overline{k}}) \leq 4$ (resp. $\sharp\mathrm{Sing}(S_{\overline{k}}) \leq 5$) if $S_{\overline{k}}$ has a non-cyclic quotient singular point (resp. log canonical but not a quotient singular point) by Theorem 1.3.12 (3) (see also Appendix A.2).

5.6.3 Converse of Theorem 1.3.12 (3)

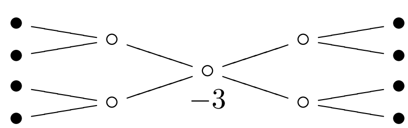
Let S be an lc del Pezzo surface of rank one over k such that $\rho_{\overline{k}}(S_{\overline{k}}) > 1$, and let $\sigma : \widetilde{S} \to S$ be the minimal resolution over k. By Theorem 1.3.12 (3), if S contains the affine plane $\mathbb{A}^2_k$, then there exists a reduced divisor $\widetilde{\Delta}$ on $\widetilde{S}$ such that the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is one of the lists (1)–(52) in Appendix A.2. Now, we shall consider this converse. According to Theorem 1.3.14, if there exists a reduced divisor $\widetilde{\Delta}$ on $\widetilde{S}$ such that the exceptional set of σ is included in $\mathrm{Supp}(\widetilde{\Delta})$, each irreducible component of $\widetilde{\Delta}_{\overline{k}}$ is a rational curve and the weighted dual graph of $\widetilde{\Delta}_{\overline{k}}$ is one of the lists (1)–(52) in Appendix A.2, then S contains the affine plane $\mathbb{A}^2_k$. Furthermore, we shall the following problem:

Problem 5.6.3 (cf. [42, Problem 1])**.** Let S be an lc del Pezzo surface of rank one over k. Assume that the singularity type of $S_{\overline{k}}$ is given as one of the graphs in [42, Appendix C], [45, Fig. 1] or Appendix A.2. Does then S contain the affine plane $\mathbb{A}^2_k$?

In the case of $k = \overline{k}$, Problem 5.6.3 is not true in general ([42, §4]) but is true if S has a singular point, which is not a cyclic quotient singularity, ([45, 46]) or S is a log del Pezzo surface of the Gorenstein index ≤ 3 ([56, 44, 42]). On the other hand, in the case of $k \neq \overline{k}$, we find some counter-examples of Problem 5.6.3 as follows:

Example 5.6.4. Let S be a Du Val del Pezzo surface of rank one, let $\sigma : \widetilde{S} \to S$ be the minimal resolution. Assume that $S_{\overline{k}}$ has only one singular point p of type E_6^+ over k (see Section 4.1, for this notation). Then the degree $d := (-K_S)^2$ of S is equal to 1 or 2. If $d = 2$, then S contains the affine plane $\mathbb{A}^2_k$ since $\widetilde{S}_{\overline{k}}$ includes a reduced effective divisor with a weighted dual graph as (24) in Appendix A.2. If $d = 1$, then S does not contain $\mathbb{A}^2_k$ since it does not contain any cylinder by Theorem 1.3.9 (3)(iv).

Example 5.6.5. Assume that there exist two elements in $\overline{k}$ such that their minimal polynomials over k are of degree 2 and 4, respectively. Let us fix the Hirzebruch surface $\mathbb{F}_3$ of degree 3 defined over k, let M be the minimal section of the structure morphism $\pi : \mathbb{F}_3 \to \mathbb{P}^1_k$. Let $F_1, \dots, F_4$ be four fibers of π and let $\{x_{i,j}\}_{1\leq i\leq 4,\, 1\leq j\leq 2}$ be eight points on $\mathbb{F}_3$, which lie in the same $\mathrm{Gal}(\overline{k}/k)$-orbit, such that $x_{i,1}$ and $x_{i,2}$ lie on the fiber F_i of π for $i = 1, \dots, 4$. Letting $\nu : \widetilde{S} \to \mathbb{F}_3$ be a blow-up at $\{x_{i,j}\}_{1\leq i\leq 4,\, 1\leq j\leq 2}$, the weighted dual graph of $\nu^*(M + F_1 + \cdots + F_4)_{\mathrm{red.}}$ is as follows:



Let $\sigma : \widetilde{S} \to S$ be the contraction of $\nu_*^{-1}(M + F_1 + \cdots + F_4)$. By construction, S is then an lc del Pezzo surface of rank one, which has a log canonical but not a quotient singular point. In particular, the singularity type of $S_{\overline{k}}$ is the same singularity type as (30) in Appendix A.2, however, S does not contain the affine plane $\mathbb{A}^2_k$ by Theorem 1.3.12 (3).

