

Phenomenology of Neutrino Mass Matrix ^a

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The search for possible mixing patterns of charged leptons and neutrinos is important to get clues of the origin of nearly maximal mixings, since there are some preferred bases of the lepton mass matrices given by underlying theories. We systematically examine the mixing patterns which could lead to large lepton mixing angles. We find out 37 mixing patterns are consistent with experimental data if taking into account phase factors in the mixing matrices. Only 6 patterns of them can explain the observed data without any tuning of parameters, while the others need particular choices for phase values.

1 Introduction

The Super-Kamiokande experiment has confirmed the neutrino oscillation in atmospheric neutrinos, which favors the $\nu_\mu \rightarrow \nu_\tau$ process with a large mixing angle $\sin^2 2\theta_{\text{atm}} \geq 0.88$ and a mass-squared difference $\Delta m_{\text{atm}}^2 = (1.6-4) \times 10^{-3} \text{ eV}^2$ ¹. On the other hand, the recent data of Super-Kamiokande favors the large mixing angle (LMA) MSW solution² for the solar neutrinos problem, but there are still four solutions allowed; the small mixing angle (SMA) MSW³, the LMA-MSW, the low Δm^2 (LOW), and the vacuum oscillation (VO) solutions⁴. As a result, the neutrino mixing matrix (MNS matrix⁵) has two possibilities: one is the matrix with single maximal mixing, which gives the SMA-MSW solution for the solar neutrino problem, and the other with bi-maximal mixing⁶, which corresponds to the LMA-MSW, LOW, and VO solutions.

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Assuming that the neutrino oscillations only account for the solar and atmospheric neutrino data, one can consider prototypes of the MNS mixing matrix U_{MNS} with single maximal mixing, and with bi-maximal mixing. Where is the origin of the above nearly maximal mixings? This is one of the most important problems in the lepton mixing. In almost all models for the fermion masses and mixing, there are some preferred bases given by underlying theories of the models. The maximal mixing angles generally follow from both the charged-lepton and neutrino mass matrices. The search for possible mixing patterns of charged leptons and neutrinos is therefore important for constructing models with maximal lepton mixings. We systematically investigate the mixing patterns where at least one of the mixing matrices has sources of maximal mixings⁷. Our analysis is not concerned with any particular structures of lepton mass matrices and hence with the mass spectrum of neutrinos.

2 Phenomenology of Mixing Matrices

When the charged-lepton and neutrino mass matrices are given, the MNS matrix is defined as

$$U_{\text{MNS}} = V_E^\dagger V_\nu, \quad (1)$$

where V 's are the mixing matrices which rotate the left-handed fields so that the mass matrices are diagonalized. The mixing matrices V_E and V_ν are generally parameterized as follows:

$$V_E = P U(23) P' U(13) U(12) P'', \quad V_\nu = \overline{P} \overline{U}(23) \overline{P}' \overline{U}(13) \overline{U}(12) \overline{P}''. \quad (2)$$

Here $U(ij)$ are the rotation matrices,

$$U(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, U(13) = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}, U(12) = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3)$$

in which $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$, and P 's are the phase matrices; $P = \text{diag}(1, e^{ia}, e^{ib})$, $P' = \text{diag}(1, 1, e^{i\delta})$, and $P'' = \text{diag}(e^{ip}, e^{iq}, e^{ir})$. The matrices $\overline{U}(ij)$, \overline{P} , \overline{P}' , and \overline{P}'' in the neutrino side take the same forms as above. Since there are six mixing angles in V_E and V_ν , it is meaningful to raise a query which angles are responsible for the observed maximal mixings in U_{MNS} . In order to answer this, we analyze the mixing patterns in a model-independent way.