Example 5.6.6. Assuming $k = \mathbb{Q}$, let m be a positive integer, let C be the plane conic over $\mathbb{Q}$ defined by $(xz = y^2) \subseteq \mathbb{P}^2_{\mathbb{Q}} = \mathrm{Proj}(\mathbb{Q}[x, y, z])$ and let $x_1, \dots, x_{2m+4}$ be points on $C_{\overline{\mathbb{Q}}}$ given by $x_i := [1 : \sqrt[2m+4]{2}\zeta^i : \sqrt[m+2]{2}\zeta^{2i}] \in \mathbb{P}^2_{\overline{\mathbb{Q}}}$ for $i = 1, \dots, 2m+4$, where $\zeta := \exp\left(\frac{\pi\sqrt{-1}}{m+2}\right)$. Noticing that the union $\sum_{i=1}^{2m+4} x_i$ is defined over $\mathbb{Q}$, let $\nu : \widetilde{S} \to \mathbb{P}^2_{\mathbb{Q}}$ be a blow-up at $x_1, \dots, x_{2m+4}$ over $\mathbb{Q}$. By construction, $\widetilde{C} := \nu_*^{-1}(C)$ is a $\mathbb{Q}$-form of $(-2m)$-curve. Hence, we obtain the contraction $\sigma : \widetilde{S} \to S$ of $\widetilde{C}$, so that S is a log del Pezzo surface of rank one over $\mathbb{Q}$. Since $S_{\overline{\mathbb{Q}}}$ has exactly one singular point, whose singularity type is the same as the singular point on the weighted projective space $\mathbb{P}(1, 1, 2m)$ over $\overline{\mathbb{Q}}$, we see that S is of the Gorenstein index m. However, S does not contain the affine plane $\mathbb{A}^2_{\mathbb{Q}}$ by Theorem 1.3.12 (3). On the other hand, the weighted projective space $\mathbb{P}(1, 1, 2m)$ over $\mathbb{Q}$ is a log del Pezzo surface of rank one and of the Gorenstein index m, and contains $\mathbb{A}^2_{\mathbb{Q}}$.

Appendix A

Classification lists

A.1 Types of weak del Pezzo surfaces

This Appendix summarizes all types of weak del Pezzo surfaces over algebraically closed fields of characteristic zero (see Subsection 2.4.2, for the definition). We mainly refer to [18, 15, 69].

Degree 8		Degree 7	
Singularities	# Lines	Singularities	# Lines
A_1	0	A_1	2

Degree 6					
Singularities	# Lines	Singularities	# Lines	Singularities	# Lines
A_2+A_1	1	A_2	2	$2A_1$	2
$(A_1)_<$	3	$(A_1)_>$	4		

Degree 5					
Singularities	# Lines	Singularities	# Lines	Singularities	# Lines
A_4	1	A_3	2	A_2+A_1	3
A_2	4	$2A_1$	5	A_1	7

Degree 4					
Singularities	# Lines	Singularities	# Lines	Singularities	# Lines
D_5	1	A_3+2A_1	2	D_4	2
A_4	3	A_3+A_1	3	A_2+2A_1	4
$4A_1$	4	$(A_3)_<$	4	$(A_3)_>$	5
A_2+A_1	6	$3A_1$	6	A_2	8
$(2A_1)_<$	8	$(2A_1)_>$	9	A_1	12

Degree 3					
Singularities	# Lines	Singularities	# Lines	Singularities	# Lines
E_6	1	A_5+A_1	2	$3A_2$	3
D_5	3	A_5	3	A_4+A_1	4
A_3+2A_1	5	$2A_2+A_1$	5	D_4	6
A_4	6	A_3+A_1	7	$2A_2$	7
A_2+2A_1	8	$4A_1$	9	A_3	10
A_2+A_1	11	$3A_1$	12	A_2	15
$2A_1$	16	A_1	21		