Now, the MNS mixing matrix is written as

$$U_{\text{MNS}} = \{U(23) P' U(13) U(12)\}^\dagger Q \overline{U}(23) \overline{P}' \overline{U}(13) \overline{U}(12) \equiv U_E^\dagger Q U_\nu, \quad (4)$$

in which $Q = P^* \bar{P} \equiv \text{diag}(1, e^{i\alpha}, e^{i\beta})$. As will be seen below, in our analysis, the phase factors in the matrix Q sometimes play important roles to have phenomenologically viable mixing angles. The mixing matrix $U(ij)$ and $\bar{U}(ij)$ are determined if the mass matrices of charged leptons and neutrinos are given. In the first approximation, we assume that these mixing angles are zeros or maximal ones, and then examine possible combinations of U_E and U_ν combined with indications of Super-Kamiokande and long baseline neutrino experiments.

Let us consider 9 types of mixing matrices for U_E and U_ν . The first three types of matrices are given by taking one of mixing angles being maximal and the others being zero:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, L = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (5)$$

where we use the notation A , S , and L for three type mixing matrices of U_E and U_ν . The second three types of matrices are given by taking one of mixing angles being zero and the others being maximal:

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, H = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, N = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \quad (6)$$

The threefold maximal mixing⁸ and the unit matrix are also added in our analyses:

$$T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} e^{-i\delta} \\ -\frac{1}{2} - \frac{1}{2\sqrt{3}} e^{i\delta} & \frac{1}{2} - \frac{1}{2\sqrt{3}} e^{i\delta} & \frac{1}{\sqrt{3}} \\ \frac{1}{2} - \frac{1}{2\sqrt{3}} e^{i\delta} & -\frac{1}{2} - \frac{1}{2\sqrt{3}} e^{i\delta} & \frac{1}{\sqrt{3}} \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (7)$$

In addition to these, one specific mixing, which is the so-called democratic type mixing⁹, is examined because this mixing is different from the above ones and may be derived from well-motivated underlying theories:

$$D = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}. \quad \begin{array}{l} s_{12} = 1/2 \\ s_{13} = 1/\sqrt{3} \\ s_{23} = 1/\sqrt{2} \end{array} \quad (8)$$

By using the above types of mixing matrices, we have 81 ($= 9 \times 9$) combinations of matrices for the MNS matrix U_{MNS} , in which the phases α , β , δ_E , and δ_ν are free parameters. Note that if at least one of the matrix elements is

zero in U_E (U_ν), we can take P' (\overline{P}') as a unit matrix without loss of generality. The phase δ_E (δ_ν) can be absorbed into the matrices P'' (\overline{P}'') and/or Q .

We examine the MNS matrices according to phenomenological constraints coming from the atmospheric neutrino experiments. The Chooz experiment¹⁰ also provides a useful guide for the classification of mixing matrices, in particular, for the $(U_{\text{MNS}})_{e3}$ element. The solar neutrino problem can be solved with both large and small mixing angle solutions, which are now predictions of our systematic search of taking 81 combinations of U_E and U_ν . We here take a convention where the mixing between the labels 2 and 3 is relevant to atmospheric neutrinos and the mixing between the labels 1 and 2 to the solar neutrino problem. We find that the 81 mixing patterns are classified into the following five categories:

- class 1: small mixing for atmospheric neutrinos
- class 2: large value of $(U_{\text{MNS}})_{e3}$
- class 3: small mixing for atmospheric neutrinos when $(U_{\text{MNS}})_{e3} \ll 1$ by tuning phase values
- class 4: consistent with experiments by tuning phase values
- class 5: consistent with experiments independently of phase values

The classes 4 and 5 are consistent with the experimental data. We have also checked the “stability” of our classification numerically by taking the fluctuations of all mixing angles in the region of $\theta_{ij} = \theta_{ij} \pm 5^\circ$ both in the charged-lepton and neutrino sectors. Due to a constraint from the Chooz experiment, one may usually assume that a bi-maximal mixing matrix takes the form of type B . It is, however, found here that the matrix N , which has a large 1-3 mixing, gives exactly the same results as the matrix B does. (The predictions for 1-2 mixing angles are also the same.) This would give a new possibility of model-building for the fermion masses and mixings.

In the category of class 5, there are the following 6 mixing patterns:

$$(U_E, U_\nu) = (A, S), (A, I), (I, A), (I, B), (D, S), (D, I). \quad (9)$$

An interesting fact we find in (9) is that without tuning of phase parameters (i.e., in class 5), a large 1-2 mixing relevant to the solar neutrino problem must come from the neutrino side (except for the cases of democratic-type mixing). This may be a natural result in view of the charged-lepton mass matrix and commonly discussed in the literature. That is, in the charged-lepton sector, the mass hierarchy between the first and second generations may be too large for the large angle solar solutions. It is, however, noted that the same result can be obtained only from a viewpoint of mixing matrices.

Class 4 contains the other 31 patterns of mixing matrices. These patterns require suitable choices of phase values to be consistent with the experimental data. The result is summarized in Table 1, which shows the values of mixing angle for atmospheric neutrinos ($\sin^2 2\theta_{\text{atm}}$) and for solar neutrinos ($\sin^2 2\theta_{\odot}$) in cases that the values of $(U_{\text{MNS}})_{e3}$ are fixed to be minimum. For each combination, we also present the relevant phases which are tuned to obtain the minimum value of $(U_{\text{MNS}})_{e3}$. In some cases, the mixing angles of $\sin^2 2\theta_{\text{atm}}$ and $\sin^2 2\theta_{\odot}$ have some uncertainties in their predictions. It is because there are remaining phase degrees of freedom even with fixed values of $(U_{\text{MNS}})_{e3}$. The mixing patterns in class 4 have different numbers of phase tuning to obtain experimentally suitable MNS matrices. For example, the types $(U_E, U_\nu) = (A, A)$ and (A, B) , which are often seen in the literature, requires only one tuning of phase values to fix all the mixing angles in U_{MNS} (see also the next section). These combinations have not been discussed so far in the literature and would provide new possibilities for constructing models where fermion masses and mixing angles are properly reproduced.

3 Texture of Lepton Mass Matrices

Let us begin with discussing the patterns in class 5. As noted in the previous section, these mixing patterns are well known in the literature, in other words, there are a lot of models of mass matrices which lead to these mixing patterns. We summarize the 6 patterns briefly assuming that neutrinos are Majorana.

The first pattern is the case $(U_E, U_\nu) = (A, S)$, which predicts bi-maximal mixing for the MNS matrix. Such a type of texture is at first derived in SO(10) grand unified models¹¹. The next one is the case $(U_E, U_\nu) = (A, I)$, which predicts single maximal mixing for the MNS matrix. These mass matrices are indeed obtained in E_7 , E_6 , and $SO(10)$ grand unified theories¹². The third one is $(U_E, U_\nu) = (I, A)$, which gives single maximal mixing for the MNS matrix. This texture can be given by, for example, R -parity violating models¹³. The fourth one is $(U_E, U_\nu) = (I, B)$, which gives bi-maximal mixing for the MNS matrix. This texture follows from radiative generation mechanisms for neutrino masses¹⁴. The fifth and sixth patterns are specific ones because they depend on the democratic lepton mass matrix⁹. All the above mixing patterns are allowed by the experimental data without any tuning of phases, $\alpha, \beta, \delta_{E, \nu}$.

Next let us discuss the mixing patterns in class 4, where the presence of phase factors is essential in the MNS matrix to have right values of mixing angles. There are 31 patterns classified into this category, but only a few mass matrix models with these mixing patterns have been proposed. These patterns thus provide potentially useful textures of lepton mass matrices.

At first, we discuss a well-known example $(U_E, U_\nu) = (A, A)$ which is derived from the mass matrices,

$$M_E \propto \begin{pmatrix} & & \\ & \lambda^2 & 1 \\ & \lambda^2 & 1 \end{pmatrix}, \quad M_\nu \propto \begin{pmatrix} & & \\ 1 & 1 & \\ 1 & 1 & \end{pmatrix}, \quad (10)$$

obtained in the models with $U(1)$ flavor symmetries^{15,16}. The mixing angles at leading order become

$$\theta_{\text{atm}} = \frac{\beta - \alpha}{2}, \quad \theta_\odot = (U_{\text{MNS}})_{e3} = 0, \quad (11)$$

that gives a SMA solution for the solar neutrino problem. The experimental constraint, $(U_{\text{MNS}})_{e3} \ll 1$, is satisfied, but for atmospheric neutrinos, tuning of phase values for $\beta - \alpha$ must be involved. In the presence of the phase matrix Q , a cancellation of two large mixing angles from U_E and U_ν can be avoided.