Degree 2					
Singularities	# Lines	Singularities	# Lines	Singularities	# Lines
E_7	1	A_7	2	$D_6 + A_1$	2
$A_5 + A_2$	3	$D_4 + 3A_1$	4	$2A_3 + A_1$	4
E_6	4	D_6	3	A_6	4
$D_5 + A_1$	5	$(A_5 + A_1)_<$	5	$(A_5 + A_1)_>$	6
$D_4 + 2A_1$	6	$A_4 + A_2$	6	$2A_3$	6
$A_3 + A_2 + A_1$	7	$A_3 + 3A_1$	8	$3A_2$	8
$6A_1$	10	D_5	8	$(A_5)_<$	7
$(A_5)_>$	8	$D_4 + A_1$	9	$A_4 + A_1$	10
$A_3 + A_2$	10	$(A_3 + 2A_1)_<$	11	$(A_3 + 2A_1)_>$	12
$2A_2 + A_1$	12	$A_2 + 3A_1$	13	$5A_1$	14
D_4	14	A_4	14	$(A_3 + A_1)_<$	15
$(A_3 + A_1)_>$	16	$2A_2$	16	$A_2 + 2A_1$	18
$(4A_1)_<$	19	$(4A_1)_>$	20	A_3	22
$A_2 + A_1$	24	$(3A_1)_<$	25	$(3A_1)_>$	26
A_2	32	$2A_1$	34	A_1	44

Degree 1					
Singularities	# Lines	Singularities	# Lines	Singularities	# Lines
E_8	1	D_8	2	A_8	3
$E_7 + A_1$	3	$A_7 + A_1$	5	$E_6 + A_2$	4
$D_6 + 2A_1$	5	$D_5 + A_3$	5	$A_5 + A_2 + A_1$	8
$2D_4$	5	$2A_4$	6	$2A_3 + 2A_1$	11
$4A_2$	12	E_7	5	D_7	5
$(A_7)_<$	7	$(A_7)_>$	8	$E_6 + A_1$	8
$D_6 + A_1$	9	$A_6 + A_1$	10	$D_5 + A_2$	10
$D_5 + 2A_1$	12	$A_5 + A_2$	12	$A_5 + 2A_1$	14
$D_4 + A_3$	11	$D_4 + 3A_1$	17	$A_4 + A_3$	12
$A_4 + A_2 + A_1$	15	$2A_3 + A_1$	16	$A_3 + A_2 + 2A_1$	19
$A_3 + 4A_1$	22	$3A_2 + A_1$	20	E_6	13
D_6	13	A_6	15	$D_5 + A_1$	18
$(A_5 + A_1)_<$	20	$(A_5 + A_1)_>$	21	$D_4 + A_2$	20
$D_4 + 2A_1$	24	$A_4 + A_2$	22	$A_4 + 2A_1$	25
$(2A_3)_<$	22	$(2A_3)_>$	23	$A_3 + A_2 + A_1$	27
$A_3 + 3A_1$	31	$3A_2$	29	$2A_2 + 2A_1$	32
$A_2 + 4A_1$	36	$6A_1$	41	D_5	27
A_5	29	$D_4 + A_1$	34	$A_4 + A_1$	36
$A_3 + A_2$	38	$(A_3 + 2A_1)_<$	43	$(A_3 + 2A_1)_>$	44
$2A_2 + A_1$	45	$A_2 + 3A_1$	50	$5A_1$	56
D_4	49	A_4	51	$A_3 + A_1$	60
$2A_2$	62	$A_2 + 2A_1$	69	$(4A_1)_<$	76
$(4A_1)_>$	77	A_3	83	$A_2 + A_1$	94
$3A_1$	103	A_2	127	$2A_1$	138
A_1	183				

A.2 Lc compactifications of the affine plane

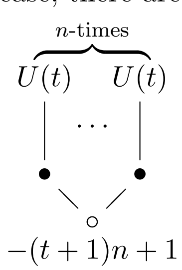
Letting the notation and the assumptions be the same as in Theorem 1.3.12 (3), this Appendix summarizes the list of configurations of all weighted dual graphs of $\widetilde{\Delta}_{\overline{k}}$, where we employ the following notation:

- For the following all weighted dual graphs, t, t' and m are arbitrary integers with $t \geq 0$, $t' \geq 0$ and $m \geq 2$.
- In (4), (5), (6) or (7), assume that n is even and let n' be the integer with $2n' = n$.
- The subgraph $U(t)$ means $t\text{-vertices}\left\{\begin{array}{c}\circ\\ |\\ \vdots\\ |\\ \circ\end{array}\right.$.
- The subgraph $L(m;t)$ means $\underbrace{\circ - \cdots - \circ}_{t\text{-vertices}} - \underset{-m}{\circ}$.
- The subgraph $R(m;t)$ means $\underset{-m}{\circ} - \underbrace{\circ - \cdots - \circ}_{t\text{-vertices}}$.
- In (3), (5), (7), (8) or (10), the subgraph A means an arbitrary admissible twig, and m_A means the integer as in Definition 5.2.6. Moreover, the subgraph ${}^tA$ means the transposal of A, and A^* means the adjoint of A (see Definition 5.2.1). On the other hand, if A can be denoted by $\underset{-m_1}{\circ} - \underset{-m_2}{\circ} - \cdots - \underset{-m_r}{\circ}$, then the subgraph $\underline{A}$ means $\underset{-m_1}{\circ} - \underset{-m_2}{\circ} - \cdots - \underset{-m_{r-1}}{\circ}$.

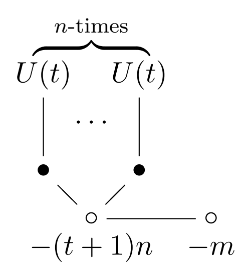
Noting that $S_{\overline{k}}$ contains exactly one singular point p_0, which is k-rational, by Theorem 1.3.12 (2)(ii) and (iii), the list is divided into three case about the singularity of p_0 (cf. Section 5.4):

Case of admitting only cyclic quotient singularities

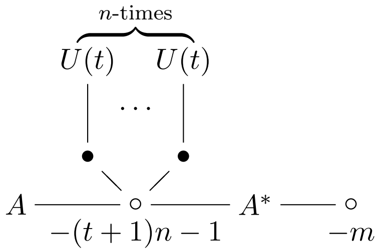
In this case, there are 10 cases (1)–(10):

(1) 

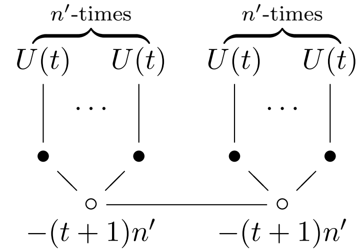
$(n \geq 3)$

(2) 

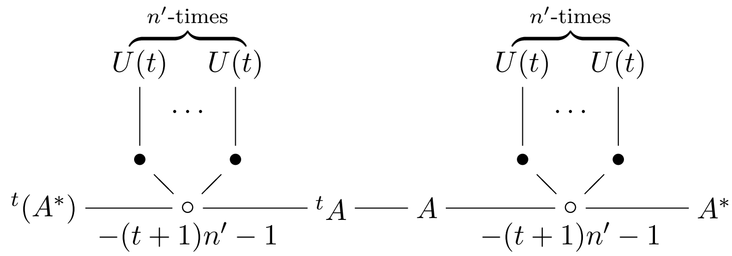
$(n \geq 2)$

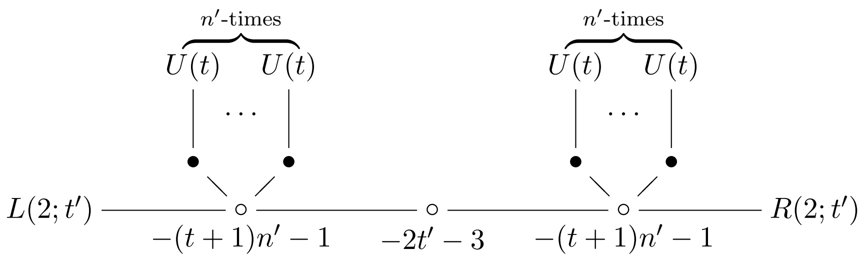
(3) 

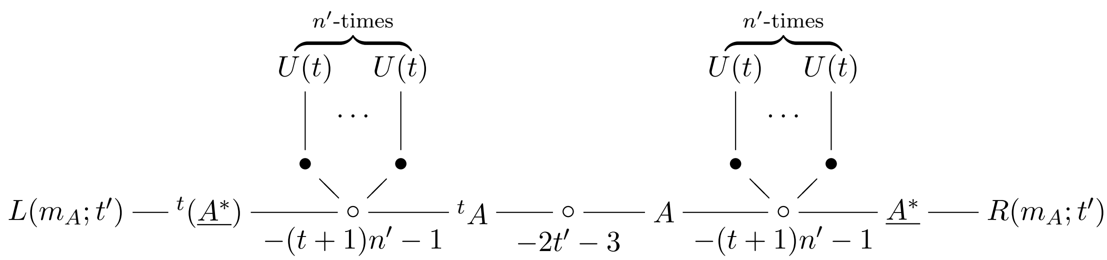
(n ≥ 2)