Another pattern for which concrete models have been constructed is the case of $(U_E, U_\nu) = (A, B)$, which can be derived from the mass matrices,

$$M_E \propto \begin{pmatrix} & & \\ & \lambda^2 & 1 \\ & \lambda^2 & 1 \end{pmatrix}, \quad M_\nu \propto \begin{pmatrix} & & \\ 1 & 1 & \epsilon \\ 1 & \epsilon & \end{pmatrix}. \quad (12)$$

These textures have been discussed in Ref.¹⁷. It is pointed out in Ref.¹⁶ that this combination of the mixing matrices is also derived from the mass matrices in Eq. (11). The mixing angle θ_{atm} is the same as Eq. (11), and a phase value, $\beta - \alpha \simeq \pi/2$ must be chosen to get maximal mixing of atmospheric neutrinos.

We here comment on the models¹⁸ which introduce the following type of mass matrices:

$$M_E \propto \begin{pmatrix} & & \\ \lambda^3 & \lambda^2 & 1 \\ \lambda^3 & \lambda^2 & 1 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}, \quad M_\nu \propto \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (13)$$

This corresponds to $(U_E, U_\nu) = (T, T)$, or to a special case $(U_E, U_\nu) = (D, D)$, where suitable MNS matrices can also be obtained by phase tuning.

As noted in the previous section, in class 4, there are several new mixing patterns which have not yet been discussed. Let us show an example of the case $(U_E, U_\nu) = (S, N)$. This mixing pattern could be derived from

$$M_E \propto \begin{pmatrix} & & \\ \lambda^4 & \lambda^2 & \\ \lambda^4 & \lambda^2 & \\ \lambda^2 & 1 & \end{pmatrix}, \quad M_\nu \propto \begin{pmatrix} 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 + \epsilon & 1 - \epsilon \\ \sqrt{2} & 1 - \epsilon & 1 + \epsilon \end{pmatrix}. \quad (14)$$

In this case, we have

$$\sin^2 2\theta_{\text{atm}} = \sin^2 2\theta_{\odot} = \left| \frac{1}{2} + \frac{1}{2\sqrt{2}} e^{i\alpha} \right|^2, \quad (U_{\text{MNS}})_{e3} = \left| \frac{1}{2} - \frac{1}{2\sqrt{2}} e^{i\alpha} \right|. \quad (15)$$

Here we would like to emphasize that a single phase tuning of α can ensure all the mixing angles to be consistent with experiments. Since the $(U_{\text{MNS}})_{e3}$ mixing in (15) is close to the Chooz bound, this pattern will be tested in the near future. Including the above example, we find several possible mixing patterns which no one has discussed so far (see Table 1). Model-construction utilizing such types of textures may be worth being performed.

4 Summary and Discussion

We have found that there are many allowed mixing patterns of charged leptons and neutrinos for the MNS matrix with bi-maximal or single maximal mixing. Among them, only 6 mixing patterns are allowed without any tuning of phase values. Interestingly, these patterns are indeed derived from the concrete models which have been proposed to account for the fermion mass hierarchy problem. The other patterns can give solutions of the observed neutrino anomalies depending on the choices of phase values. In this class, physically significant mixing patterns might be the ones which need a fewer numbers of phase tuning to have definite predictions consistent with experiments. We have found that 9 combinations satisfy this criterion, a single phase tuning required. They have not been studied enough in mass matrix models and would give new possibilities of model-construction. The phases to be tuned are not completely unphysical unlike the quark sector, but some of them could be connected to Majorana phases and CP violation in the lepton sector. Combined with these effects, the measurements of mixing angles $\sin^2 2\theta_{\odot}$ and $(U_{\text{MNS}})_{e3}$ will be important to select possible mixing patterns.