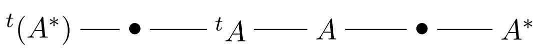
(4) 

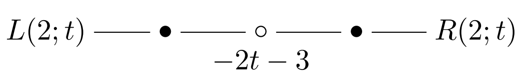
(n′ ≥ 2)

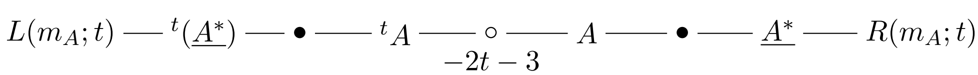
(5) 

(6) 

(7) 

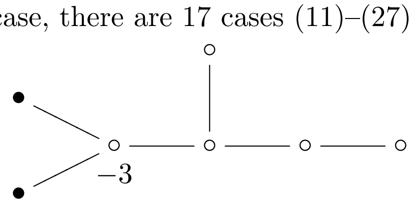
(8) 

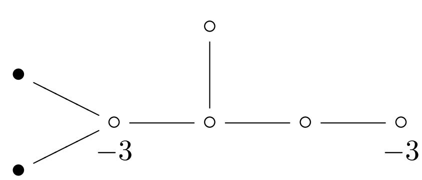
(9) 

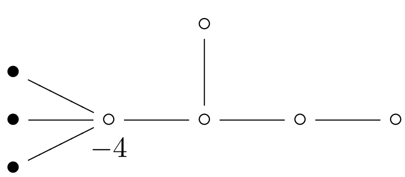
(10) 

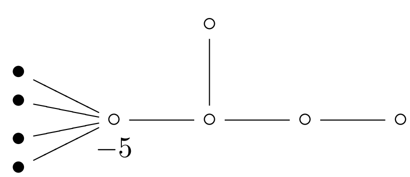
Case of admitting non-cyclic quotient singularities

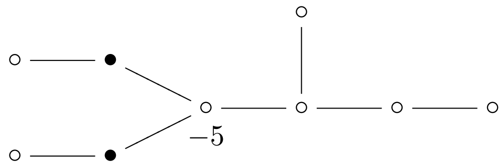
In this case, there are 17 cases (11)–(27):

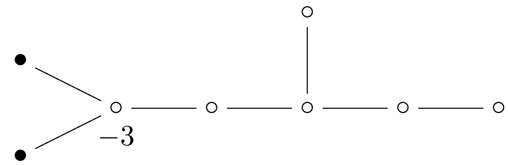
(11) 

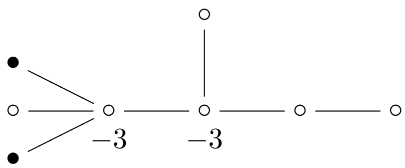
(12) 

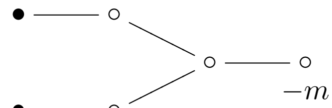
(13) 

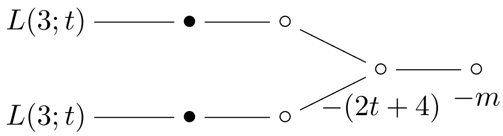
(14) 

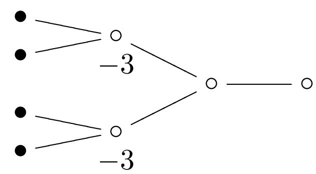
(15) 

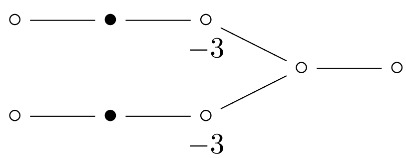
(16) 

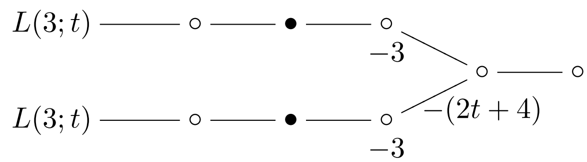
(17) 

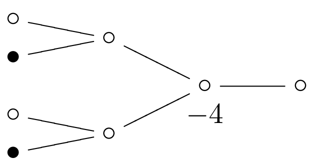
(18) 

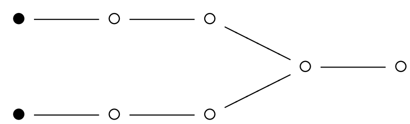
(19) 

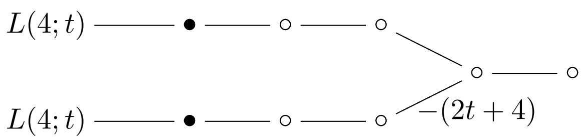
(20) 

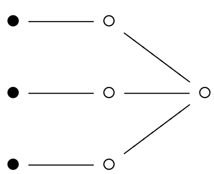
(21) 

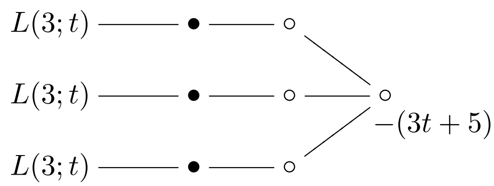
(22) 

(23) 

(24) 

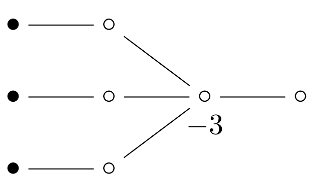
(25) 

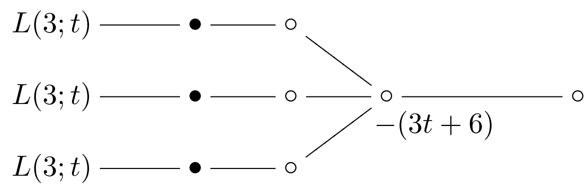
(26) 

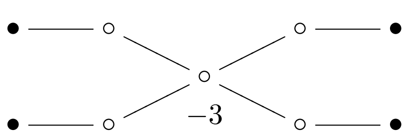
(27) 

Case of admitting log canonical but not quotient singularities

In this case, there are 25 cases (28)–(52):

(28) 

(29) 

(30) 

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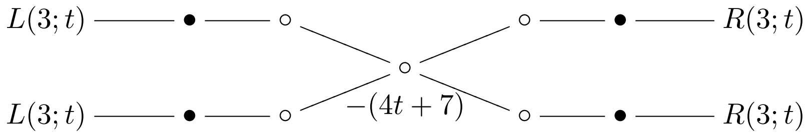

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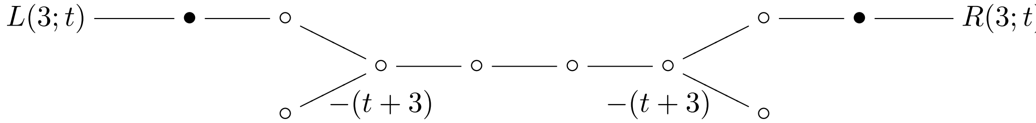

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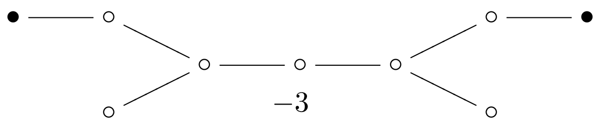

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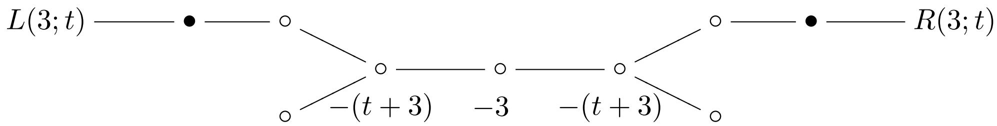

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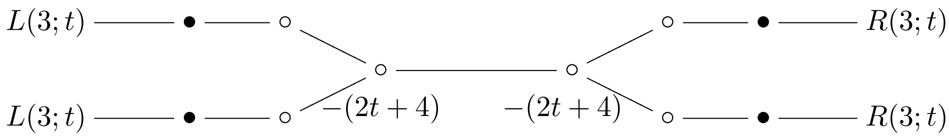

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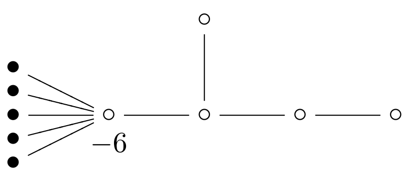

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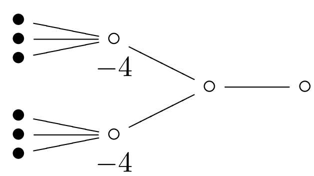