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References

1. Super-Kamiokande Collaboration, Y. Fukuda et al., Phys. Lett. **B433** (1998) 9; **B436** (1998) 33; Phys. Rev. Lett. **81** (1998) 1562; **82** (1999) 2644.
2. Y. Suzuki, Invited talk at *Neutrino 2000*, Sudbury, June 2000.
3. L. Wolfenstein, Phys. Rev. **D17** (1978) 2369; S.P. Mikheyev and A.Yu. Smirnov, Yad. Fiz. **42** (1985) 1441.
4. J.N. Bahcall, P.I. Krastev and A.Yu. Smirnov, Phys. Rev. **D58** (1998) 6016; G.L. Fogli, E. Lisi, D. Montanino and A. Palazzo, hep-ph/0008012; M.C. Gonzalez-Garcia and C. Peña-Garay, hep-ph/0009041; M.C. Gonzalez-Garcia, M. Maltoni, C. Peña-Garay and J.W.F. Valle, hep-ph/0009350.
5. Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. **28** (1962) 870.
6. V. Barger, S. Pakvasa, T.J. Weiler and K. Whisnant, Phys. Lett. **B437** (1998) 107; A.J. Baltz, A.S. Goldhaber and M. Goldhaber, Phys. Rev. Lett. **81** (1998) 5730.
7. N. Haba, J. Sato, M. Tanimoto and K. Yoshioka, hep-ph/0101334.
8. P.F. Harrison, D.H. Perkins and W.G. Scott, Phys. Lett. **B349** (1995) 137.
9. H. Fritzsch and Z. Xing, Phys. Lett. **B372** (1996) 265; **B440** (1998) 313; K. Kang and S. K. Kang, hep-ph/9802328; K. Kang, S. K. Kang, J. E. Kim and P. Ko, Mod. Phys. Lett. **A12** (1997) 1175; M. Fukugita, M. Tanimoto and T. Yanagida, Phys. Rev. **D57** (1998) 4429; M. Tanimoto, Phys. Rev. **D59** (1999) 017304-1; M. Tanimoto, T. Watari and T. Yanagida, Phys. Lett. **B461** (1999) 345; E.Kh. Akhmedov, G.C. Branco, F.R. Joaquim and J.I. Silva-Marcos, hep-ph/0008010.
10. The CHOOZ Collaboration, M. Apollonio et al., Phys. Lett. **B420** (1998) 397.
11. Y. Nomura and T. Yanagida, Phys. Rev. **D59** (1999) 017303; C.H. Albright and S.M. Barr, Phys. Lett. **B461** (1999) 218.
12. J. Sato and T. Yanagida, Phys. Lett. **B430** (1998) 127; C.H. Albright, K.S. Babu and S.M. Barr, Phys. Rev. Lett. **81** (1998) 1167; M. Bando and T. Kugo, Prog. Theor. Phys. **101** (1999) 1313.
13. F. Takayama and M. Yamaguchi, Phys. Lett. **B476** (2000) 116.
14. C. Jarlskog, M. Matsuda, S. Skadhauge and M. Tanimoto, Phys. Lett. **B449** (1999) 240; P.H. Frampton and S. Glashow, Phys. Lett. **B461** (1999) 95.
15. J.K. Elwood, N. Irges and P. Ramond, Phys. Rev. Lett. **81** (1998) 5064; N. Irges, S. Lavignac and P. Ramond, Phys. Rev. **D58** (1998) 035003.
16. F. Vissani, **JHEP** 9811 (1998) 025 ; J. Sato and T. Yanagida, Phys. Lett. **B493** (2000) 356; T. Yanagida and J. Sato, Nucl. Phys. Proc. Suppl.77

- (1999) 293.
17. Q. Shafi and Z. Tavartkiladze, Phys. Lett. **B482** (2000) 145; hep-ph/0101350.
 18. K.S. Babu and S.M. Barr, Phys. Lett. **B381** (1996) 202; S.M. Barr, Phys. Rev. **D55** (1997) 1659; M. Strassler, Phys. Lett. **B376** (1996) 119; N. Haba, Phys. Rev. **D59** (1999) 035011; K. Yoshioka, Mod. Phys. Lett. **A15** (2000) 29; J. Hisano, K. Kurosawa and Y. Nomura, Nucl. Phys. **B584** (2000) 3.