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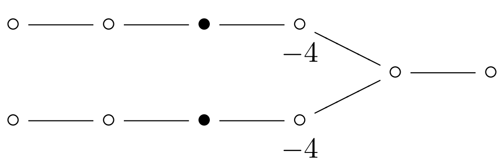

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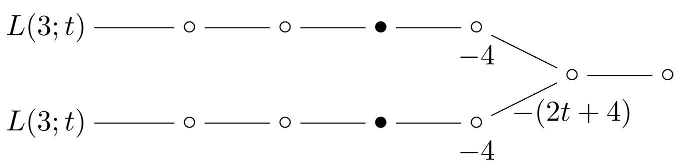

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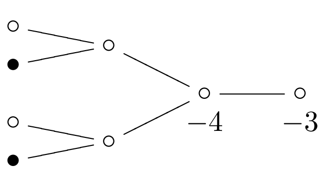

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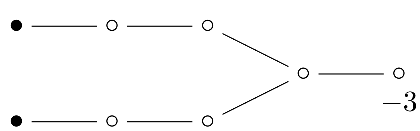

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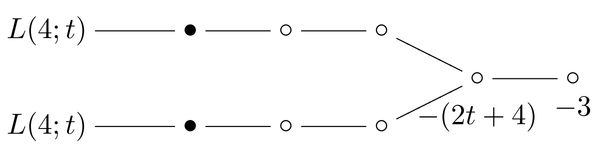

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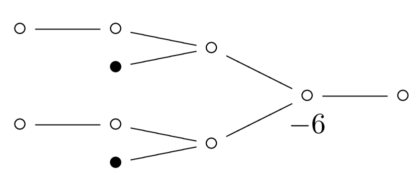

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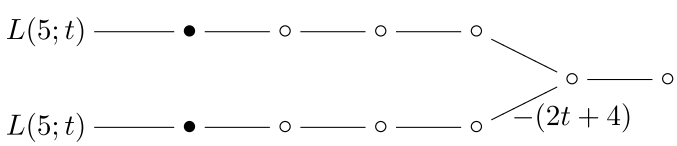

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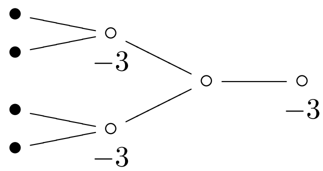

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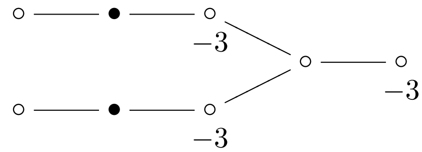

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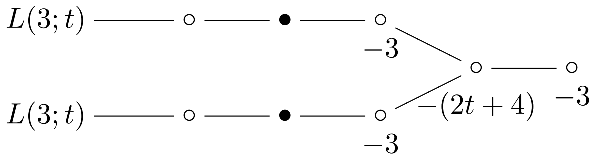

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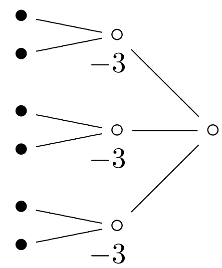

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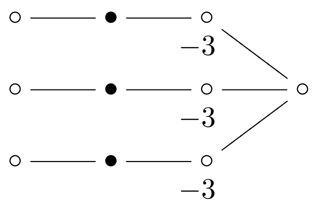

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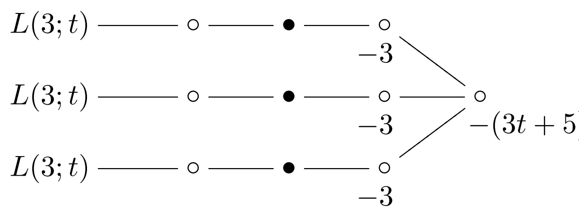

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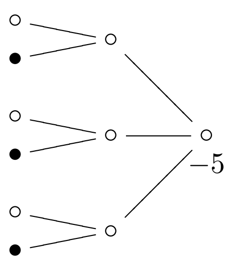

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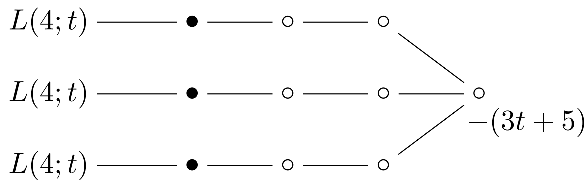

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