| U_E-U_ν | $\sin^2 2\theta_{\text{atm}}$ | $\sin^2 2\theta_\odot$ | $(U_{\text{MNS}})_{e3}$ | (# of) phases |
|-------------|-------------------------------|------------------------|-------------------------|---------------------------|
| $A-A$ | 0 – 1 | 0 | 0 | (0) |
| $A-B$ | 0 – 1 | 1 | 0 | (0) |
| $S-D$ | 8/9 | 0 | 0 | α (1) |
| $S-T$ | 8/9 | 1/4 – 1 | 0 | $\alpha + \delta_\nu$ (1) |
| $S-N$ | 0.73 | 0.73 | 0.15 | α (1) |
| $L-T$ | 8/9 | 1/4 – 1 | 0 | $\beta + \delta_\nu$ (1) |
| $L-N$ | 0.73 | 0.73 | 0.15 | β (1) |
| $L-D$ | 8/9 | 3/4 | 0 | β (1) |
| $B-T$ | 8/9 | 1/4 – 1 | 0 | α, β (2) |
| $B-H$ | 0.73 | 0.23 – 0.96 | 0.15 | β (1) |
| $B-L$ | 0.73 | 0.73 | 0.15 | β (1) |
| $B-D$ | 8/9 | 15/16 | 0 | α, β (2) |
| $H-T$ | 8/9 | 1/16 – 1 | 0 | α, β (2) |
| $H-B$ | 0.73 | 0.23 – 0.96 | 0.15 | $\alpha - \beta$ (1) |
| $H-N$ | 1 | 1 | 0 | α, β (2) |
| $H-A$ | 0.73 | 0.73 | 0.15 | $\alpha - \beta$ (1) |
| $H-D$ | 8/9 | 15/16 | 0 | $\alpha - \beta$ (1) |

Table 1: The mixing patterns in class 4. The values of mixing angles are shown in case of $(U_{\text{MNS}})_{e3}$ being minimal. The last column denotes the (number of) relevant phases which are needed for tuning $(U_{\text{MNS}})_{e3}$. The uncertainties in $\sin^2 2\theta_{\text{atm}}$ and $\sin^2 2\theta_\odot$ are fixed by additional phase tunings.

| U_E-U_ν | $\sin^2 2\theta_{\text{atm}}$ | $\sin^2 2\theta_\odot$ | $(U_{\text{MNS}})_{e3}$ | (# of) phases |
|-------------|-------------------------------|------------------------|-------------------------|---|
| $N-T$ | 8/9 | 1/4 - 1 | 0 | α, β (2) |
| $N-H$ | 0.73 | 0.23 - 0.96 | 0.15 | β (1) |
| $N-L$ | 0.73 | 0.73 | 0.15 | β (1) |
| $N-D$ | 8/9 | 15/16 | 0 | α, β (2) |
| $T-T$ | 0 - 1 | 0 - 1 | 0 | $\alpha + \delta_\nu, \beta + \delta_\nu$ (2) |
| $T-B$ | 1 | 1/9 - 1 | 0 | $\delta_E, \alpha - \beta$ (2) |
| $T-H$ | 1 | 1/9 - 1 | 0 | δ_E, β (2) |
| $T-N$ | 8/9 - 1 | 8/9 | 0 | α, β (2) |
| $T-A$ | 1 | 8/9 | 0 | $\delta_E, \alpha - \beta$ (2) |
| $T-L$ | 1 | 8/9 | 0 | δ_E, β (2) |
| $T-D$ | 0 - 1 | 0 - 1 | 0 | α, β (2) |
| $D-T$ | 0 - 1 | 1/4 - 1 | 0 | $\alpha + \delta_\nu$ (1) |
| $D-N$ | 1/36 - 0.96 | 0.73 | 0.15 | α (1) |
| $D-D$ | 0 - 1 | 0 | 0 | α (1) |

Table 1 (continued.)